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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

BASICITY PROPERTIES OF PERTURBED TRIGONOMETRIC SYSTEMS IN SOME NON-STANDARD SPACES OF FUNCTIONS

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GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the theme.

Solving many problems of mechanics and mathematical physics by the method of separation into Fourier variables, it is necessary to study basicity, completeness and minimality properties of eigen and adjoint systems of appropriate differential operator in the considered Banach space. Often, in spectral theory of operators there arise exponent type

$$\left\{e^{i[nt+\gamma_1(t)]}; e^{-i[kt+\gamma_2(t)]}\right\}_{n=0;k=1}^{\infty} \quad (1)$$

and sine type

$$\{\sin(nt+\alpha(t)\}_{n=0}^{\infty} \quad (2)$$

systems, here $\gamma_i(t),\ i=1,2$ and $\alpha(t)$ are the functions given in the interval $[-\pi,\pi]$ and $[0,\pi]$, respectively. In general, the basicity properties of the functions of such systems in various Banach spaces have been studied for a long time and have always attracted serious mathematical interest.

The basicity properties of the classic exponent sine and cosine systems were fully reflected in various textbooks and in the monographs of A.Zigmund, N.K.Bari, R.Edwards and others Back in 1934 A.Ingam[1] and N.Levinson[2] in their works have studied basicity properties of more complicated phase exponential systems in L_p Lebesgue spaces.

N.Wiener and R.Pelin's work had a stimulating effect on studying basicity properties of trigonometric systems. Afterwards, the basicity problems of spaces (1), (2) with various form complications of the phase in L_p spaces were studied in Y.U.Kazmin, S.M.Ponomaryev, A.A.Barmenkov,

[1] Ingam, A.A. A note on Fourier transforms // J.London Math.Soc., – 1934. 9, – p.29-32.
[2] Levinson, N. Gap and Density Theorems // Bull. Amer. Math.Soc., – 1941. 47:7, – p.543-547.

A.M.Sedletsky, E.I.Moiseev, G.G.Devdariani, B.T.Bilalov and others works.

The completeness and minimality problems of such systems in the space of continuous functions were studied by A.V.Bitsadze, E.I.Moiseev, A.M.Sedletsky, V.A.Molodenkov, A.P.Khromov, B.T.Bilalov. The completeness and minimality of discontinuous phase exponent and trigonometric systems in the space of piecewise continuous functions were studied in the works of W.Shepherd, C.Tranter and V.F.Salmanov.

Basicity problems of (1), (2) type systems in $W_p^1(a,b)$ Sobolev spaces have begun to be studied in E.I.Moiseev's[3] work. B.T.Bilalov[4] has studied the basicity of the systems $\{\sin\lambda_n t\}_{n=1}^{\infty}$ and $\{\cos\lambda_n t\}_{n=1}^{\infty}$ in the space $W_p^1(0,\pi)$. Basicity problems of these systems in a Sobolev space was studied by V.F.Salmanov[5] by another method. Based on this method, T.R.Muradov and V.F.Salmanov have found a necessary and sufficients condition for the basicity of the system (2) in the weighted Sobolev space $W_{p,\rho}^1(0,\pi)$.

The development of functional analysis and the demands of theory of partial differential equations led to creation of new type functional spaces called non-standard ones. So, starting from the 30s of the last century for the purpose of studying smoothness problems of partial differential equations, parametric spaces of differentiable functions began to be studied.

Parametric spaces have began to be studied for the first time by J. Morrey (afterwards called Morrey spaces), and then have been studied and developed by S.Kompanato, V.P.Il'in,

[3] Moiseev, E.I. On differential properties of expansions in a system of sines and cosines // Diffen. Urav., – 1996. v.32, № 1, – p.117-126.
[4] Билалов, Б.Т. Некоторые вопросы аппроксимации / Б.Т.Билалов. – Баку: Элм, – 2016. – 379 с.
[5] Salmanov, V.F. On the basicity of a system of cosince in the space $W_p^1(0,\pi)$ // Proc. of NAS of Azerb., – 2007. v.XXVII, – p.81-86

S.T.Zorko, V.S.Guliyev, A.M.Najafov, V.I.Burenkov, G.D.Fazio, M.Ragua, D.Fan, S.Lu, D.Yang, Y.Giga, T.Miyakama, M.A.Ragusa, L.Tang, J.Xu, Y.Y.Mammadov, J.J. Hasanov and others.

Since the 90s of the last century, for the functions from the space $W_{loc}^{1,p}(R^n)$ to be Jacobian local integrable, T.İvaniec and C.Sbordon in their work have introduced the spaces that are the modification of the Lebesgue space and afterwards called the grand Sobolev space and denoted as $L_{p)}$. In what follows there spaces were studied and developed by many mathematicians including A.Florenza, G.E.Karadzhov, A.Florenza, M.C.Formica and L.Donofrio, V.M.Kokilashvili, A.Meskhi, H.Rafero, L.Sh.Kadimova, A.M.Najafov and others as Lebesgue, small Lebesgue, grand Lebesgue-Morrey, grand-grand Lebesgue-Morrey, grand Sobolev Morrey, grand-grand Sobolev-Morrey and small-small Sobolev-Morrey spaces.

