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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

SPECTRAL PROPERTIES OF THE SYSTEM OF EIGEN AND ASSOCIATED FUNCTIONS OF DİRAC’S DİSCONTİNUOUS OPERATOR

Specialty: 1211.01 – Differential equations

Field of science: Mathematics

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GENERAL CHARACTERIRISTICS OF THE WORK

Rationale and development degree of the topic. In the last centenry in connection with the development of quantum mechanics, the spectral theory of differential operators received special development.

The most important studies of spectral theory are: the study of spectrum, basicity of the systems of root functions of the given operator in various function spaces; equiconvergence of spectral expansion of the function in the system of root functions of the operator under consideration with expansion of the same function in Fourier trigonometric series; absolute and uniform convergence of spectral expansion of functions from the class not coinciding with the domain of definition of the operator studied, etc.

The study of spectral theory of ordinary differential equations begins with classic works of J.Liouville, Sh.Sturm, further contribution to the development of the theory was made by V.A.Steklov, Ya.D.Tamarkin, D.Birkhoff, M.L.Rasulov and others.

When studying non-self-adjoint problems it was clarified that the system of eigen functions of a non-self-adjoint operator may not form a basis in L_2 and may even be incomplete in L_2. As a result, the given system must be supplemented with associated functions. In problems of this type, eigen and associated functions are not orthogonal in L_2 and neither their closedness nor their minimality do not provied its basicity. Consequently, the study of non-self-adjoint problems created a need for new methods for their research.

The fact of completeness of a specially constructed system of root vectors in L_2 (M.V. Keldysh canonic system) for a wide class of boundary value problems was first proved by M.V.Keldysh. Further, this issue was studied in the works of V.B.Lidski, M.A.Naimark, V.N.Visitey and A.S.Markus, J.E.Allahverdiyev, M.G.Gasymov and M.G.Javadov, A M. Krall, A.A. Shkalikov and other authors.

The Riesz basicity of the system of root functions in L_2 of strongly regular boundary problems was proved by V.P. Mikhailov and G.M. Keselman. And basicity with brackets of the system of root functions of a differential operator with regular boundary conditions was established by A.A. Shkalikov.

One of the more general results on equiconvergence for ordinary differential operators with regular boundary conditions and rather smooth coefficients was obtained by Ya.D. Tamarkin. For an operator with summable coefficients, similar results were proved by M.Stone.

The resolvent method is on the base of above works and the obtained equiconvergences are equiconvergences with brackets. Since eighties of the last century, a new method suggested by V.A.Ilin, a method for studying differential operators is used. He has revealed that in the presence of infinitely many multiple eigen values, the basicity and equiconvergence property unlike the completeness property: 1) essentially depends on the choice of root functions; 2) is defined not only by a specific type of boundary conditions, the values of the coefficients of a differential operator also influenced on these properties and these properties change with any small change in the values of coefficients in the metric of the classes in which these coefficients are specified.

Thus, in this situation it is impossible to formulate basicity and equiconvergence conditions in the terms of boundary conditions.

Under certain natural conditions V.A. Ilin proved theorems on equiconvergence and basicity for a system of root functions of an ordinary differential operator.

Further research was continued by V.A.Ilin[1] himself and his followers V.V.Tikhomirov, I.S.Lomov, N.B. Kerimov, V.D.Budaev, I.Yo, V.I. Komornik, L.V.Kritskov, V.M.Kurbanov and others.

Unconditional basicity of a systems of root functions of the Dirac operator was studied in the works of V.M.Kurbanov, V.M.Kurbanov[2] and A.M.Abdullaeva.

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[1] Ильин, В.А. Необходимые и достаточные условия базисности и равносходимости с тригонометрическим рядом спектральных разложений I //-Москва: Дифференциальные уравнения, -1980. т.16, №5, -с.771-794

[2] Kurbanov, V.M., Abdullaeva, A.M. Bessel property and basicity of the system of root vector-functions of Dirac operator with summable coefficient // -Zagreb: Operators and Matrics., -2018. v.12, №4, -pp.943-954

In these works, the criteria of Bessel property and unconditional basicity in L_2 of the system of root vector functions of the Dirac operator have been studied. In their works, A.I.Ismaylova, V.M.Kurbanov and A.I.Ismaylova study componentwise uniform equiconvergence on a compact, uniform equiconvergence, the Riesz property of the root vector functions of the Dirac operator.

The basicity of the system of root vector functions of the Dirac operator with specific boundary conditions has been studied in the works of P.Jakov and B.Mityagin, I. Troshin and M. Yamomota, L.L.Oridoroga and S.Khassi.

Dirac operator with a potential from L_p, $p \geq 1$, was studied in the works of A.M.Savchuk, A.A.Shkalikov, A.M.Savchuk, I.V.Sadovnichiy and for the case of strongly regular boundary conditions the Riesz basicity, in the case of regular (but not strongly regular) boundary conditions the Riez basicity from half-spaces were proved. 2×2 type Diract system with potentials from the class L_1 and strongly regular boundary conditions was studied and the Riesz basicity was established in the works of A.A.Lunev, M.M.Malamud.

Proceeding from what has been said above, further study of differential operators including Dirac type operator by V.A. Il'in method is of interest.

Object and subject of the study. The main object of the study is a discontinuous Dirac operator. The subject of the study is studying basicity, componentwise uniform equiconvergence and absolute and uniform convergence of spectral expansion in root vector functions of the Dirac discontinuous operator.

