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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

BOUNDEDNESS OF FRACTIONAL MAXIMAL AND FRACTIONAL INTEGRAL OPERATORS IN GENERALIZED WEIGHTED MORREY SPACES ON HEISENBERG GROUP

Specialty: 1202.01 – Analysis and functional analysis

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the theme and development degree. The research conducted in the dissertation is dedicated to the boundedness of the fractional maximal operator, the fractional integral operator and the fractional maximal commutator in generalized Morrey spaces on Heisenberg group.

Additionally, the boundedness of the fractional maximal operator, the fractional integral operators, the higher order maximal commutator, and the higher order commutator commutator generated by the fractional integral operator, in generalized weighted Morrey phases on Heisenberg group are investigated in the dissertation.

In recent years, the theory of functional spaces in Heisenberg group and more general nilpotent groups has attracted significant attention from researchers. This attention arises primarily in the study of solutions to variable coefficient differential equations in manifolds.

In the 1970s, there arose a significant need to study problems in Harmonic Analysis on Heisenberg group in the investigation of certain problems in complex analysis with several variables. Consequently, serious research efforts were initiated on Harmonic Analysis in Heisenberg group. Esteemed mathematicians such as E. Stein, G. Folland, C. Fefferman, R. Beals, S. Tangavelu, L. Grafakos, S. Vodopyanov, V.S. Guliyev, V. Kokilashvili, A. Meskhi, and others have made substantial contributions in this field.

In the study of local properties of solutions to differential equations classical Morrey spaces play an important role. Morrey spaces were introduced by C.B. Morrey in 1938 in connection with the study of local behavior of solutions to second-order elliptic partial differential equations. In 1969, C. Petre included the boundedness theorem for Riesz potentials in Morrey spaces proved by Spanne. In 1988, F. Chiarenza and M. Frasca established the boundedness of the Hardy-Littlewood maximal operator in Morrey spaces.

In the 1990s extensive research on generalized Mori phases was initiated. In 1991, T. Mizuhara introduced generalized Morrey space and obtained the boundedness of some classical operators in this space. In 1994, E. Nakai proved the boundedness of the maximal operators, Riesz potentials, and the singular integral operators in generalized Morrey spaces. In the same year, V.S. Guliyev introduced local and complementary local Morrey-type spaces on the homogeneous Lie groups in his doctoral dissertation and investigated the boundedness of fractional integral operators and singular integral operators in these spaces on Lie groups.

Significant results on Morrey and generalized Morrey spaces have been obtained in the works of D. Adams, M. Taylor, S. Campanato, J. Xiao, D. Yang, Y. Giga, A.J. Mazzucato, G.D. Fazio, E. Nakai, V.P. Ilin, V.S. Guliyev, L. Softova, V.I. Burenkov, M. Ragusa, D. Palagachev, A. Gogotashvili, Y. Sawano, L. Tang, D. Yang, Jingshi Xu, B.T. Bilalov, R.M. Rzayev, R.Ch. Mustafayev, J.J. Hasanov, R.A. Bandaliyev, T.S Gadjiev, and others.

Weighted Morrey spaces were introduced by Y. Komori and S. Shirai in 2009. In this work, the boundedness of important operators in harmonic analysis, such as the maximal operator, Riesz potential, and singular integral operator, in Morrey spaces was investigated. In 2012 V.S. Guliyev, on the other hand, introduced the generalized weighted Morrey spaces that unify both generalized Morrey spaces and weighted Morrey spaces, and investigated the boundedness of classical operators and their commutators in generalized weighted Morrey spaces.

The object and subject of the study. The object and subject of the dissertation work are the fractional maximal operators, the fractional integral operators, the fractional maximal commutators, and the commutator of the fractional integral operator defined in Heisenberg group.

Goal and objectives of the study. The main purpose and objective of the dissertation work is to obtain conditions for the boundedness of the fractional maximal operator, the fractional integral operator, the fractional maximal commutator, and the commutator of the fractional integral operator, in generalized Morrey and generalized weighted Morrey spaces on Heisenberg group.

Research methods. In the dissertation work were utilized methods from functional analysis, the theory of real variable functions, the theory of functional spaces in Heisenberg group, harmonic analysis, embedding theorems, and operator theory.

The main theses to be defended.

1. The study of the boundedness of the fractional maximal operator and fractional maximal commutator in generalized Morrey spaces in Heisenberg group.
2. The study of boundedness of the fractional integral operator in generalized Morrey spaces in Heisenberg group.
3. The study of boundedness of the fractional maximal operator and the fractional maximal commutator in generalized weighted Morrey spaces in Heisenberg group.
4. The study of boundedness of fractional integral operator and its commutator in generalized weighted Morrey spaces in Heisenberg group.

Scientific novelty of the research.

1. Necessary and sufficient conditions have been obtained for the boundedness of the fractional maximal operator and the fractional maximal commutator in generalized Morrey spaces on Heisenberg group.
2. Necessary and sufficient conditions for the boundedness of the fractional integral operator in generalized Morrey spaces have been derived on Heisenberg group.
3. Sufficient conditions have been obtained for the boundedness of the fractional maximal operator and fractional maximal commutator in generalized weighted Morrey spaces on Heisenberg group.
4. Adequate sufficient conditions have been obtained for the boundedness of fractional integral operator and its commutator in generalized weighted Morrey spaces on Heisenberg group.

