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ON SITUATION ABOVE PRIME-CHAIN-PRIMAL VARIETES IN THE INTERPRETABILITY LATTICE

1. Preliminaries. The notion of interpretability of varieties was introduced by W.D. Neumann [1], and studied by O.C. Garcia and W. Taylor in their memoir [2]. For varietes $\mathcal{V}$ and $\mathcal{W}$ (not necessarily of the same type) $\mathcal{V}$ is interpretable in $\mathcal{W}$, in notation $\mathcal{V} \leq \mathcal{W}$, iff for each $\mathcal{V}$-operation $f_t(x_1,\dots,x_n)$ there exists a $\mathcal{W}$-term $F_t(x_1,\dots,x_n)$ such that if $\mathcal{A} = \langle A; G_s \rangle_{s\in S} \in \mathcal{W}$ then $\langle A; F_t^A \rangle \in \mathcal{V}$. Notice that the constants in the language of $\mathcal{V}$ must be interpreted as constants in the language of $\mathcal{W}$. As proved in [1], the class of all varieties forms alattice, $\mathbb{L}\mathbb{I}$, under the order relation which arises naturally from the quasiorder $\leq$, after the identification of mutually interpretable varieties. A persistent problem in the study of the lattice $\mathbb{L}\mathbb{I}$ has been whether any (naturally occurring) element is completely meet-irreducible. W. Taylor [3] proved that the zero element of $\mathbb{L}\mathbb{I}$ (i.e., the variety of sets) is not completely meet-irreducible, and just the same for the variety of all commutative groupoids. In [4] I proved the same result for the variety of bounded distributive lattices, and the aim of present note is to generalize this result for varieties generated by some pre-primal algebras.

Let A be a finite set. A clone on A is a set of finitary operations on A that contains the projections and is closed under superposition of functions [5]. It is well-known [6], that the set of all clones on A forms a complete lattice with respect to the set-theoretic inclusion; this lattice is an atomic and dual atomic lattice with finitely many atoms (i.e. minimal clones) and dual atoms (i.e. maximal = pre-primal clones). If $\rho \subseteq A^2$ is a partial order with least and greatest elements then $F = Pol_A \rho$ ("polymorphism" of ρ) is a maximal clone and all operations of F are monotone functions; these clone are monotone clones.

Let $\mathbf{C}_n = \langle C_n; \leq \rangle$ be the n-element clain. An algebra $\mathcal{A}_n = \langle C_n; f_i \rangle$ is pre-primal with respect to $\mathbf{C}_n$ if every fundamental operation f_i of $\mathcal{A}_n$ preserves the order of $\mathbf{C}_n$ and the clone of $\mathcal{A}_n$, consisting of all monotone functions, is identical with the clone of term operations of $\mathcal{A}_n$. The variety $\mathcal{V}(\mathcal{A}_n)$ generated by $\mathcal{A}_n$ will be called (finite) chain-primal variety. For example, $\mathcal{V}(\mathcal{A}_2)$, i.e. $\underline{2}$-primal variety ($\underline{2}$ is the two-element chain) is the variety of bounded distributive lattices. The general studies of varieties generated by pre-primal algebras, see K.Denecke [7]. Moreover, D.Lau [8] showed among others that all finite lattice ordered sets define finitely generated monotone clones; in particular, *the clone of* term functions of $\mathcal{A}_n$ is finitely generated by the operations max, min, and all unary monotone functions on $\mathbf{C}_n$ (for the case $n = 3$, see H.Machida [9]). Notice that G.Tardos found an eight-element bounded poset $\mathbf{T}$ such that the pre-primal clone of $\mathbf{T}$ is not finitely generated.

2. Result and comments.

Theorem. *Let p be a prime number. The variety $\mathcal{V}(\mathcal{A}_p)$ is not completely meet-irreducible in the lattice* $\mathbf{LI}$.

Corollary. *The class of varieties $(\mathcal{W} | \mathcal{V}(\mathcal{A}_p) < \mathcal{W})$ is a Mal'tcev class but it is not a strong Mal'tcev class.*

In particular, for $p = 2$ we obtain the result of [4].