Goal and objectives of the study. To study the Riesz property, unconditional basicity in $L_2^2(0,2\pi)$ (the Riesz basicity) of the system of root vector-functions of the discontinuous Dirac operator. To prove a theorem on equivalent basicity in $L_p^2(0,2\pi)$, $1 < p < \infty$. To study absolute and uniform convergence of birthogonal expansion in root-functions of the given operator. To prove necessary and sufficient condition for uniform convergence of biorthogonal expansion of a vector function from the class

$W_{p,2}^{1,m}$, $p \geq 1$, and to estimate uniform convergence rate. To study componentwise uniform equiconvergence of a spectral expansion with ordinary trigonometric series on a compact. To prove a necessary condition for componentwise equiconvergence on a compact, establish necessary and sufficient condition for componentwise uniform equiconvergence.

Research methods. In the work, methods of spectral theory of differential operators, functional and harmonic analysis were used.

The main theses to be defended.

- Riesz inequality for the system of root vector-functions of Dirac's discontinuous operator.
- Necessary and sufficient conditions of unconditional basicity of the system of eigen and associated vector-functions of the discontinuous Dirac operator in $L_2^2(0, 2\pi)$.
- Theorem on equivalent basicity of the system of eigen and associated vector-functions of the Dira c discontinuous operator in $L_p^2(0, 2\pi)$, $1 < p < \infty$.
- Theorem on absolute and uniform convergence of spectral expansions of functions from the class $W_{p,2}^{1,m}(0,2\pi)$, $p \geq 1$.
- Necessary condition for componentwise uniform equiconvergence of spectral expansion with an ordinary trigonometric series on a compact.
- Necessary and sufficient condition for componentwise uniform equiconvergence of spectral expansion with an ordinary trigonometric series on a compact for the case of zero potential.
- Theorem on componentwise equiconvergence of spectral expansion with an ordinary trigonometric series on a compact for Dirac's discontinuous operator with the potential $L_p(0, 2\pi) \otimes C^{2\times 2}$, $p > 2$.

Scientific novelty of the study. In the work the following results have been obtained:

- A necessary condition of Riesz property for root vector-functions of Dirac's discontinuous operator, was obtained.

- The criterion of Riesz property of the system of root vector-functions of Dirac's discontinuous operator in $L_p^2(0,2\pi)$, $1<p\le 2$ was obtained.
- The criterion of unconditional basicity of the system of root vector-functions of Dirac's discontinuous operator in $L_2^2(0,2\pi)$ was obtained.
- A theorem on equivalent basicity of the system of root (eigen and associated) vector-functions of Dirac's discontinuous operator in $L_p^2(0,2\pi)$, $1<p<\infty$ was proved.
- A theorem on absolute and uniform convergence for componentwise uniform eguiconvergence of expansions of functions from the class $W_{p,2}^{1,m}(0,2\pi)$, $p\ge 1$ was proved.
- Necessary, necessary and sufficient conditions for componentwise uniform eguiconvergence of spectral expansion with an ordinary trigonometric series on a compact were proved.
- A theorem on componentwise uniform equiconvergence of spectral expansion with an ordinary trigonometric series on a compact was proved for Dirac's discontinuous operator with the potential $L_p(0,2\pi)\otimes C^{2\times 2}, p>2$.

Theoretical and practical value of the study is of theoretical character. The results of the dissertation work can be used in spectral theory of differential equations, when justifying the solution of the problems of mathematical physics, theory of approximation of functions by the Fourier method.

Approbation and application. The main results of the dissertation work were reported: at the Republican Scientific conference devoted to the 100-th anniversary of professor A.Sh.Habibzade "Functional analysis and its applications" (Baku, 2016); at the International Scientific conference devoted to the 55 years of Sumgait State University "Theoretical and applied problems of mathematics" (Sumgait, 2017); at the Republican Scientific conference devoted to 95-th anniversary of the national leader Heydar Aliyev "Actual problems of Mathematics and Mechanics"

(Baku, 2018); Proceedings of the XIII International Conference devoted to 55 years of the Faculty of Mathematics and Computer Science, (Makhachkala, 2019); Proceedings of the International conference of Voronezh Spring Mathematical School Pontryagin Readings-XXII, (Voronezh, 2021); in the seminar of the Department of “Mathematical analysis” (head: prof. B.A.Aliyev) of Azerbaijan State Pedagogical University.

Author’s personal contribution. All the conclusions and the obtained results belong to the author.

Author's publications. The main results of the work were published in 10 works the list of which is at the end of the author’s thesis.

The name of the organization where the work was performed. The dissertation work was performed in the Department of “Mathematical analysis” of Azerbaijan State Pedagogical University.

Structure and volume of the dissertation (in digns, indicating the volume of each structural unit separately). The volume of the dissertation work - 202629 signs (title page- 364 signs, contents 1656 signs, introduction -46609 signs, chapter I-86000 signs, chapter II-66000 signs, conclusion -2000 signs). The list of references consists of 107 name.

CONTENT OF THE DISSERTATION

In the work we consider Dirac’s discontinuous operator on a finite interval with a summable complex-valued potential.

The work consists of introduction, two chapters, conclusion and a list of references. Each chapter was divided into sections.

Chapter I of the dissertation work has been devoted to the Riesz property, basicity, abcolute and uniform convergence.