It is natural that after that the study of basicity properties of systems in non-standard spaces became an urgent issue. We can mention the works of B.T.Bilalov, A.Guliyeva, B.T.Bilalov and Z.G.Huseynov devoted to the basicity of the classical exponent system in Morrey and variable exponent Lebesgue space as the first works in this direction. The works of T.B. Gasymov, A.A. Guliyeva and T.B. Gasymov, G.V. Maharramova also are among these works.

Afterwards, B.T.Bilalov and his followers began to study the basicity properties of exponential and trigonometric systems with more complicated phase in Morrey and grand Lebesgue spaces.

In this dissertation work we study the basicity, completeness and minimality properties of the classic and perturbed exponential and trigonometric systems of piecewise smooth functions in Sobolev spaces, grand-Sobolev spaces, Sobolev-Morrey and weighted Sobolev-Morrey spaces.

Goals and objectives of the study. Studying the basicity, completeness and minimality properties of classic and perturbed exponential and trigonometric systems of piecewise smooth functions in Sobolev space, grand-Sobolev spaces, Sobolev-Morrey and weighted Sobolev-Morrey spaces.

Research methods. The methods of approximation theory, of functional analysis, of functions theory and of mathematical analysis were used to justify the obtained results.

The main clauses to be defended.

- Studying the basicity of the discontinuous space exponent system (1) in $KW_p^1(-\pi,\pi)$ Sobolev space of piecewise smooth functions;
- Constructing one separable subspace of grand-Sobolev space and studying the basicity properties of some trogonometric and exponent systems in this subspace;
- Constructing separable subspace of Sobolev-Morrey space and studying the basicity problems of some trigonometric and exponent systems in this subspace;
- Constructing a certain separable subspace of one weighted Sobolev-Morrey space and stadying the basicity of classic and trigonometric and exponent systems in this subspace.

Scientific novelty of the study. The following main results were obtained:

- Necessary and sufficient conditions for the basicity of discontinuous space (1) exponents system in the space $KW_p^1(-\pi,\pi)$ of piecewise smooth functions;
- Constructing one separable subspace of grand-Sobolev space and finding necessary and sufficient conditions for the basicity of some trigonometric and exponential systems in this subspace;
- One sepable subspace of grand-Sobolev space was constructed and necessary and sufficient conditions for the basicity of some trigonometric and exponential systems in this subspace were found;
- One separable subspace of Sobolev-Morrey space was structured, necessary and sufficient conditions for the basicity of

some trigonometric and exponent systems in this subspace were found;

- One separable subspace of Sobolev-Morrey space was structured and sufficient conditions for the basicity of classic trigonometric systems, necessary and sufficient conditions for the basicity of the exponent systems were found.

Theoretical and practical importance of the research.

The dissertation work is mainly of theoretical character. The obtained results are of particular scientific interest. Furthermore, the results obtained in the work can be used in justification of the finding the solution of boundary value problems stated for a class of partial differential equations by the Fourier method.

Approbation and application. The results of the dissertation work were reported in the seminars of the department of "Nonharmonic analysis" (head: corr. -member of ANAS B.T.Bilalov) of IMM and also "Operators, Functions, and Systems of Mathematical Physics" an International conference dedicated to the 70-th anniversary of the birth of Hamlet Isaxanli (KHAZAR University, 2018), "Modern Problems of Innovative Technologies in Oil and Gas Production and Applied Mathematics" International conference dedicated to the 90-th anniversary of academician Azad Khalil Mirzajanzade (Baku, 2019), "Complex Analysis and Approximation Theory" in International conference, (Ufa, 2019), "Spectral Theory and its Applications" International Workshop dedicated to the 80th anniversary of an academician Mirabbas Geogja Gasymov (Baku, 2019), in the IV Republican conference "Applied Problem of Mathematics and New İnformation Technologies" (Sumgayt, 2021) 5th İnternational E-conference on Mathematical Advanced and Applications, İCOMAA -2022 (Istanbul, 2022).

Applicant's personal contribution. All the results obtained in the work belong to the applicant.

Author's publications.

The author's 6 papers (2 of them SCOPUS) and 6 conference materials (5 of them international, 2 abroad) were published in journals recommended by the High Attestation Commission under the President of the Republic of Azerbaijan.

The name of organization where the work was executed

The work was executed in the department of "General Mathematics" Nakhchivan State University.

The total volune of the dissertation work indicating separately the volume of each structural unit. The dissertation work consists of title page 439 signs, introduction 1651 signs, contents 39378 signs, chapter I 82000 signs, chapter II 92000 signs, conclusions 2000 signs and a list of references with 89 names. Total volume of the work is 217468 signs.

THE CONTENT OF THE DISSERTATION WORK

In the introduction, the rationale of the work was justified and the problems raised have been commented.

The dissertation work consists of introduction, two chapters, conclusion and a list of references.