Theoretical and practical value. The obtained results in the work carry theoretical importance. The newly derived results hold particular interest in the theory of functional spaces in Heisenberg group. One of the achievements in harmonic analysis in recent years is attributed to the successful application of the ideas and techniques of the theory of fractional maximal and fractional integral operators determined in Heisenberg group to analysis in this or other fields. These ideas and methods have been and continue to be applied to specific topics such as partial differential equations, functional analysis, theory of functions, approximation theory problems, probability theory, harmonic analysis in homogeneous group, and other areas of mathematics.

Approbation and application. The results obtained in the dissertation have been reported at the seminars of the departments of "Mathematical Analysis" (head: corresponding member of ANAS, Prof. V.S. Guliyev), "Functional Analysis" (head: Prof. H.I. Aslanov), and "Non-harmonic Analysis" (head: corresponding member of ANAS, Prof. B.T. Bilalov) at the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

In addition, the results of the dissertation work were presented at the following conferences: The international conference "Operators in Mori-Type spaces and Their Applications" dedicated to the 60th anniversary of Professor Vagif S. Guliyev (Kırşehir 2017), the international conference "Modern Problems of Mathematics and Mechanics" dedicated to the 80th anniversary of Academician Akif Hajiyev (Baku 2017), the international conference "Actual Problems of Mathematics and Mechanics" dedicated to the 100th anniversary of Professor Goshgar Ahmadov (Baku 2017), the international conference "Modern Problems of Mathematics and Mechanics" dedicated to the 60th anniversary of the Institute of Mathematics and Mechanics (Baku 2019), the international conference on the occasion of the 110th anniversary of Academician Ibrahim Ibrahimov on the topic "Modern Problems of Mathematics and Mechanics" (Baku 2022), the 8th international conference on "Control and Optimization with Industrial Applications" (Baku 2022), and the international conference "Modern Problems of Mathematics and Mechanics" dedicated to the 100th anniversary of National Leader Heydar Aliyev (Baku 2023).

Author's personal contribution. The obtained results and proposals are attributed to the author.

Author's publications. In the recommended publishing houses by the HCC under the President of the Republic of Azerbaijan, the author has published 5 articles, 2 of which are included in the Web of Science Core Collection database and 3 in the SCOPUS database. Additionally, 7 theses (a total of 12 works) have been published related to dissertation work.

The name of the organization where the dissertation work is carried out. The dissertation work was performed at the department of "Mathematical Analysis" of the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately) The total volume of the dissertation work is 223607 characters (title page - 404 characters, table of contents - 1744 characters, introduction - 46654 characters, the first chapter - 108000 characters, the second chapter - 66000 characters, conclusion - 805 characters), and it consists of 124 pages. The bibliography used in the dissertation work consists of 102 references.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation consists of an introduction, two chapters, and a bibliography of the literature used.

The operators we investigate in the dissertation are defined in Heisenberg group, and in the first and second paragraphs of the first chapter of the dissertation some concepts about Heisenberg group and the generalized Morri spaces and the basic information, that will be needed in the proof of obtained results, are included.

Let us denote the points of the space R^{2n+1} as $u=(x,y,t)$, where $x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n)$, $t\in R$.

Heisenberg group H_n is a set where for any $u=(x,y,t)$ and $v=(x',y',t')$ in H_n

$$u \otimes v = uv = \left(x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y) \right).$$

Heisenberg norm of an arbitrary element $u = (x, y, t) \in H_n$ is given by

$$|u|_H \equiv |u| = \left(\left(|x|^2 + |y|^2 \right)^2 + t^2 \right)^{\frac{1}{4}}$$

With this norm, we define the open Heisenberg ball centered at $u = (x, y, t)$ with a radius r by $B(u, r) = \left\{ v \in H_n : \left| u^{-1} v \right| < r \right\}$.

The Haar measure of the sphere $B(u, r)$ is denoted as $|B(u, r)_H|$ or simply $|B(u, r)|$.

$Q = 2n + 2$ is the homogeneous dimension of Heisenberg group H_n.

Definition 1. Let $1 \le p < \infty$ and φ be a positive measurable function on $H_n \times (0, \infty)$. Generalized Morrey space $M_{p,\varphi}(H_n)$ is a space of all functions $f \in L_p^{loc}(H_n)$ with the finite norm

$$\| f \|_{WM_{p,\varphi}(H_n)} = \sup_{u \in H_n, r > 0} \frac{r^{-\frac{Q}{p}}}{\varphi(u, r)} \| f \|_{WL_p(B(u,r))}$$

In Addition, weak generalized Morrey space $WM_{p,\varphi}(H_n)$ is a space of all functions $f \in WL_p^{loc}(H_n)$ with the finite norm

$$\| f \|_{WM_{p,\varphi}(H_n)} = \sup_{u \in H_n, r > 0} \frac{r^{-\frac{Q}{p}}}{\varphi(u, r)} \| f \|_{WL_p(B(u,r))}.$$

Remark 1. We denote by Ω_p the sets of all positive measurable functions φ on $H_n \times (0, \infty)$ such that for all $t > 0$.

$$\sup_{u\in H_n}\left\|\frac{r^{-\frac{Q}{p}}}{\varphi(u,r)}\right\|_{L_\infty(t,\infty)}<\infty \quad \text{and} \quad \sup_{u\in H_n}\left\|\frac{1}{\varphi(u,r)}\right\|_{L_\infty(0,t)}<\infty$$

respectively.