We remark that R.McKenzie showed that *BA*, the variety of Boolean algebras, has the (unique) cover in $\mathbf{LI}$.

The method of proving of Theorem pushes us to following

Conjecture. *Theorem is true for every variety generated by the pre-primal algebra with respect to a finite lattice with prime number of elements.*

At this time is this conjecture is still unanswered in general.

3. Proof.

We shall construct varieties $\mathcal{W}_i > \mathcal{V}(\mathcal{A}_p)$, $i \in \omega$ with $Spec(\mathcal{W}_i) = \{1\}$ such that

$$\bigwedge_{i\ \mathbf{LI}} \mathcal{W}_i = \mathcal{V}(\mathcal{A}_p).$$

Construction of $\mathcal{W}_i$. Notice that in general we repeat the construction in [4]. Every variety $\mathcal{W}_i = \mathbf{HSP}(\mathcal{P}_i^*)$ is generated by the following algebra $\mathcal{P}_i^*$ where * means adding all constants, $\mathcal{P}_i = \langle P_i; F_1, F_2 \rangle$, $\langle P_i; F_1 \rangle$ is the modular lattice with all unary monotone operations whose ranges belong to tops of diamonds (see the left-side picture), and $\langle P_i; F_2 \rangle$ is the chain (on the right-side picture). Moreover the tops of the diamonds are left fixed and the atoms of the diamonds are ordered as arbitrary chains between corresponding tops.

Obviously $Spec(\mathcal{W}_i) = \{1\}$ for every i, thus in $\mathbf{LI}$

$$\mathcal{W}_i \not\leq \mathcal{V}(\mathcal{A}_p).$$

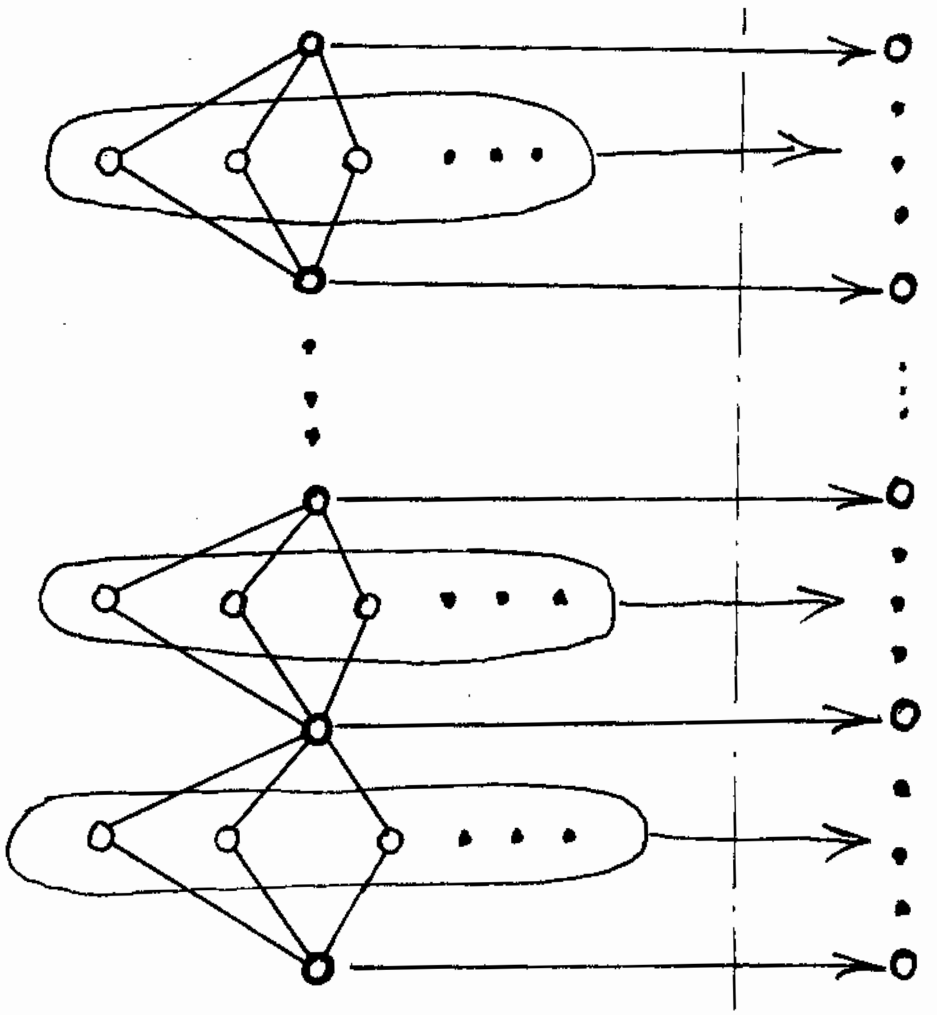
On the other side, $\mathcal{W}_i \geq \mathcal{V}(\mathcal{A}_p)$ for every $i = p, p+1, p+2, \ldots$. Thus