In chapter I the basicity of discontinuous space exponent system of a piecewise smooth functions in the Sobolev space $KW_p^1(-\pi,\pi)$ and the basicity some trigonometric and exponents systems in a separable subspace of grand-Sobolev space was studied. The main results of this chapter are in the works [1,2,3,7,8,9,11,12].

In section 1.1. a brief review of notions and facts to be used in the work is given.

In section 1.2. when the $\gamma_i(t)$ phase functions are piecewise linear functions in the form of

$$\gamma_i(t)=\begin{cases}\alpha_i t+\beta_1, -\pi<t<0\\ \alpha_i t+\beta_2,\ 0<t<\pi,\ i=1,2\end{cases}$$

the basicity of the system (1) in the Sobolev space of piecewise functions $KW_p^1(-\pi,\pi)$ are studied.

So, we denote $I_1=(-\pi,0), I_2=(0,\pi),\ I=I_1\cup I_2$. By $f\,|_M$ we denote the contraction of the function f to the set M (The set M is any subset of the domain of definition of the function $f(t)$). Now we determine the space $KW_p^1(-\pi,\pi)$ of piecewise differentiable functions:

$$f\in KW_p^1(-\pi,\pi)\Leftrightarrow f\,|_{I_k}\in W_p^1(I_k),\ k=1{,}2\,.$$

We determine the norm in $KW_p^1(-\pi,\pi)$ as follows:

$$\|f\|_{KW_p^1(-\pi,\pi)}=\sqrt{\|f\|^2_{W_p^1(I_1)}+\|f\|^2_{W_p^1(I_2)}}\,,$$

here

$$\|f\|_{W_p^1(a,b)}=\|f\|_p+\|f'\|_p\,,$$

$\|\cdot\|_p$ is ordinary $L_p(a,b)$ norm, i.e.

$$\|f\|_{L_p}=\left(\int_a^b|f(t)|^p\,dt\right)^{\frac{1}{p}},\ p>1\,.\|f\|_{L_p}=\left(\int_a^b|f(t)|^p\,dt\right)^{\frac{1}{p}},\ p>1$$

By $L_p(-\pi,\pi)$ we denote straight sum L_p of the space C^2 (C is a complex plane):

$$\mathsf{L}_p=L_p(-\pi,\pi)\oplus C^2\,.$$

Determine the norm in L_p as

$$\|\hat{u}\|_{\mathsf{L}_p}=\|u\|_{L_p}+|\lambda|+|\mu|$$

here $\hat{u}=(u,\lambda,\mu)\in\mathsf{L}_p,\ p>1$.

We use the following lemma.

Lemma 1. The operatordetermined in the form

$$A\hat{u}(t)=\begin{cases}\lambda+\int_{-\pi}^{t}u(\tau)d\tau,\ -\pi\le t<0,\\ \mu+\int_0^t u(\tau)d\tau,\ 0\le t\le\pi,\end{cases}$$

is an isomorphism acting from the space L_p to the space $KW_p^1(-\pi,\pi)$ In other words, the spaces L_p and $KW_p^1(-\pi,\pi)$ are isomorphic.

Theorem 1. Let $-\frac{1}{q}<\frac{\beta_1-\beta_2}{\pi}<\frac{1}{p}$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and $\omega=\alpha_1+\alpha_2+1+\frac{\beta_2-\beta_1}{\pi}$. Then the system

$$e^-(\cdot)\cup e^+(\cdot)\cup\left\{e^{i[(n+\alpha_1)t+\beta(t)]};e^{-i[(n+\alpha_2)t+\beta(t)]}\right\}_{n\in N}$$

forms a basis in the space $KW_p^1(-\pi,\pi)$ if and only if the condition

$$-\frac{1}{q}<\omega<\frac{1}{p}$$

is satisfied, here

$$e^-(t)=\begin{cases}1, & -\pi\le t<0,\\ 0, & 0\le t\le\pi,\end{cases}$$

$$e^+(t)=\begin{cases}0, & -\pi\le t<0,\\ 1, & 0\le t\le\pi,\end{cases}$$

are piecewise constant functions, $\alpha_i,\beta_i\in R$, $-\alpha_i\notin N$, $i=1,2$, are real parameters.

Corollary 1. Let $-\frac{1}{2q}<\frac{\beta}{\pi}<\frac{1}{2p}$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and $\omega=2\alpha+1-\frac{2\beta}{\pi}$. Then the system

$$e^-(\cdot)\cup e^+(\cdot)\cup\left\{e^{i[(n+\alpha \operatorname{sign} n)t+\beta \operatorname{sign} n \operatorname{sign} t]}\right\}_{n\in N}$$

forms a basis in the space $KW_p^1(-\pi,\pi)$ if and only if the inequality

$$-\frac{1}{q}<\omega<\frac{1}{p}$$

is satisfied, here

$$e^-(t) = \begin{cases} 1, -\pi \le t < 0, \\ 0, \ 0 \le t \le \pi, \end{cases}$$

$$e^+(t) = \begin{cases} 0, -\pi \le t < 0, \\ 1, \ 0 \le t \le \pi, \end{cases}$$

$\alpha, \beta \in R, -\alpha \notin N$ are real parameteres.