Definition 2. A function $\varphi:(0,\infty)\to(0,\infty)$ is said to be almost increasing (respectively almost decreasing) if there exists a constant $C>0$ such that

$\varphi(r)\le C\varphi(s)$ (respectively $\varphi(r)\ge C\varphi(s)$) for $r\le s$.

Let $1\le p<\infty$. We denote $\mathcal{G}_p$ the set of all almost decreasing functions $\varphi:(0,\infty)\to(0,\infty)$ such that $r\in(0,\infty)\to r^{\frac{Q}{p}}\varphi(r)\in(0,\infty)$ is almost increasing.

The following lemma shows that both generalized Morrey spaces and weak generalized Morrey spaces are not trivial, and it is used for proving some boundedness theorems in the dissertation.

Lemma 1. L*et* $\varphi\in\mathcal{G}_p$, $1\le p<\infty$, $B_0=(u_0,r_0)$ *and* χ_{B_0} *is the characteristic function of the ball* B_0 *then* $\chi_{B_0}\in M_{p,\varphi}(H_n)$. *Moreover, there exist* $c>0$ *and* $C>0$ *such that*

$$\frac{c}{\varphi(r_0)}\le\left\|\chi_{B_0}\right\|_{WM_{p,\varphi}(H_n)}\le\left\|\chi_{B_0}\right\|_{M_{p,\varphi}(H_n)}\le\frac{C}{\varphi(r_0)}.$$

Remark 2. In the dissertation by $A\lesssim B$ we mean that $A\le CB$ with some positive constant C independent on appropriate quantities. If $A\lesssim B$ and $A\gtrsim B$, we write $A\approx B$ and say that A and B are equivalent.

In paragraph 1.3 we investigate the weak and strong boundedness of fractional maximal operator M_α in generalized Morrey spaces $M_{p,\varphi}(H_n)$. Additionally, we study the boundedness of fractional maximal commutator $M_{b,\alpha}$ and commutator $[b,M_\alpha]$

generated by the fractional maximal operator M_α on the generalized Morrey spaces.

Let $f, b \in L_1^{loc}(H_n)$. The fractional maximal operator M_α, the fractional maximal commutator $M_{b,\alpha}$ and the commutator $[b, M_\alpha]$ generated by the function b and the fractional maximal operator M_α are defined by

$$M_\alpha f(u) = \sup_{r>0} |B(u,r)|^{-1+\frac{\alpha}{Q}} \int_{B(u,r)} |f(v)| dv,$$

$$M_{b,\alpha} f(u) = \sup_{r>0} |B(u,r)|^{-1+\frac{\alpha}{Q}} \int_{B(u,r)} |b(u)-b(v)||f(v)| dv,$$

$$[b, M_\alpha] f(u) = b(u) M_\alpha f(u) - M_\alpha (bf)(u),$$

where $0 \le \alpha < Q$.

For $b \in L_1^{loc}(H_n)$ we define functions

$$b^-(u) = \begin{cases} 0, & b(u) \ge 0 \\ |b(u)|, & b(u) < 0 \end{cases}$$

and $b^+(u) = |b(u)| - b^-(u)$.

Definition 3. The space $BMO(H_n)$

is a space of all functions $b \in L_1^{loc}(H_n)$ by the finite

$$\|b\|_* = \sup_{u \in H_n, r>0} \frac{1}{|B(u,r)|} \int_{B(u,r)} |b(v) - b_{B(u,r)}| dv < \infty,$$

where

$$b_{B(u,r)} = \frac{1}{|B(u,r)|} \int_{B(u,r)} b(v) dv.$$

In this paragraph, we obtain the following main theorems.

Theorem 1. *Let* $1 \le p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $\varphi_1 \in \Omega_p$ *and* $\varphi_2 \in \Omega_q$.

1. *Then the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{q}} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(u,s) s^{\frac{Q}{p}} \le C\varphi_2(u,r) \tag{1}$$

for all $r>0$, *where* C *does not depend on* u *and* r, *is sufficient for the boundedness of the operator* M_α *from* $M_{p,\varphi_1}(H_n)$ *fo* $WM_{q,\varphi_2}(H_n)$. *Moreover, if* $p>1$, *the condition (1) is sufficient for the boundedness of the operator* M_α *from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$ *f.*

2. *If* $\varphi_1 \in \mathcal{G}_p$, *then the condition*

$$r^\alpha \varphi_1(r) \le C\varphi_2(r) \tag{2}$$

for all $r>0$, *where* C *does not depend on* r, *is necessery for the boundedness of the operator* M_α *from* $M_{p,\varphi_1}(H_n)$ *to* $WM_{q,\varphi_2}(H_n)$ *and from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$.