Let the point $\{\xi_i\}_{i=0}^{m}$, $a=\xi_0<\xi_1<\ldots<\xi_m=b$ partition the interval $G=(a,b)$. Denote $G_l=(\xi_{l-1},\xi_l)$, $l=\overline{1,m}$. By A_l we denote a class of absolutely continuous two component vector-functions on

$\overline{G_l}$. We define the class $A(a,b)$ as follows: if $f(x)\in A(a,b)$, then for each $l=\overline{1,m}$ there exists such a vector-function $f_l(x)\in A_l$ that $f(x)=f_l(x)$ for $\xi_{l-1}<x<\xi_l$.

Let $L_p^2(G)$, $p\geq 1$, be a space of two-component vector-function $f(x)=(f_1(x),f_2(x))^T$, with the norm

$$\|f\|_p\equiv\|f\|_{p,2}\equiv\|f\|_{p,2,G}=\left(\int_G|f(x)|^p dx\right)^{1/p}=\left\{\int_G\left(\sum_{j=1}^2|f_j(x)|^2\right)^{p/2}dx\right\}^{1/p}$$

In the case $p=\infty$ the norm is determined by the equality

$$\|f\|_\infty\equiv\|f\|_{\infty,2}=\sup_{x\in\overline{G}} vrai\,|f(x)|.$$

It is clear that for arbitrary vector-functions $f(x)=L_p^2(G)$, $g(x)=L_q^2(G)$, where $p^{-1}+q^{-1}=1$, $1\leq p\leq\infty$, it was determined $(f,g)=\int_G\langle f(x),g(x)\rangle dx,$

where $\langle f(x),\,g(x)\rangle=\sum_{i=1}^2 f_1(x)\,\overline{g_i(x)}$. $L_p(G)\otimes C^{2\times 2}$, $1\leq p\leq\infty$, is a space of matrix-functions of dimension 2×2, whose elements belong to the space $L_p(G)$.

Let us consider the Dirac operator

$$Ly=B\frac{dy}{dx}+P(x)y,\ x\in\bigcup_{\ell=1}^m G_\ell$$

where

$$B=(b_{ij})_{ij=1}^2,b_{i,3-i}=(-1)^{i-1},b_{ii}=0,y(x)=(y_1(x),y_2(x))^T,$$

$$P(x)=diag\,(p(x),q(x)),$$

moreover $p(x)$ and $q(x)$ are complex valued functions summable on G.

Following V.A.Ilin, we will understand the root vector-functions of the operator L irrespective to the form of boundary conditions and “sewing” conditions, more exactly: under the eigen vector-function of the operator *L*, responding to the complex eigen value λ, we will understand any identically non zero complex valued function $\overset{0}{y}(x) \in A(a.b)$ satisfying almost everywhere in G the equation $L\overset{0}{y} = \lambda \overset{0}{y}$. Similarly under the associated vector-function of order $r, r \geq 1$, responding to λ and the eigen-vector-function $\overset{0}{y}(x)$, we will understand any complexvalued vector-function $\overset{r}{y}(x) \in A(a,b)$ satisfying almost everywhere in G the equation $L\overset{r}{y} = \lambda \overset{r}{y} + \overset{r-1}{y}$.

Let $\{u_k(x)\}_{k=1}^{\infty}$ be an arbitrary domain composed of the root (eigen and associated) vector-functions of the operator *L*, $\{\lambda_k\}_{k=\infty}^{\infty}$ be a system of eigen functions corresponding to it. Furthermore, each vector-function $u_k(x)$ enters into the system $\{u_k(x)\}_{k=1}^{\infty}$ together with all appropriate associated functions of the lowest order. This means that each element $u_k(x)$ of the system $\{u_k(x)\}_{k=1}^{\infty}$ almost everywhere in G satisfies either the equation

$$Lu_k = \lambda_k u_k \tag{1}$$

(in this case $u_k(x)$ is an eigen vector-function) or the equation

$$Lu_k = \lambda_k u_k + u_{\nu(k)}, \tag{2}$$

where the number $\nu(k)$ is uniquely determined by the number k and $\nu(k_1) \neq \nu(k_2)$ for $k_1 \neq k_2$ (in this case $\lambda_k = \lambda_{\nu(k)}$, $u_k(x)$ is an associated vector-function of order $r \geq 1$ and $u_{\nu(k)}(x)$ is an associated vector-function of order $r-1$). In the case when the length of the chain of associated functions is bounded, in equality (2) we should take $\nu(k) = k-1$, $u_{\nu(k)} = \theta_k u_{k-1}$. This time θ_k equals eiter 0 (in this case $u_k(x)$ is an eigen vector-function), or 1 (in this case $u_k(x)$ is an associated function $\lambda_{k-1} = \lambda_k$).

In 1.1 we give some notions and main facts from theory of bases.

Definition 1. *The system* $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_q^2(0,2\pi)$ *is said to be a Riesz system (or satisfies the Riesz inequality) if there exists such a constant* $M(p)$ *that for an arbitrary* $f(x) \in L_p^2(0,2\pi), 1 < p \le 2,$ *the following inequality is fulfilled*

$$\sum_{k=1}^{\infty} |(f,\varphi_k)|^q \le M \|f\|_p^q,$$

where $p^{-1} + q^{-1} = 1.$ *Note that for* $p = q = 2$ *the system* $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_2^2(0,2\pi)$ *becomes a Bessel system.*

Definition 2. The system $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_p^2(0,2\pi),\ p \ge 1,$ is said to be p close in $L_p(0,2\pi)$ to the system $\{\psi_k(x)\}_{k=1}^{\infty} \subset L_p^2(0,2\pi),$ if $\sum_{k=1}^{\infty} \|\varphi_k - \psi_k\|_p^p < \infty$ is fulfilled.