Remark 1. The system $e^-(t) \cup e^+(t) \cup t \cup \{e^{int}\}_{n \neq 0}$ is a basis in the space $KW_p^1(-\pi, \pi)$.

In section 1.3. we study the basicity, completeness and minimality properties of trigonometric systems in grand Sobolev spaces. As we have noted the grand Lebesgue space $L^{p)}$ was introduced to study Jacobian in an open set. This space has wide applications in partial differential equations, in interpolation theory and others.

Note that this space is not separable. Therefore here the basicity and approximation problems should be studied in an another way. M.I.Ismaylov has introduced a separable subspace $M^{p)}$ of functions of this space continuous with respect to a shift operator. This space is of great importance from the point of view of theory of differential functions. M.I.Ismayilov has studied the basicity of some trigonometric and exponent systems in the space $M^{p)}$.

Let $1 < p < \infty$. The space $L^{p)}(a,b)$ consisting of measurable functions determined in the interval $(a,b) \subset R$ and satisfying the condition

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{b-a} \int_a^b |f|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} < \infty$$

is called a grand-Lebesgue space.

Denote by $\tilde{M}^{p)}$ the set of all functions continuous with respect to shift operator, i.e. as $\delta \to 0$ satisfying the condition

$$\left\| \tilde{f}(\cdot + \delta) - \tilde{f}(\cdot) \right\|_{p)} \to 0$$

here

$$\tilde{f}(t) = \begin{cases} f(t), \; t \in (a,b), \\ 0, \; t \notin (a,b). \end{cases}$$

It is clear that $\tilde{M}^{p)}$ is a manifold in $L^{p)}(a,b)$. We denote the closure of this set with the norm $L^{p)}$ by $M^{p)}$. M.I.Ismaylov has shown that $M^{p)}$ forms a separable subspace.

By $W^1_{p)}(a,b)$ a space of functions whose derivative and itself are form the space $L^{p)}(a,b)$ and equipped with the norm

$$\|f\|_{W^1_{p)}} = \|f\|_{p)} + \|f'\|_{p)}.$$

We determine grand-Sobolev space as follows:

$$W^1_{p)}(a,b) = \left\{ f \mid f, \, f' \in L^p(a,b), \|f\|_{p)} + \|f'\|_{p)} < +\infty \right\}.$$

It is easy to show that it is a Banach space and is not separable. By $\tilde{M}W^1_{p)}(a,b)$ we denote a set of functions from $W^1_{p)}(a,b)$ and as $\delta \to 0$ satisfying the condition $\left\|\tilde{f}'(\cdot+\delta) - \tilde{f}'(\cdot)\right\|_{p)} \to 0$ here

$$\tilde{f}(t) = \begin{cases} f(t), \; t \in (a,b), \\ 0, \; t \notin (a,b). \end{cases}$$

As can be seen, the set $\tilde{M}W^1_{p)}(a,b)$ becoms a manifold in $W^1_{p)}(a,b)$. By $MW^1_{p)}(a,b)$ we denote its closure with respect to the norm $W^1_{p)}$.

The following lemma is valid.

Lemma 2. The operatpor $A(f,\lambda) = \lambda + \int\limits_a^t f(\tau)d\tau$ forms an isomorphism between the space $M^{p)}(a,b) \oplus C$ and the space $MW^1_{p)}(a,b)$, here C is a complex plane $1 < p < +\infty$.

The following theorems are proved by means of this lemma.

Theorem 2. Let $2\operatorname{Re}\beta+\frac{1}{p}\notin Z$, $\beta\neq 1$, $1<p<\infty$. Then $\left[\operatorname{Re}\beta+\frac{1}{2p}-\frac{1}{2}\right]=0$ is a necessary and sufficient condition for the system

$$1\cup\{\sin(n-\beta)t\}_{n\geq 1}$$

to be a basis in the space $MW^1_{p)}(0,\pi)$. And also for $\left[\operatorname{Re}\beta+\frac{1}{2p}-\frac{1}{2}\right]>0$ this system is not minimal in $MW^1_{p)}(0,\pi)$; For $\left[\operatorname{Re}\beta+\frac{1}{2p}-\frac{1}{2}\right]<0$ is minimal but not complete in $MW^1_{p)}(0,\pi)$ $[\cdot]$ is an entire part of the number.

Remark 2. For $\beta=1$ the system

$$1\cup\{t\}\cup\{\sin nt\}_{n\geq 1}$$

is a basis in $MW^1_{p)}(0,\pi)$.

Theorem 3. Let $2\operatorname{Re}\beta+\frac{1}{p}\notin Z$, $1<p<\infty$. Then $\left[\operatorname{Re}\beta+\frac{1}{2p}\right]=0$ is a necessary and sufficient condition for the system

$$1\cup\{\cos(n-\beta)t\}_{n\geq 1}$$

to be a basis in the space $MW^1_{p)}(0,\pi)$. And also, for $\left[\operatorname{Re}\beta+\frac{1}{2p}\right]>0$ this system is complete but is not minimal in $MW^1_{p)}(0,\pi)$ for $\left[\operatorname{Re}\beta+\frac{1}{2p}\right]<0$ it is minimal but not complete in $MW^1_{p)}(0,\pi)$.