3. *If* $\varphi_1 \in \mathcal{G}_p$ *satisfies the condition*

$$\sup_{r<t<\infty} t^\alpha \varphi_1(t) \le C r^\alpha \varphi_1(r)$$

for all $r>0$, *where* C *does not depend on* r, *then the condition (2) is necessary and sufficient for the boundedness of the oprator* M_α *from* $M_{p,\varphi_1}(H_n)$ *to* $WM_{q,\varphi_2}(H_n)$ *and moreover, if* $p>1$, *then it is necessary and sufficient for the boundedness of the oprator* M_α *from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$.

Remark 3. The first part of Theorem 1 was proved in the work of Guliyev, V.S., Akbulut, A., Mammadov, Y.Y.[1].

Theorem 2. *Let* $1 \le p < q < \infty$, $0 < \alpha < \dfrac{Q}{p}$ *and* $\varphi \in \Omega_p$.

[1] Guliyev, V.S., Akbulut, A., Mammadov, Y.Y. Boundedness of fractional maximal operator and their higher order commutators in generalized Morrey spaces on Carnot groups // Acta Math. Sci. Ser. B Engl. Ed. – 2013, 33 (5), – p. 1329-1346.

1. *If* φ *satisfies the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(u,s) s^{\frac{Q}{p}} \le C\varphi(u,r) \tag{3}$$

for all $r>0$ *where* C *does not depend on* u *and* r*, then the condition*

$$r^{\alpha}\varphi(u,r) + \sup_{r<t<\infty} t^{\alpha}\varphi(u,t) \le C\varphi(u,r)^{\frac{p}{q}} \tag{4}$$

where C *does not depend on* u *and* r*, is sufficient for the boundedness of the operator* M_{α} *from* $M_{p,\varphi}(H_n)$ *to* $WM_{q,\varphi^{\frac{p}{q}}}(H_n)$*. Moreover, if* $p>1$*, then (4) is is sufficient for the boundedness of the operator* M_{α} *from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$*.*

2. *If* $\varphi \in \mathcal{G}_p$*, then the condition*

$$r^{\alpha}\varphi(r) \le C\varphi(r)^{\frac{p}{q}} \tag{5}$$

for all $r>0$*, where* C *does not depend on* r*, is necessary for the boundedness of the operator* M_{α} *from* $M_{p,\varphi}(H_n)$ *to* $WM_{q,\varphi^{\frac{p}{q}}}(H_n)$ *and from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$*.*

3. *If* $\varphi \in \mathcal{G}_p$ *satisfies the condition*

$$\sup_{r<t<\infty} t^{\alpha}\varphi(t) \le Cr^{\alpha}\varphi(r)$$

for all $r>0$*, where* C *does not depend on* r*, then (5) is necessasry and sufficient for the boundedness of the operator* M_{α} *from* $M_{p,\varphi}(H_n)$ *to* $WM_{q,\varphi^{\frac{p}{q}}}(H_n)$*. Moreover, if* $p>1$*, then (5) is necessasry and sufficient for the boundedness of the operator* M_{α} *from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$*.*

Teorem 3. *Let* $1<p<\infty$, $0\le\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $\varphi_1\in\Omega_p$, $\varphi_2\in\Omega_q$ *and* $b\in BMO(H_n)$.

1. *Then the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{q}}\left(1+\ln\frac{t}{r}\right)\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(u,s)s^{\frac{Q}{p}}\le C\varphi_2(u,r) \qquad (6)$$

for all $r>0$, *where* C *does not depend on* u *and* r, *is sufficient for the boundedness of the commutator* $M_{b,\alpha}$ *from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$.

2. *If* $\varphi_1\in\mathcal{G}_p$, *then the condition (2) is necessary for the boundedness of the commutator* $M_{b,\alpha}$ *from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$.

3. *If* $\varphi_1\in\mathcal{G}_p$ *satisfies the condition*

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)t^{\alpha}\varphi_1(t)\le Cr^{\alpha}\varphi_1(r)$$

for all $r>0$, *where* C *does not depend* r, *then the condition (2) is necessary and sufficient for the boundedness of the commutator* $M_{b,\alpha}$ *from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$.

Remark 4. The first part of Theorem 1 was proved in the work of Guliyev, V.S., Akbulut, A., Mammadov, Y.Y.[1].

Corollary 1. *Let,* $1<p<\infty$, $0\le\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $\varphi_1\in\Omega_p$, $\varphi_2\in\Omega_q$, $b\in BMO(H_n)$ *and* $b^-\in L_\infty(H_n)$. *If the pair* (φ_1,φ_2) *satisfy the condition (6), then the commutator* $[b,M_\alpha]$ *is bounded from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$.