$$\wedge \mathcal{W}_i = \otimes \mathcal{W}_i \geq \mathcal{V}(\mathcal{A}_p),$$

where $\otimes$ means the nonindexed product of varieties.

It remains to show that $\wedge \mathcal{W}_i \leq \mathcal{V}(\mathcal{A}_p)$. For this we will show that there is a p- element chain (with all monotone functions on it) in the variety $\wedge \mathcal{W}_i$.

Now we fix a nonprincipal ultrafiller $\mathcal{F}$ on ω and define a map $\varphi : \Pi P_i \to \{0 = e_0, e_1, \ldots, e_{p-1} = 1\}$ by following:



$$\varphi(a_1,a_2,\ldots)=\begin{cases}0, & \text{if} \quad \{i \mid h(a_i)\le \ln 3(i+1)\}\in \mathcal{F}\\ e_1, & \text{if} \quad \{i \mid \ln 3(i+1)< h(a_i)\le \ln 3^2(i+1)\}\in \mathcal{F}\\ \ldots & \\ e_{p-2}, & \text{if} \quad \{i \mid \ln 3^{p-2}(i+1)< h(a_i)\le \ln 3^{p-1}(i+1)\}\in \mathcal{F}\\ 1, & \text{in other cases}\end{cases},$$

where h is the height- function in the modular lattice $\langle P_i;F_1\rangle$. Then $Ker\varphi$ is a congruence on $\otimes\mathcal{P}_i^*$ (remainding: both orders on $\mathcal{P}_i$ are the same on the tops of diamonds). Consequently, the algebra $\otimes\mathcal{P}_i^*/Ker\varphi$ is the p- element chain with all monotone operations on this chain.

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İNTERPRETASİYA QƏFƏSİNDƏ SADƏ-ZƏNCİRVARİ-PRİMAL ÇOXOBRAZLILARDAN YUXARIVƏZİYYƏT HAQQINDA

Məqalənin əsas nəticəsi:

Teorem. Fərz edək ki, $\mathbf{C}_p$ - p-sadə sayda elementləri və bütün monoton funksiyaları olan sonlu zəncirdir. Onda onun doğurduğu çoxobrazlı interpretasiya qəfəsində sonsuz Λ-ayrılmayan deyil.

Buradan bir nəticə kimi, güclü Maltsev sinifləri olmayan Maltsev sinifləri üçün yeni nümunələr tapılmışdır.

Мамедов О.М.

О СИТУАЦИИ НАД ПРОСТЫМИ-ЦЕПНО - ПРИМАЛЬНЫМИ МНОГООБРАЗИЯМИ В ИНТЕРПРЕТАЦИОННОЙ РЕШЕТКЕ

Основным утверждением статьи является следующая

Теорема. Пусть $\mathbf{C}_p$ - конечная цепь с простым числом p элементов и со множеством всех монотонных функций над ней. Многообразие, порожденное такой алгеброй $\mathbf{C}_p$, не является бесконечно Λ- неразложимой в решетке интерпретационных типов многообразий.

Отсюда в качестве следствия получаем новые примеры классов Мальцева, не являющихся сильными классами Мальцева.