In 1.4. the same problems are considered for exponent systems.

Theorem 4. Let $2\operatorname{Re}\beta+\frac{1}{p}\notin Z$, $\beta\neq 0$, $1<p<\infty$. Then $\left[2\operatorname{Re}\beta+\frac{1}{p}\right]=0$ is a sufficient and necessary condition for the system

$$1\cup\{e^{i(n-\beta \operatorname{sign} n)t}\}_{n\in Z}$$

to be a basis in the space $MW^{1}_{p)}(-\pi,\pi)$. And also, for $\left[2\operatorname{Re}\beta+\frac{1}{p}\right]<0$ this system is not complete; For $\left[2\operatorname{Re}\beta+\frac{1}{p}\right]>0$ this system is not minimal.

In chapter II at first we study the completeness, minimality and basicity properties of trigonometric systems in Sobolev-Morrey spaces. The $L^{p,\alpha}$ parameter spaces were first introduced by Morrey for studying smoothness problems of partial differential equations. The main results of this chapter in the author’s works [4,5,6,10].

It is known that this space is also a non-separable Banach space. B.T.Bilalov, A.D.Guliyeva introduced the subspace $M^{p,\alpha}$ being a separable subspace of functions continuous with respect to the shift operator of this space and proved the basicity of the classic exponent ssystem $\{e^{int}\}_{n\in Z}$ in this space.

Assume that $1\le p<\infty, 0\le\alpha\le 1$. The Banach space of all measurable functions determined in the interval $(a,b)\subset R$ and satisfying the condition

$$\|f\|_{L^{p,\alpha}(a,b)}=\sup_{I\subset(a,b)}\left(|I|^{\alpha-1}\int_I|f(t)|^p dt\right)^{\frac{1}{p}}<\infty$$

is called $L^{p,\alpha}(a,b)$ Morrey space, here $\sup$ is taken over all $I \subset (a,b)$ intervals. Note that when $0 \le \alpha_1 < \alpha_2 \le 1$ is satisfied, the inclusion $L^{p,\alpha_1} \subset L^{p,\alpha_2}$ is valid. It can be easily seen that $L^{p,1}(a,b) = L_p(a,b)$ and $L^{p,0}(a,b) = L_\infty(a,b)$. It is known that for $1 \le p < +\infty$ and $\alpha \in (0,1)$ the $L^{p,\alpha}(a,b)$ Morrey space is not separable and the space $C[a,b]$ continuous functions is not everywhere dense.

By $\tilde{M}^{p,\alpha}$ we denote the set all functions $f \in L^{p,\alpha}(a,b)$ continuous with respect the shift operator $\tilde{M}^{p,\alpha}$:

$$\tilde{M}^{p,\alpha}(a,b) = \{ f \in L^{p,\alpha}(a,b) :$$

$$: \left\| \tilde{f}(\cdot + \delta) - \tilde{f}(\cdot) \right\|_{L^{p,\alpha}} \to 0, \quad for \;\; \delta \to 0 \},$$

here

$$\tilde{f}(t) = \begin{cases} f(t), \; t \in (a,b), \\ 0, \;\; t \notin (a,b). \end{cases}$$

It is clear that $\tilde{M}^{p,\alpha}(a,b)$ is a manifeld in the space $L^{p,\alpha}(a,b)$. The closure of this set with respect to the norm $L^{p,\alpha}$ is denoted by $M^{p,\alpha}$ i.e. $\overline{\tilde{M}^{p,\alpha}} = M^{p,\alpha}$. For $1 \le p < \infty, 0 \le \alpha \le 1$ the space $M^{p,\alpha}(a,b)$ is a separable Banach space and $C_0^\infty(a,b)$ infinitely differentiable and with a finite support in (a,b) is dense here. Note that in the definition of $\tilde{M}^{p,\alpha}(a,b)$ the continution of the function $f(\cdot)$ outside of the interval (a,b) is understood as a zero.

In section 2.1. we construct a separable subspace of Sobolev-Morrey space and study the basicity properties of some trigonometris systems in this subspace. Assume that $0 \le \alpha \le 1, p \ge 1$. By $W^1_{p,\alpha}(a,b)$ we denote a space of functions with derivatives and itself from the space $L^{p,\alpha}(a,b)$ and equipped with the norm

$$\|f\|_{W^1_{p,\alpha}} = \|f\|_{L^{p,\alpha}} + \|f'\|_{L^{p,\alpha}}$$

We can show that this is a Banach space.