Theorem 4. *Let* $1<p<q<\infty$, $0<\alpha<\frac{Q}{p}$, $\varphi\in\Omega_p$ *and* $b\in BMO(H_n)$.

1. *If* φ *is surjective for every* $u\in H_n$ *and satisfies the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}}\left(1+\ln\frac{t}{r}\right)\operatorname*{ess\,inf}_{t<s<\infty}\varphi(u,s)s^{\frac{Q}{p}} \le C\varphi(u,r) \qquad (7)$$

for all $r>0$*, where* C *does not depend on* u *and* r*, then the condition*

$$r^{\alpha}\varphi(u,r)+\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)t^{\alpha}\varphi(u,t)\le C\varphi(u,r)^{\frac{p}{q}} \qquad (8)$$

for all $r>0$*, where* C *does not depend on* u *and* r*, is sufficient for the boundedness of the* $M_{b,\alpha}$ *from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$.

2. *If* $\varphi\in\mathcal{G}_p$*, then the condition (5) is necessary for the boundedness of the* $M_{b,\alpha}$ *from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$.
3. *If* $\varphi\in\mathcal{G}_p$ *satisfies the condition*

$$\sup_{r<t<\infty}\left(1+\frac{t}{r}\right)t^{\alpha}\varphi(t)\le Cr^{\alpha}\varphi(r)$$

for all $r>0$*, where* C *does not depend on* r*, then the condition (5) is necessary and sufficient for the boundedness of the* $M_{b,\alpha}$ *from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$.

Corollary 2. *Let* $1<p<q<\infty$, $0<\alpha<\frac{Q}{p}$, $\varphi\in\Omega_p$, $b\in BMO(H_n)$ *and* $b^{-}\in L_{\infty}(H_n)$*. If* φ *is surjective for every* $u\in H_n$ *and satisfies the conditions (7) and (8), then* $[b,M_{\alpha}]$ *is bounded from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$.

In paragraph 1.4 we investigate the weak and strong boundedness of the fractional integral operator I_{α} in the generalized Morrey spaces $M_{p,\varphi}(H_n)$.

Let $f \in L_1^{loc}(H_n)$. The fractional integral operator I_α is defined as

$$I_\alpha f(u) = \int_{H_n} \frac{f(v)}{|u^{-1}v|^{Q-\alpha}} dv,$$

where $0 < \alpha < Q$.

We get the following main results in this paragraph.

Theorem 5. Let $1 \le p < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $\varphi_1 \in \Omega_p$ *and* $\varphi_2 \in \Omega_q$.

1. *Then the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(u,s) s^{\frac{Q}{p}}}{t^{\frac{Q}{q}}} \frac{dt}{t} \le C\varphi_2(u,r) \tag{9}$$

for all $r > 0$, where C does not depend on u and r, is sufficient for the boundedness of the operator I_α from $M_{p,\varphi_1}(H_n)$ to $WM_{q,\varphi_2}(H_n)$. Moreover, if $p > 1$, then the condition (9) is sufficient for the bounedeness of the operator I_α from $M_{p,\varphi_1}(H_n)$ to $M_{q,\varphi_2}(H_n)$.

2. *If $\varphi_1 \in \mathcal{G}_p$, then the condition (2) is necessary for the boundedness of the operator I_α from $M_{p,\varphi_1}(H_n)$ to $WM_{q,\varphi_2}(H_n)$ and from $M_{p,\varphi_1}(H_n)$ to $M_{q,\varphi_2}(H_n)$.*

3. *If $\varphi_1 \in \mathcal{G}_p$ satisfies the regularity condition*

$$\int_r^\infty t^{\alpha-1}\varphi_1(t)dt \le Cr^\alpha \varphi_1(r)$$

for all $r > 0$, where C does not depend on r, then the condition (2) is necessary and sufficient for the boundedness of the operator I_α from $M_{p,\varphi_1}(H_n)$ to $WM_{q,\varphi_2}(H_n)$. Moreover, if $p > 1$, then *the condition (2) is necessary and sufficient for the boundedness of the operator* I_α *from* $M_{p,\varphi_1}(H_n)$ *to* $M_{q,\varphi_2}(H_n)$.

Remark 5. The first part of Theorem 1 was proved in the work of Guliyev, V.S., Eroglu, A., Mammadov Y.Y.[2].

Teorem 6. *Let* $1 \le p < q < \infty$, $0 < \alpha < \dfrac{Q}{p}$ *and* $\varphi \in \Omega_p$.

1. *If the fuction* φ *satisfies the condition (3), then the coindition*

$$r^\alpha \varphi(u,r) + \int_r^\infty t^{\alpha-1}\varphi(u,t)dt \le C\varphi(u,r)^{\frac{p}{q}} \qquad (10)$$

for all $r > 0$, *where* C *does not depend on* u *and* r, *is sufficient for the boundedness of the operator* I_α *from* $M_{p,\varphi}(H_n)$ *to* $WM_{q,\varphi^{\frac{p}{q}}}(H_n)$. *Moreover, if* $p > 1$, *then the condition (10) is sufficient for the boundedness of* I_α *from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$.