Definition 3. *Two sequences of the elements in banach space X are called equivalent if there exists a linear bounded and boundedly inversible in X operator taking one of these sequences to another one.*

$L_p(G) \otimes C^{2\times 2}, 1 \le p \le \infty$ is a space of matrix-functions of dimension 2×2 whose elements belong to the space $L_p(G)$.

Necessary conditions of Riesz property of the systems of root vector functions of the operator L are established in 1.2.

Theorem 1. [1] *Let the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_p(G), 1 < p \le 2,$ *the lengths of the root vectors of vector-functions be bounded. Then for the system* $\{\varphi_k(x)\}_{k=1}^{\infty},$ *where* $\varphi_k(x) = u_k(x)\|u_k\|_{q,2}^{-1}$, *to satisfy the Riesz inequality, it is necessary*

$$\sum_{|\operatorname{Re}\lambda_k - \nu| \le 1} \frac{|u_k(x)|^q}{\|u_k\|_{q,2}^q} \le K_1\left(1 + \sup_{|\operatorname{Re}\lambda_k - \nu| \le 1} |\operatorname{Im}\lambda_k|\right), \ x \in \overline{G} \qquad (3)$$

where ν *is an arbitrary real number;* K_1 *is a constant independent of* ν *;* $u_k(a)=u_k(a+0)$, $u_k(b)=u_k(b-0)$; $u_k(\xi_i)$ *equals to any of the values* $u_k(\xi_i-0)$, $u_k(\xi_i+0)$, $i=\overline{1,m-1}$; *the summation is carried out only over the eigen vector-functions.*

Corollary 1. *Subject to the conditions of theorem 1, the lengths of the chains of vector-functions be uniformly bounded. Then for the Riesz property of the system* $\{\varphi_k(x)\}_{k=1}^{\infty}$, *where* $\varphi_k(x)=u_k(x)\|u_k\|_{q,2}^{-1}$, *it is necessary*

$$\sum_{|\mathrm{Re}\,\lambda_k-\nu|\le 1} 1 \le K_2\left(1+\sup_{|\mathrm{Re}\,\lambda_k-\nu|\le 1}|\mathrm{Im}\,\lambda_k|\right) \tag{4}$$

where K_2 *is a constant independent of* ν, *and the summation is carried out with taking into account the multiplicity of the number* λ_k *.*

Theorem 2. [1] *Let the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_p(G)$, $1<p\le 2$, *and the a priori estimate*

$$\left\|u_{\nu(k)}\right\|_{q,2,G_l} \le C_0\left(1+|\mathrm{Im}\,\lambda_k|\right)^{1/p}\|u_k\|_{q,2,G_l} \tag{5}$$

be fulfilled, where C_0 *is independent of the order of associated functions* $l=\overline{1,m}$, $p^{-1}+q^{-1}=1$. *Then for the Riesz property of the system* $\{\varphi_k(x)\}_{k=1}^{\infty}$, *where* $\varphi_k(x)=u_k(x)\|u_k\|_{q,2}^{-1}$ *it is necessary to satisfy the inequality (3), where the summation is carried out over all root vector-functions.*

Section 1.3 considers Dirac's discontinuous operator in the interval $(0,2\pi)$. The criteria of Riesz property, unconditional basicity in $L_2^2(0,2\pi)$ of the system of root vector-functions are established and a theorem on equivalent basicity in $L_p^2(0,2\pi)$, $1<p<\infty$ is proved. This section is divided into 2 subsections.

The following theorem is the main result of 1.3.1

Theorem 3 (the Riesz property criterion). [5] *Let the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_1(0,2\pi)$*, the length the chain of the root vector-functions be uniformly bounded and there exist such a constant* C_0 *that*

$$\left|\operatorname{Im}\lambda_k\right| \le C_0, \quad k = 1,2,... \tag{6}$$

Then for the Riesz property of the system $\left\{u_k(x)\|u_k\|_q^{-1}\right\} \subset L_q^2(0,2\pi)$ *it is necessary and sufficient the existence of such a constant* M_1 *that*

$$\sum_{\left|\operatorname{Re}\lambda_k - \tau\right| \le 1} 1 \le M_1 , \tag{7}$$

where τ *is an arbitrary real number.*

Note that for the Dirac operator at $P(x) \in L_p(0,2\pi) \otimes C^{2\times 2}$, $1 < p \le 2$, the Riesz property criterion was earlier established in the papers of V.M.Kurbanov and A.I.Ismaylova.

Let L^* be an operator formally conjugated to $L: L^* = Bd/dx + P^*(x)$, where $P^*(x)$ is a matrix-function conjugated to $P(x)$. Denote by $\{\upsilon_k(x)\}_{k=1}^{\infty}$ a system biorthogonally conjugated to $\{u_k(x)\}_{k=1}^{\infty}$ and assume that it consists of root vector-functions of the operator L^*.