Now, denote the set of functions whose derivative is continuous with respect to the shift operator by $\widetilde{M}W^1_{p,\alpha}(a,b)$:

$$\widetilde{M}W^1_{p,\alpha}(a,b) =$$
$$= \{ f : f \in W^1_{p,\alpha}(a,b), \left\| \tilde{f}'(\cdot+\delta) - \tilde{f}'(\cdot) \right\|_{L^{p,\alpha}} \to 0, \delta \to 0 \ olduqda \}$$

It is clear that the set $\widetilde{M}W^1_{p,\alpha}(a,b)$ is a manifild of the space $W^1_{p,\alpha}(a,b)$. By $MW^1_{p,\alpha}(a,b)$ we denote the closure of $\widetilde{M}W^1_{p,\alpha}(a,b)$ with respect to the norm $W^1_{p,\alpha}$:

$$MW^1_{p,\alpha}(a,b) = \overline{\widetilde{M}W^1_{p,\alpha}(a,b)}.$$

Let $\mathsf{M}_{p,\alpha}$ be a straight sum of the space $M^{p,\alpha}(a,b)$ and the complex plane C:

$$\mathsf{M}_{p,\alpha} = M^{p,\alpha} \oplus C.$$

We denote the norm in the space $\mathsf{M}_{p,\alpha}$ as follows:

$$\|\hat{u}\|_{m_{p,\alpha}} = \|u\|_{L^{p,\alpha}} + |\lambda|, \ \forall \hat{u} = (u, \lambda) \in \mathsf{M}_{p,\alpha}.$$

The following lemmas are valid.

Lemma 3. The operator $(A\hat{u})(t) = \lambda + \int_a^t u(\tau)d\tau$ is an isomorphism acting from the space $L^{p,\alpha} \oplus C$ to the space $W^1_{p,\alpha}$.

This lemma yields that the space $W^1_{p,\alpha}(a,b)$ is also a non-separable space.

Lemma 4. The operator $(A\hat{u})(t) = \lambda + \int_a^t u(\tau)d\tau$ is an isomorphism acting from the space $\mathsf{M}_{p,\alpha}$ to the space $MW^1_{p,\alpha}$.

So, $\mathrm{M}_{p,\alpha}$ is a separable subspace. We prove the following theorem by means of these lemma.

Theorem 5. Let $2\operatorname{Re}\beta+\frac{\alpha}{p}\notin Z$, $\beta\neq 1$, $1<p<+\infty, 0<\alpha<1$. Then the system

$$1\cup\{\sin(n-\beta)\}_{n\geq 1}$$

forms a basis in the space $MW^1_{p,\alpha}(0,\pi)$ if and only if $\left[\operatorname{Re}\beta+\frac{\alpha}{2p}-\frac{1}{2}\right]=0$. Furthermore, for $\left[\operatorname{Re}\beta+\frac{\alpha}{2p}-\frac{1}{2}\right]<0$ this system is minimal but not complete in $MW^1_{p,\alpha}(0,\pi)$; For $\left[\operatorname{Re}\beta+\frac{\alpha}{2p}-\frac{1}{2}\right]>0$ this system is complete and not minimal in $MW^1_{p,\alpha}(0,\pi)$-.

Remark 3. For $\beta=1$ we can show that the system

$$1\cup\{t\}\cup\{\sin nt\}_{n\geq 1}$$

is a basis in the space $MW^1_{p,\alpha}(0,\pi)$.

The similar result was obtained for a cosine type system as well.

Theorem 6. Let $\operatorname{Re}\beta+\frac{\alpha}{2p}\notin Z$, $1<p<\infty, 0<\alpha<1$. Then the system

$$1\cup\{\cos(n-\beta)t\}_{n\geq 1}$$

forms a basis in the space $MW^1_{p,\alpha}(0,\pi)$ if and only if $\left[\operatorname{Re}\beta+\frac{\alpha}{2p}\right]=0$. Furthermore, for $\left[\operatorname{Re}\beta+\frac{\alpha}{2p}\right]<0$ this system is minimal but not complete in the space $MW^1_{p,\alpha}(0,\pi)$;

Furthermore, for $\left[\operatorname{Re}\beta+\frac{\alpha}{2p}\right]>0$ this system is not minimal but complete in the space $MW_{p,\alpha}^{1}(0,\pi)$.

In section 2.2. we study the basicity completeness and minimality properties of some perturbed exponents systems in the space $MW_{p,\alpha}^{1}(-\pi,\pi)$. So,

Theorem 7. Let, $1<p<+\infty$, $2\operatorname{Re}\beta+\frac{\alpha}{p}\in Z$, $0<\alpha<1$. Then $\left[2\operatorname{Re}\beta+\frac{\alpha}{p}\right]=0$ is a necessary and sufficient condition for the system

$$1\cup\{t\}\cup\{e^{i(n-\beta signn)t)}\}_{n\neq 0}$$

to be a basis in the space $MW_{p,\alpha}^{1}(-\pi,\pi)$. And also, for $\left[2\operatorname{Re}\beta+\frac{\alpha}{p}\right]<0$ this system is not complete in $MW_{p,\alpha}^{1}(-\pi,\pi)$; For $\left[2\operatorname{Re}\beta+\frac{\alpha}{p}\right]>0$ this system is not minimal in $MW_{p,\alpha}^{1}(-\pi,\pi)$.