2. *If* $\varphi \in \mathcal{G}_p$, *then the condition (5) is necessary for the boundedness of the operator* I_α *from* $M_{p,\varphi}(H_n)$ *to* $WM_{q,\varphi^{\frac{p}{q}}}(H_n)$ *and from* $M_{p,\varphi}(H_n)$ *to* $M_{q,\varphi^{\frac{p}{q}}}(H_n)$.

3. *If* $\varphi \in \mathcal{G}_p$ *satisfies the regularity condition*

$$\int_r^\infty t^{\alpha-1}\varphi(t)dt \le Cr^\alpha \varphi(r)$$

for all $r > 0$, *where* C *does not depend on* r, *then the condition (5) is necessary and sufficient for the boundedness of the operator*

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[2] Guliyev, V.S., Eroglu, A., Mammadov Y.Y. Riesz potential in generalized Morrey spaces on the Heisenberg group // Problems in mathematical analysis, – 2013, No. 68. J. Math. Sci. (N. Y.) 189 (3), – p. 365-382.

I_α from $M_{p,\varphi}(H_n)$ to $WM_{q,\varphi^{\frac{p}{q}}}(H_n)$ and moreover, if $p>1$, then the condition (5) is necessary and sufficient for the boundedness of the operator I_α from $M_{p,\varphi}(H_n)$ to $M_{q,\varphi^{\frac{p}{q}}}(H_n)$.

In the first paragraph of the second chapter of the dissertation work it were given some concepts about generalized weighted Morrey spaces and the essential information required for the second and third paragraphs of the second chapter.

By a weight function, briefly weight, on H_n we mean a locally integrable function on H_n which takes values in $(0,\infty)$ almost everywhere.

For a weight ω and a measurable set E, we define a function $\omega(E)=\int_E \omega(u)du$.

Let $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$ and $B\subset H_n$ is an arbitrary ball. A weight function ω is in the Muckenhoupt-Weeden class $A_{p,q}(H_n)$ if

$$[\omega]_{A_{p,q}}=\sup_B[\omega]_{A_{p,q}(B)}=\sup_B\left(\frac{1}{|B|}\int_B\omega(u)^q du\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B\omega(u)^{-p'}du\right)^{\frac{1}{p'}}<\infty$$

Let $1<q<\infty$. A weight function ω is in the Muckenhoupt-Weeden class $A_{1,q}(H_n)$ if

$$[\omega]_{A_{1,q}}=\sup_B[\omega]_{A_{1.q}(B)}=\sup_B\left(\frac{1}{|B|}\int_B\omega(u)^q du\right)^{\frac{1}{q}}\left(\operatorname*{ess\,sup}_{u\in B}\frac{1}{\omega(u)}\right)<\infty$$

Definition 4. Let $1\le p<\infty$, φ be a positive measurable function on $H_n\times(0,\infty)$ and ω be a weight function. generalized weighted Morrey space, denoted by $M_{p,\varphi}(H_n,\omega)\equiv M_{p,\varphi}(\omega)$, is the the space of functions $f\in L^{loc}_{p,\omega}(H_n)$ with the finite norm

$$\| f \|_{M_{p,\varphi}(\omega)} = \sup_{u \in H_n, r>0} \varphi(u,r)^{-1} \omega(B(u,r))^{-\frac{1}{p}} \| f \|_{L_{p,\omega}(B(u,r))}.$$

Moreover, generalized weighted Morrey spaces denoted by $WM_{p,\varphi}(H_n,\omega) \equiv WM_{p,\varphi}(\omega)$ is the space of functions $f \in WL_{p,\omega}^{loc}(H_n)$ with the finite norm

$$\| f \|_{WM_{p,\varphi}(\omega)} = \sup_{u \in H_n, r>0} \varphi(u,r)^{-1} \omega(B(u,r))^{-\frac{1}{p}} \| f \|_{WL_{p,\omega}(B(u,r))}.$$

In paragraph 2.2 we study the weak and strong boundedness of the fractional maximal operator M_α restrictions in the generalized weighted Morrey spaces $M_{p,\varphi}(H_n)$. In addition, we study the boundedness of the higher order maximal commutator $M_{b,\alpha,k}$ and commutator $[b, M_\alpha]$ generated by the fractional maximal operator M_α in generalized weighted Morrey spaces $M_{p,\varphi}(H_n)$.

Let $f, b \in L_1^{loc}(H_n)$. The higher order maximal commutator $M_{b,\alpha,k}$ is defined by

$$M_{b,\alpha,k} f(u) = \sup_{r>0} |B(u,r)|^{-1+\frac{\alpha}{Q}} \int_{B(u,r)} |b(u)-b(v)|^k |f(v)| dv,$$

where $0 \le \alpha < Q$ and k is a positive integer number.

In the second paragraph we have obtained the following results.