The main results of 1.3.2 are theorems on unconditional basicity and equivalent basicity

Theorem 4. (on unconditional basicity). [5] *Let the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_1(0,2\pi)$ *one of the systems* $\{u_k(x)\}_{k=1}^{\infty}$ *and* $\{\upsilon_k(x)\}_{k=1}^{\infty}$ *be complete in* $L_2^2(0,2\pi)$ *the length of chains of the root vector-functions be uniformly bounded and condition (6) be fulfilled. Then a necessary and sufficient condition for unconditional basicity in* $L_2^2(0,2\pi)$ *of each of these systems is the existence of constants* M_1 *and* M_2 *providing the validity of the inequality (7) and*

$$\|u_k\|_2 \|\upsilon_k\|_2 \le M_2, \; k = 1,2,... \tag{8}$$

Note that theorem 4 is an analog of V.A.Ilin's theorem for Schrodingers discontinuous operator.

Remark 1. *Under the conditions of theorem 4, the fulfillment of inequalities (7) and (8) is a necessary and sufficient condition for the Riesz basicity of each of these systems* $\left\{u_k(x)\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ *and* $\left\{\upsilon_k(x)\|\upsilon_k\|_2^{-1}\right\}_{k=1}^{\infty}$ *in* $L_2^2(0,2\pi)$.

Theorem 5 (on equivalent basicity). [5] *Let* $1 < p \le 2$*, the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_1(0,2\pi)$*, conditions (6), (7), (8) be fulfilled and the system* $\left\{u_k(x)\|u_k\|_p^{-1}\right\}_{k=1}^{\infty}$ be p *close to some basis* $\{\psi_k(x)\}_{k=1}^{\infty}$ in $L_p^2(0,2\pi)$. *Then the systems* $\left\{u_k(x)\|u_k\|_p^{-1}\right\}_{k=1}^{\infty}$ *and* $\left\{\upsilon_k(x)\|u_k\|_p\right\}_{k=1}^{\infty}$ *are the bases in* $L_p^2(0,2\pi)$ *and* $L_q^2(0,2\pi)$*, respectively and these systems are equivalent to the basis* $\{\psi_k(x)\}_{k=1}^{\infty}$ *and to its birthogonally conjugated system respectively.*

Remark 2. *Changing the roles to the systems* $\{u_k(x)\}_{k=1}^{\infty}$ *and* $\{\upsilon_k(x)\}_{k=1}^{\infty}$ *in theorem 5, we obtain basicity of the system* $\{u_k(x)\}_{k=1}^{\infty}$ *in* $L_p^2(0,2\pi)$ *for* $p \ge 2$.

Note that under such generalized understanding of root functions, V.A.Ilin has first established necessary and sufficient condition of unconditional basicity (the Riesz basicity) in L_2 of the system of root functions of the operator $L = -d^2/dx + q(x)$ and these results were taken to the case of discontinuous operator L. These studies served as a starting point for the works of many authors in studying the Bessel property, unconditional basicity and basicity of the system of root functions of differential operators or higher order (N.B.Kerimov, V.D.Budayev, I.S.Lomov, I.Yo, V.M.Kurbanov and others).

For a Dirac operator with a potential from the class L_2 the criteria of Bessel property and unconditional basicity was established by V.M.Kurbanov.

The works of V.M.Kurbanov, E.D.Ibadov, A.I.Ismaylova, G.R.Gadjiev, A.M.Abdullayeva have been devoted to the unconditional basicity of the system of root vector-functions of Dirac type operators.

The works of V.V.Kornev, A.P.Khromov; T.Sh.Abdullaev, I.M.Nabiev; P.Djakov, B.Mityagin; A.A.Lunyov, M.M.Malamud; Kh.R.Mamedov, O.Akcay; Ya.V.Mykytnyk, D.V. Puyda have been devoted to the basicity property and other spectral properties of the root vector-functions of the Dirac operator (with boundary conditions).

In 1.4 we prove theorems on absolute and uniform convergence of biorthogonal expansion of a vector-function from the class $W_{p,2}^{1,m}$, $p \geq 1$, in root vector-functions of Dirac's discontinuous operator and prove uniform convergence rate on $[0,2\pi]$.

We introduce $W_{p,2}^{1,m}([0,2\pi]) \equiv W_{p,2}^{1,m}(\overline{G}) \equiv W_{p,2}^{1}(\overline{G}, \{\xi_i\}_{i=0}^{m})$, $p \geq 1$, class of two-component vector-functions satisfying the following properties: if $f(x) = (f_1(x), f_2(x))^T \in W_{p,2}^{1,m}(\overline{G})$, then for each l, $l = \overline{1,m}$, there exists a vector-functions $f_l(x) = (f_{l1}(x), f_{l2}(x))^T$, $f_{lj}(x) \in W_p^1(G_l)$, $j = 1,2$, that $f(x) = f_l(x)$ for $\xi_{l-1} < x < \xi_l$; $f(\xi_i) = f(\xi_i + 0)$ $i = \overline{0, m-1}$, $f(\xi_m) = f(\xi_m - 0)$. And $W_p^1(G_l)$ is an ordinary Sobolev class, $W_{p,2}^{1,1}(\overline{G}) \equiv W_{p,2}^{1}(\overline{G})$ is a Sobolev class of two-component vector-functions determined on $\overline{G} = [0,2\pi]$.

We determine the norm of the element $f \in W_{p,2}^{1,m}(\overline{G})$ by the equality $\| f \|_{W_{p,2}^{1,m}(\overline{G})} = \sum_{l=1}^{m} \| f_l \|_{W_{p,2}^{1}(G_l)} = \sum_{l=1}^{m} \left(\| f_l \|_p + \| f_l' \|_p \right)$.