Theorem 8. Let $1<p<+\infty$, $2\operatorname{Re}\beta+\frac{\alpha}{p}\notin Z$, $\beta\neq 0$, $0<\alpha<1$. Then $\left[2\operatorname{Re}\beta+\frac{\alpha}{p}\right]=0$ is a necessary and sufficient condition for the system

$$1\cup\{e^{i(n-\beta signn)t)}\}_{n\in Z}$$

to be basis in the space $MW_{p,\alpha}^{1}(-\pi,\pi)$. And also for $\left[2\operatorname{Re}\beta+\frac{\alpha}{p}\right]<0$ this system is not complete in $MW_{p,\alpha}^{1}(-\pi,\pi)$;

And also for $\left[2\operatorname{Re}\beta+\frac{\alpha}{p}\right]>0$ this system is not minimal in $MW_{p,\alpha}^{1}(-\pi,\pi)$.

In section 2.2. we study basicity, completeness and minimality properties of classic trigonometric systems in exponential weight Sobolev, Morrey spaces. The weighted Morrey spaces $L_{\nu}^{p,\alpha}(a,b)$ was determined by B.T.Bilalov, A.A.Huseynli, S.R.El-Shabravi as follows:

$$L_{\nu}^{p,\alpha}(a,b)=\{f:f\nu\in L^{p,\alpha}(a,b)\}$$

here $\nu(t)$ is a weight function determined in (a,b) .
The norm in this space is determined in as follows:

$$\|f\|_{p,\alpha,\nu}=\|\nu f\|_{p,\alpha}.$$

Let the function $f(\cdot)$ be determined in $[a,b]$
This function is continued to the interval $[2a-b,2b-a]$ as follows:

$$\tilde{f}(t)=\begin{cases} f(2a-x),\ x\in[2a-b,a) \\ f(x),\ x\in[a,b] \\ f(2b-x),\ x\in(b,2b-a]. \end{cases}$$

We denote the set of functions continuous with respect to the shift operator in $L_{\nu}^{p,\alpha}(a,b)$ by $\tilde{M}_{\nu}^{p,\alpha}$:

$$\tilde{M}_{\nu}^{p,\alpha}(a,b)=\{f\in L_{\nu}^{p,\alpha}(a,b):$$
$$:\left\|\hat{f}(\cdot+\delta)-\hat{f}(\cdot)\right\|_{p,\alpha,\nu}\to 0,\ as\ \delta\to 0\}.$$

It is clear that it is a manifold in $M_{\nu}^{p,\alpha}$. Its closure with respect to the norm $L_{\nu}^{p,\alpha}$ is denoted as $M_{\nu}^{p,\alpha}(a,b)$. It is shown that the $C_{0}^{\infty}(a,b)$ - finitely differentiable functions with a compact support is everywhere dense i.e. $M_{\nu}^{p,\alpha}(a,b)$ becomes a separable subspace. Sometimes the space $M_{\nu}^{p,\alpha}$ is called $L_{\nu}^{p,\alpha}$ Zorko subspace of the Morrey space.

In the similar way we determine the Sobolev Morrey space and its Zorko type subspace. So, we will call the space of functions with its derivatives and itself contained in the weighted Morrey space $L_{\nu}^{p,\alpha}(a,b)$ a weighted Sobolev-Morrey space $W_{p,\alpha,\nu}^{1}(a,b)$:

$$W_{p,\alpha,\nu}^{1}(a,b)=\{ f : f \in L_{\nu}^{p,\alpha}(a,b) \text{ və } f' \in L_{\nu}^{p,\alpha}(a,b)\} .$$

We denote the norm in this space as follows

$$\| f\|_{W_{p,\alpha,\nu}^{1}} =\| f\|_{p,\alpha,\nu} +\| f'\|_{p,\alpha,\nu}$$

We can show that this space is a non-separable Banach space.

Denote the set of functions with a continuous derivative with respect to the shift operator by $\tilde{M}W_{p,\alpha,\nu}^{1}(a,b)$:

$$\tilde{M}W_{p,\alpha,\nu}^{1}(a,b)=\{ f : \left\| \tilde{f}'(\cdot+\delta)-\tilde{f}'(\cdot)\right\| \to 0, \ \ as \ \ \delta \to 0\} .$$

It is clear that $\tilde{M}W_{p,\alpha,\nu}^{1}(a,b)$ is a manifold in the space $W_{p,\alpha,\nu}^{1}(a,b)$. We denote it by $W_{p,\alpha,\nu}^{1}$:

$$MW_{p,\alpha,\nu}^{1}(a,b) \equiv \overline{\tilde{M}W_{p,\alpha,\nu}^{1}(a,b)} .$$

It should be noted that in our work the weight function is taken in the form of a power function. $L_{p,\nu}(a,b)=\{ f : \|\nu f\|_{L_p} <+\infty\}$ is a weighted Lebesgue space. In this space, the norm is given as $\| f\|_{p,\nu} =\|\nu f\|_{L_p}$.

For obtaining the main results at first we prove such an inclusion property for weighted Morrey space:

Lemma 5. Let $p \in (1;\infty)$, $\lambda \in \left(\dfrac{\alpha-1}{p},\infty\right)$. Then for arbitrary $p_0 \in (1,\beta)$

$$L_{\nu}^{p,\alpha}(a,b) \subset L_{p,\nu}(a,b) \subset L_{p_0}(a,b) ,$$

here $\nu(t)=|t-a|^{\lambda}$, $\beta=\min\left\{p,\frac{p}{\lambda+1},2\right\}$.