Theorem 7. *Let* $1 \le p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $\omega \in A_{p,q}(H_n)$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ *and the pair* (φ_1, φ_2) *satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(u,s) \left(\omega^p(B(u,s))\right)^{\frac{1}{p}}}{\left(\omega^q(B(u,t))\right)^{\frac{1}{q}}} \le C \varphi_2(u,r)$$

for all $r>0$*, where the constant* C *does not depend on* u *and* r*. Then for* $p>1$ *the operator* M_α *is bounded from from* $M_{p,\varphi_1}\left(H_n,\omega^p\right)$ *to* $M_{q,\varphi_2}\left(H_n,\omega^q\right)$*, and for* $p=1$ *the operator* M_α *is bounded from* $M_{1,\varphi_1}(H_n,\omega)$ *to* $WM_{q,\varphi_2}\left(H_n,\omega^q\right)$*. That is, for* $p>1$

$$\left\|M_\alpha f\right\|_{M_{q,\varphi_2}\left(\omega^q\right)} \le C\|f\|_{M_{p,\varphi_1}\left(\omega^p\right)},$$

where the constant C *does not depend on* f*, and for* $p=1$

$$\left\|M_\alpha f\right\|_{WM_{q,\varphi_2}\left(\omega^q\right)} \le C\|f\|_{M_{1,\varphi_1}(\omega)},$$

where the constant C *does not depend on* f*.*

Teorem 8. *Let* $1<p<\infty$, $0\le\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $\omega\in A_{p,q}(H_n)$, $b\in BMO(H_n)$, $\varphi_1\in\Omega_p$, $\varphi_2\in\Omega_q$ *and the pair* (φ_1,φ_2) *satisfy the condition*

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)^k \frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(u,s)\left(\omega^p(B(u,s))\right)^{\frac{1}{p}}}{\left(\omega^q(B(u,t))\right)^{\frac{1}{q}}} \le C\varphi_2(u,r) \quad (11)$$

for all $r>0$*, where the constant* C *does not depend on* u *and* r*. Then the commutator* $M_{b,\alpha,k}$ *is bounded from* $M_{p,\varphi_1}\left(H_n,\omega^p\right)$ *to* $M_{q,\varphi_2}\left(H_n,\omega^q\right)$*. Moreover,*

$$\left\|M_{b,\alpha,k} f\right\|_{M_{q,\varphi_2}\left(\omega^q\right)} \le C\|b\|_*^k\|f\|_{M_{p,\varphi_1}\left(\omega^p\right)},$$

where the constant C *does not depend on* f*.*

Corollary 3. *Let* $1<p<\infty$, $0\le\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $\omega\in A_{p,q}(H_n)$, $b\in BMO(H_n)$, $b^-\in L_\infty(H_n)$, $\varphi_1\in\Omega_p$, $\varphi_2\in\Omega_q$ *and* (φ_1,φ_2) *satisfy the condition (11). Then the commutator* $[b,M_\alpha]$ *is bounded from* $M_{p,\varphi_1}\left(H_n,\omega^p\right)$ *to* $M_{q,\varphi_2}\left(H_n,\omega^q\right)$*.*

In paragraph 2.3 we consider the weak and strong boundedness of the fractional integral operator I_α in generalized weighted Morrey spaces $M_{p,\varphi}(H_n,\omega)$. At the same time, we study boundeddness of the higher order commutator $[b,I_\alpha]^k$ generated by the fractional integral operator I_α in generalized weighted Morrey spaces $M_{p,\varphi}(H_n,\omega)$.

Let $f,b\in L_1^{loc}(H_n)$. The higher order commutator $[b,I_\alpha]^k$ generated by the function b and the fractional integral operator I_α is defined by

$$[b,I_\alpha]^k f(u)=\int\limits_{H_n}(b(u)-b(v))^k\frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}}dv,$$

where $0<\alpha<Q$ and k is a positive integer number.

Teorem 9. *Let* $1\le p<\infty$, $0<\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $\omega\in A_{p,q}(H_n)$, $\varphi_1\in\Omega_p$, $\varphi_2\in\Omega_q$ *and* (φ_1,φ_2) *satisfy the condition*

$$\int\limits_r^\infty\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(u,s)\left(\omega^p(B(u,s))\right)^{\frac{1}{p}}}{\left(\omega^q(B(u,t))\right)^{\frac{1}{q}}}\frac{dt}{t}\le C\varphi_2(u,r)$$

for all $r>0$*, where the constant* C *does not depend on* u *and* r*. Then, the operator* I_α *is bounded from* $M_{p,\varphi_1}(H_n,\omega^p)$ *to* $M_{q,\varphi_2}(H_n,\omega^q)$ *for* $p>1$*, and from* $M_{p,\varphi_1}(H_n,\omega^p)$ *to* $WM_{q,\varphi_2}(H_n,\omega^q)$ *for* $p=1$*. That is, for* $p>1$,

$$\|I_\alpha f\|_{M_{q,\varphi_2}(\omega^q)}\le C\|f\|_{M_{p,\varphi_1}(\omega^p)},$$

where the constant C *does not depend on* f*, and for* $p=1$

$$\|I_\alpha f\|_{WM_{q,\varphi_2}(\omega^q)}\le C\|f\|_{M_{1,\varphi_1}(\omega)},$$

where the constant C *does not depend on* f.