Note that for $p(x), q(x) \in L_1(G_l), \ l=\overline{1,m}$, there exists one-sided limits $u_k(0+)$, $u_k(2\pi-0), u_k(\xi_l \pm 0), l=\overline{1,m-1}$. Further, by $u_k(\xi_l), \ l=\overline{0,m-1}$, and $u_k(2\pi)$ we will mean one-sided limits $u_k(\xi_l+0), l=\overline{0,m-1}$, and $u_k(2\pi-0)$ respectively.

Assume that the system $\{u_k(x)\}_{k=1}^{\infty}$ is complete and minimal in $L_2^2(0,2\pi)$. Then there exists a unique system $\{\upsilon_k(x)\}_{k=1}^{\infty} \subset L_2^2(0,2\pi)$ biorthogonally conjugated to the system $\{u_k(x)\}_{k=1}^{\infty}$. Let the system $\{\upsilon_k(x)\}_{k=1}^{\infty}$ consist of eigen and associated vector-functions formalty conjugated to L of the operator $L^* = B\frac{d}{dx} + P^*(x)$, where $P^*(x) = \overline{P(x)}$ is a matrix function conjugated to $P(x)$. This means that the function $\upsilon_k(x)$ satisfies almost everywhere in G the equation $L^*\upsilon_k = \overline{\lambda_k}\upsilon_k + \theta_{k+1}\upsilon_{k+1}$.

For the vector-function $f(x) \in W_{p,2}^{1,m}(\overline{G})$ we determine the numbers $\alpha_k(f), \ k=1,2,...$:

$$\overline{\alpha_k(f)} = \sum_{i=1}^{m} \left[\langle B\upsilon_k(\xi_i-0), f(\xi_i-0)\rangle - \langle B\upsilon_k(\xi_i+0), f(\xi_i+0)\rangle\right] + \\ + \langle B\upsilon_k(2\pi-0), f(2\pi)\rangle - \langle B\upsilon_k(+0), f(0)\rangle$$

and compose a biorthogonal series

$$\sum_{k=1}^{\infty} f_k u_k(x), \qquad f_k = (f, \upsilon_k)$$

of the vector-function $f(x)$.

Introduce the partial sum of order ν of this biorthogonal series

$$\sigma_\nu(x,f) = \sum_{|\lambda_k|<\nu} f_k u_k(x)$$

and the residual in the form $R_\nu(x,f) = f(x) - \sigma_\nu(x,f)$.

The main results of this section are in the following theorem.

Theorem 6. [9] *Let* $p(x),\ q(x)\in L_r(0,2\pi),\quad r>1,$ $f(x)\in W_{p,2}^{1,m}(\overline{G}),\ \ p>1,$ *and for the system of root vector-functions* $\{u_k(x)\}_{k=1}^{\infty}$ *and eigen values* $\{\lambda_k\}_{k=1}^{\infty}$ *the following conditions be fulfilled:*

1) The system $\{u_k(x)\}_{k=1}^{\infty}$ *is complete and minimal* $L_2^2(0,2\pi)$;

2) For any $k=1,2,\dots$

$$\left|\operatorname{Im}\lambda_k\right|\le const; \tag{9}$$

3) For any $\tau\in(-\infty,\infty)$

$$\sum_{\left|\operatorname{Re}\lambda_k-\tau\right|\le 1} 1 \quad \le const \tag{10}$$

4) The biorthogonal system $\{\upsilon_k\}$ *consists of the root vector-functions of the formally conjugated operator* L^**;*

5) There exists such a constant C_0*, that*

$$\left\|u_k\right\|_{2,2}\left\|\upsilon_k\right\|_{2,2}\le C_0\,,\quad k=1,2,\dots\,. \tag{11}$$

Then the following statements are valid:

a) for uniform convergence of the series

$$\sum_{k=1}^{\infty}\left|f_k\right|\left|u_k(x)\right|,\qquad x\in\overline{G}=[0,2\pi] \tag{12}$$

it is necessary and sufficient a uniform on $[0,2\pi]$ *convergence of the series*

$$\sum_{|\lambda_k|\ge 1}\left|\lambda_k\right|^{-1}\left|\alpha_k(f)\right|\,\left|u_k(x)\right|; \tag{13}$$

в) for uniform convergence on $[0,2\pi]$ *of biorthogonal expansion*

$$\sum_{k=1}^{\infty} f_k\ u_k(x) \tag{14}$$

it is necessary and sufficient a uniform on $[0,2\pi]$ *convergence of the series*

$$\sum_{|\lambda_k|\ge 1}\lambda_k^{-1}\,\alpha_k(f)u_n(x); \tag{15}$$

c) if $\alpha_k(f)=0$ *for* $k \ge k_0$ *(*k_0 *is some fixed integer), then biorthogonal series (14) of the vector-function* $f(x)$ *converges absolutely and uniformly on* $[0,2\pi]$ *and the for the residual* $R_\nu(x, f)$ *the following estimates are fulfilled:*

$$\sup_{x\in\overline{G}}|R_\nu(x, f)| \le const\, \nu^{-\beta}\left\{ \|f\|_{W^{1,m}_{p,2}(\overline{G})} + \|Pf\|_{r,2} \right\}, \quad (16)$$

for $\nu \ge \max\{1, |\lambda_{k_0}|\}$;

$$\sup_{x\in\overline{G}}|R_\nu(x, f)| = o\left(\nu^{-\beta}\right),\ \nu \to +\infty, \quad (17)$$

where $\frac{1}{p}+\frac{1}{q}=1,\ \frac{1}{r}+\frac{1}{r'}=1,\ \beta=\min\left\{\frac{1}{2}, \frac{1}{q}, \frac{1}{r'}\right\},$ *const is independent of* f *, and the symbol «o» depends on the vector-function* $f(x)$.