By $M_{\nu}^{p,\alpha}(a,b)$ we denote the straight sum of the space $M_{\nu}^{p,\alpha}(a,b)$ with the complex plane C:

$$M_{\nu}^{p,\alpha}(a,b)=M_{\nu}^{p,\alpha}(a,b)\oplus C.$$

Determine the norm of this space as follows:

$$\|\hat{u}\|_{M_{\nu}^{p,\alpha}(a,b)}=\|u\|_{p,\alpha,\nu}+|\mu|,$$

here

$$\hat{u}=(u,\mu)\in m_{\nu}^{p,\alpha}(a,b).$$

We can show that this space is a Banach space.

Now we give the next lemma:

Lemma 6. Let $p\in(1,+\infty)$, $\lambda\in\left[\frac{\alpha-1}{p},\frac{\alpha}{p}+\frac{1}{p'}\right)$, $\nu(t)=|t-a|^{\lambda}$ və $p'=\frac{p}{p-1}$. Then the operator

$$A(u,\mu)=\mu+\int_a^t u(\tau)d\tau$$

is an isomorphism acting from the space $\mathrm{M}_{\nu}^{p,\alpha}(a,b)$ to the space $MW_{p,\alpha,\nu}^{1}(a,b)$.

It becomes clear from this lemma that $MW_{p,\alpha,\nu}^{1}(a,b)$ is a separable Banach space.

Now we pass to the main theorems:

Theorem 9. Let $p\in(1,+\infty)$, $p'=\frac{p}{p-1}$, $\nu(t)=t^{\lambda}$. Then:

1) For $\lambda\in\left(\frac{\alpha-1}{p},\frac{\alpha}{p}+\frac{1}{p'}\right)$ the system $1\cup\{\cos nt\}_{n\geq1}$ is a basis in the space $MW_{p,\alpha,\nu}^{1}(0,\pi)$.

2) For $\lambda \in \left(\frac{\alpha - 1}{p}, \frac{\alpha}{p} + \frac{1}{p'} \right)$ the system $1 \cup \{\cos nt\}_{n \geq 1}$ $W^1_{p,\alpha,v}(0,\pi)$ is minimal in the space;

Theorem 10. Let $p \in (1,+\infty)$, $p' = \frac{p}{p-1}$, $\nu(t) = t^{\lambda}$. Then:

1) For $\lambda \in \left(\frac{\alpha - 1}{p}, \frac{\alpha}{p} + \frac{1}{p'} \right)$ the system $1 \cup t \cup \{\sin nt\}_{n \geq 1}$ is a basis in the space $MW^1_{p,\alpha,v}(0,\pi)$.

2) For $\lambda \in \left(\frac{\alpha - 1}{p}, \frac{\alpha}{p} + \frac{1}{p'} \right)$ the system $1 \cup t \cup \{\sin nt\}_{n \geq 1}$ is minimal in the space $W^1_{p,\alpha,v}(0,\pi)$;

Theorem 11. Let $p \in (1,+\infty)$, $p' = \frac{p}{p-1}$, $\nu(t) = |t + \pi|^{\lambda}$. Then:

1) $t \cup \{e^{\mathrm{int}}\}_{n \in z}$ is a necessary and sufficient condition for the system $MW^1_{p,\alpha,v}\left(-\pi,\pi\right)$ to be a basis in the space

$$\lambda \in \left(\frac{\alpha - 1}{p}, \frac{\alpha}{p} + \frac{1}{p'} \right).$$

2) For $\lambda \in \left(\frac{\alpha - 1}{p}, \frac{\alpha}{p} + \frac{1}{p'} \right)$ the system $t \cup \{e^{\mathrm{int}}\}_{n \in z}$ is minimal in the space $W^1_{p,\alpha,v}(-\pi,\pi)$.

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CONCLUSION

In this dissertation work we study the basicity, completeness and minimality properties of the classic and perturbed exponential and trigonometric systems of piecewise smooth functions in Sobolev spaces, grand-Sobolev spaces, Sobolev-Morrey and weighted Sobolev-Morrey spaces..

The following main results were obtained:

- Necessary and sufficient conditions for the basicity of discontinuous space (1) exponents system in the space $KW_p^1(-\pi, \pi)$ of piecewise smooth functions;
- Constructing one separable subspace of grand-Sobolev space and finding necessary and sufficient conditions for the basicity of some trigonometric and exponential systems in this subspace;
- One sepable subspace of grand-Sobolev space was constructed and necessary and sufficient conditions for the basicity of some trigonometric and exponential systems in this subspace were found;
- One separable subspace of Sobolev-Morrey space was structured, necessary and sufficient conditions for the basicity of some trigonometric and exponent systems in this subspace were found;
- One separable subspace of Sobolev-Morrey space was structured and sufficient conditions for the basicity of classic trigonometric systems, necessary and sufficient conditions for the basicity of the exponent systems were found.

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