Teorem 10. *Let* $1 < p < \infty$, $0 < \alpha < \dfrac{Q}{p}$, $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{Q}$, $\omega \in A_{p,q}(H_n)$, $b \in BMO(H_n)$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ *and the pair* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right)^k \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(u,s)\left(\omega^p(B(u,s))\right)^{\frac{1}{p}}}{\left(\omega^q(B(u,t))\right)^{\frac{1}{q}}} \frac{dt}{t} \le C\varphi_2(u,r)$$

for all $r > 0$, *where the constant* C *does not depend on* u *and* r. *Then the commutator* $[b, I_\alpha]^k$ *is bounded from* $M_{p,\varphi_1}(H_n, \omega^p)$ *to* $M_{q,\varphi_2}(H_n, \omega^q)$. *Moreover,*

$$\left\| [b, I_\alpha]^k f \right\|_{M_{q,\varphi_2}(\omega^q)} \le C \|b\|_*^k \|f\|_{M_{p,\varphi_1}(\omega^p)},$$

where the constant C *does not depend on* f.

CONCLUSION

The dissertation work is dedicated to the boundedness of the fractional maximal operator, fractional integral operator, fractional maximal commutator, and the commutator of fractional integral operator in generalized Morrey spaces and generalized weighted Morrey spaces on Heisenberg group, which are important operators in harmonic analysis.

The main results of the dissertation work are as follows:

1. Necessary and sufficient conditions have been found for the boundednes of the fractional maximal operator and fractional maximal commutator in generalized Morrey spaces on Heisenberg group.

2. Necessary and sufficient conditions have been obtained for the boundedness of the fractional integral operator in generalized Morrey spaces on Heisenberg group.

3. Sufficient conditions have been found for the boundedness of the fractional maximal operator and fractional maximal commutator in generalized weighted Morrey spaces on Heisenberg group.

4. Sufficient conditions have been obtained for the boundedness of the fractional integral operator and its commutator in generalized weighted Morrey spaces on Heisenberg group.

The main results of the dissertation work were published in the following works:

1. Eroglu, A. Azizov, J.V. A note of the fractional integral operators in generalized Morrey spaces on the Heisenberg group // – Baku: Transactions of NAS of Azerbaijan, Issue Mathematics, – 2017, 37(1), – p. 86-91.
2. Azizov, J.V. Characterizations for the fractional integral operators in generalized Morrey spaces on Carnot groups // International conference on “Operators in Morrey-type spaces and applications” dedicated to 60-th birthday of professor V.S.Guliyev, -Kırşehir – Turkey, -10-13 July, -2017, -p.113.
3. Azizov, J.V, Characterizations for the fractional maximal operators in generalized Morrey spaces on Heisenberg groups // International conference “Modern problems of mathematics and mechanics” devoted to the 100-th anniversary of corr.- member of ANAS Goshgar Ahmedov, – Baku: –02-03 November, – 2017 – p. 68.
4. Eroglu, A. Guliyev, V.S. Azizov, J.V. Characterizations for the fractional integral operators in generalized Morrey spaces on Carnot groups // Mat. Zametki, – 2017, (102), no. 5, – p. 789-804; translation in Math. Notes, – 2017, 102, no. 5-6. – p. 722-734.
5. Eroglu, A. Azizov, J.V., Guliyev, V.S. Fractional maximal operator and its commutators in generalized Morrey spaces on Heisenberg group // – Baku: Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., – 2018, 44 (2), – p. 304-317.
6. Azizov, J.V. Fractional maximal commutator in generalized Morrey spaces on Heisenberg group // International conference “Modern problems of mathematics and mechanics” devoted of the 60th anniversary of the Institute of Mathematics and Mechanics of ANAS, – Baku: –23-25 October, – 2019. – p. 142-143.
7. Azizov, J.V. Fractional integral operator and its higher order commutators in generalized weighted Morrey spaces on

Heisenberg group // – Baku: Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, 39 (4), 25-36.
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9. Azizov, J.V. Fractional maximal operator and its higher order commutators in generalized weighted Morrey spaces on Heisenberg group // – Baku: Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, – 2020, 40 (1), – p. 66-78.
10. Azizov, J.V. The boundedness of fractional integral operators in generalized weighted Morrey spaces on Heisenberg groups // International conference “Modern problems of mathematics and mechanics” dedicated to the 110-th anniversary of the academician Ibrahim Ibrahimov, – Baku: – 29 June-01 July, – 2022, – p. 58-59.
11. Azizov, J.V. The boundedness of commutator of fractional integral operators in generalized weighted Morrey spaces on Heisenberg groups // The 8th International conference on “Control and optimization with industrial applications”, – Baku: COIA 2022, – 26-28 August, – 2022, v. I, – p. 132-134.
12. Azizov, J.V. Boundedness of fractional maximal operator in generalized weighted Morrey spaces on Heisenberg groups // The international conference “Modern problems of mathematics and mechanics” dedicated to the 100-th anniversary of the national leader Heydar Aliyev, – Baku: – 26-28 April, – 2023, – p. 124-126.

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