In this section subject to the conditions 1)-5) of theorem 6 the cases $r>1,\ p=1;\ r=1,\ p>1$ have been also considered. Note that the results similar to the results of theorem 6 (item c)) for an ordinary Dirac operator earlier have been obtained in the paper of V.M.Kurbanov and A.I.Ismaylova.

Chapter II studies componentwise uniform equiconvergence of expansions in root vector-functions of a Dirac operator with an ordinary trigonometric series on a compact.

Definition 4. *The system of vector-functions* $\{\psi_k(x)\}_{k=1}^{\infty} \subset L_p^2(0,2\pi)$ *is said to be closed in* $L_p^2(0,2\pi)$*, if only vector-function* $f(x) \in L_p^2(0,2\pi)$ *can be approximated in the metric* $L_p^2(0,2\pi)$ *with any accunacy degree of finite linear combination of the elementars of the system* $\{\psi_k(x)\}_{k=1}^{\infty}$.

Definition 5. *The system of vector-functions*

$\{\psi_k(x)\}_{k=1}^{\infty} \subset L_p^2(0,2\pi)$ *is said to be minimal in* $L_p^2(0,2\pi)$, *if none of the elements of this system is a limit in the metric* $L_p^2(0,2\pi)$ *of finite linear combinations of other elements of this system.*

We will study expansions in a biorthogonal series in the system $\{u_k(x)\}_{k=1}^{\infty}$, satisfying the conditions A_p:

1) for some fixed $p \geq 1$ the system of vector-functions $\{u_k(x)\}_{k=1}^{\infty}$ is closed and minimal in $L_p^2(0,2\pi)$.

2) the system of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfies the two inequalities:

$$\left|\operatorname{Im}\lambda_k\right| \leq C_1, \quad k = 1,2,... \tag{18}$$

$$\sum_{t \leq |\lambda_k| \leq t+1} 1 \leq C_2, \quad \forall t \geq 0 \tag{19}$$

The first from two conditions A_p provides the uniqueness of the system $\{\upsilon_k(x)\}_{k=1}^{\infty} \subset L_q^2(0,2\pi)$, biorthogonally conjugated with the system $\{u_k(x)\}_{k=1}^{\infty}$, i.e. the following biorthogonal condition is fulfilled

$$(u_k, \upsilon_j) = \int_0^{2\pi} \sum_{l=1}^{2} u_k^l(x)\overline{\upsilon_j^l(x)}\, dx = \delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases},$$

where $u_k(x) = \left(u_k^1(x), u_k^2(x)\right)^T, \quad \upsilon_k(x) = \left(\upsilon_k^1(x), \upsilon_k^2(x)\right)^T$.

The second one of conditions allows to assume that all the elements of the system $\{u_k(x)\}_{k=1}^{\infty}$ were numbered in non-descending order of the varianble $|\lambda_k|$.

For an arbitrary $f(x) \in L_p^2(0,2\pi)$ we compose a partial sum of order a n of biorthogonal expansion in the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_n(x, f) = \sum_{k=1}^{n} (f, \upsilon_k) u_k(x), \qquad x \in G. \tag{20}$$

For each *j=1,2* we consider the *j*- th component of the partial sum (20)

$$\sigma_n^j(x, f) = \sum_{k=1}^{n} (f, \upsilon_k) u_k^j(x), \qquad x \in G, \tag{21}$$

and compare (21) with a modified partial sum of the Fourier trigonometric series corresponding to the *j-th* component $f_j(x)$ of the vector-function $f(x)$

$$S_\nu(x, f_j) = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin \nu(x - y)}{x - y} f_j(y) dy. \tag{22}$$

or order $\nu = |\lambda_n|$.

Definition 6. *We say that the j-th component of expansion of the vector-function* $f(x) \in L_p^2(0, 2\pi)$ *in a biorthogonal series by the system* $\{u_k(x)\}_{k=1}^{\infty}$ *uniformy equiconverges on any compact of the set* $G = \bigcup_{l=1}^{m} G_l$ *with expansion corresponding to the j-th component* $f_j(x)$ *of the vector-function* $f(x)$ *in Fourier trigonometric series if on any compact* $K \subset G$

$$\lim_{n \to \infty} \left\| \sigma_n^j(\cdot, f) - S_{|\lambda_n|}(\cdot, f_j) \right\|_{C(K)} = 0.$$

Necessary condition of componentwise uniform equiconvergence on any compact $K \subset G$ with a trigonometric Fourier series was formulated and proved in 2.1 and 2.2.

Theorem 7. [7] *Let the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_1(0, 2\pi)$ *and the system* $\{u_k(x)\}_{k=1}^{\infty}$ *satisfy for fixed* $p \geq 1$ *two conditions of* A_p. *Then for each j-th (j=1,2) component of expansion of an arbitrary vector-function* $f(x) \in L_p^2(0, 2\pi)$ *in biorthogonal series by the system* $\{u_k(x)\}_{k=1}^{\infty}$ *equiconverge uniformly on any compact* $K \subset G$ *with expansion in trigonometric Fourier series of the corresponding j-th component* $f_j(x)$ *of the vector-function* $f(x)$, *it is necessary that for any compact* $K_0 \subset G$ *there* *exist a constant* $C(K_0)$*, providing the validity for all the numbers of the inequlities*

$$\|u_k\|_{L_p^2(K_0)} \|\upsilon_k\|_{L_q^2(0,2\pi)} \le C(K_0), \qquad (23)$$

in which $q = \dfrac{p}{(p-1)}$ $(q = \infty \ \ for \ \ p = 1)$.

2.3 is devoted to the proof of necessary and sufficient condition of componentwise uniform equiconvergence for $P(x) \equiv 0$.

2.4 considers Dirac's discontinuous operator with a potential from the class $L_p(0,2\pi) \otimes C^{2\times 2}$, $p > 2$, necessary and necessary conditions of componentwise equiconvergence on compact with a trigonometric series of expansions in a biorthogonal series of an arbitrary vector-function $f(x) \in L_2^2(0,2\pi)$ by the system of root vector-functions of the given operator are established.

Let the considered system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the condition B_2:

1) the system $\{u_k(x)\}_{k=1}^{\infty}$ is complete and minimal in $L_2^2(0,2\pi)$

2) the system of eigen values $\{\lambda_k\}_{k=1}^{\infty}$ satisfy the two inequalities (18) and (19)

3) the system $\{\upsilon_k(x)\}_{k=1}^{\infty} \subset L_2^2(0,2\pi)$, biorthogonally conjugated to the system $\{u_k(x)\}_{k=1}^{\infty}$, consist of the root vector-functions a formally conjugated operator

$$L^* = B\frac{d}{dx} + \overline{P(x)}, \quad i.e.\ L^*\upsilon_k = \overline{\lambda_k}\upsilon_k + \theta_k\upsilon_{k+1}.$$

This section is divided into three subsections. The following theorem on componentwise uniform convergence on a compact is the main result of this section.

Theorem 8. [10] *Let the potential* $P(x)$ *belong to the class* $L_p(0,2\pi)\otimes C^{2\times 2}$, $p>2$, *and the system of the root vector-functions* $\{u_k(x)\}_{k=1}^{\infty}$ *satisfy the condition* B_2. *Then for*

$$\lim_{n\to\infty}\left\| \sigma_n^j(\cdot, f) - S_{|\lambda_n|}(\cdot, f_j)\right\|_{C(K)} = 0 \tag{24}$$

to be fulfilled for any arbitrary vector-function $f(x)\in L_2^2(0,2\pi)$ *on any compact* $K\subset G$, *it is necessary and sufficient that for any compact* $K_0\subset G$ *there exists a constant* $C(K_0)$, *providing the validity of the following inequality for all the numbers* k

$$\|u_k\|_{L_2^2(K_0)}\|\upsilon_k\|_{L_2^2(0,2\pi)} \le C(K_0). \tag{25}$$

The localization principle follows from this theorem:

Theorem 9. *If the potential* $P(x)$ *of the operator* L *and the system of the root vector-functions* $\{u_k(x)\}_{k=1}^{\infty}$ *satisfy the same requirements that in theorem 8, then subject to the condition (25) for biorthogonal expansion of the vector-function* $f(x)\in L_2^2(0,2\pi)$ *the componentwise localization principle in* G *is valid: the convergence or divergence of the* j*-th component of the indicated biorthogonal expansion at the point* $x_0\in G$ *depends on the behavior of the point* x_0 *in small vicinity corresponding only to the* $j-$ *th component* $f_j(x)$ *of the expanded vector-function* $f(x)$ *(but is independent on the behavior of another component).*

Note that the sufficient part of theorem 8 was earlier proved for an ordinary Dirac operator in the paper of V.M.Kurbanov and A.I.Ismaylova.

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CONCLUSIONS

The dissertation work was performed of the study is studying basicity, componentwise uniform equiconvergence and absolute and uniform convergence of spectral expansion in root vector functions of the Dirac discontinuous operator.

In the work the following results have been obtained:

- Necessary condition of Riesz property was established for the root vector-functions of Dirac's discontinuous operator.
- The criterion of Riesz property of the system of root vector-functions of Dirac's discontinuous operator in $L_p^2(0,2\pi)$, $1 < p \le 2$ was established.
- The criterion of unconditional basicity of the system of root vector-functions of Dirac's discontinuous operator in $L_2^2(0,2\pi)$ was established.
- A theorem on equivalent basicity of the system of root vector functions of Dirac's discontinuous operator in $L_p^2(0,2\pi)$, $1 < p < \infty$ was proved.
- A theorem on absolute and uniform convergence of expansions of functions from the class $W_{p,2}^{1,m}(0,2\pi)$, $p \ge 1$was proved.
- Necessary and sufficient conditions for componentwise uniform equiconvergence of spectral expansion with an ordinary trigonometric series on a compact was proved.
- A theorem on componentwise uniform equiconvergence of spectral expansion with an ordinary trigonometric series on a compact was proved for Dirac's operator with the potential $L_p(0,2\pi)\otimes C^{2\times 2}$, $p > 2$.

The main results of the dissertation work are in the following works:

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