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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

CONSTRUCTION OF TRANSFORMATION OPERATORS FOR A PENCIL OF 4-TH ORDER DIFFERENTIAL OPERATORS WITH TRIPLE CHARACTERISTICS

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GENERAL CHARACTERISTICS OF WORK

Relevance and degree of development of the topic.

The dissertation is devoted to the construction of transformation operators for fourth-order differential equations with triple characteristics and the study of their spectral properties by means of transformation operators.

The spectral theory of ordinary differential operators of the second order was studied in sufficient detail by M. Jolent, I. Mlodek, M. G. Gasymov, Kh. Sh. Guseinov, I. M. Guseinov, I. M. Nabiev and other authors. In studying these issues, a special role was played by transformation operators, which transform solutions of one equation into solutions of another equation.

The transformation operators introduced by J. Delsarte in the 1940s have two important properties:

1) The kernel of the transformation operator is triangular;

2) The coefficients of a differential operator are expressed through the kernel of the transformation operator.

These properties of transformation operators allow them to be applied to the spectral theory of ordinary differential operators. Therefore, since the 1940s, the spectral theory of differential operators of the Sturm-Liouville and Dirac types has been studied using transformation operators. In this direction, one can note the fundamental works of B. M. Levitan, A. Yu. Povzner, V. A. Marchenko, B. Ya. Levin.

After applying transformation operators to the spectral theory of Sturm-Liouville operators, there were attempts to apply them to the spectral theory of higher-order differential operators. However, these attempts were accompanied by serious difficulties. It turned out that in this case the Goursat problem for the kernel of the transformation operator is generally not correct, and in order to prove the existence of the transformation operator, it is necessary to impose the analyticity condition on the coefficients of the equation. In the works of M.K. Fage, representations of solutions of a high-order differential equation were found. However, for these representations the two above properties of the transformation operators do not hold. Therefore, the scope of application of these ideas is not so wide. From this point of view, the construction of transformation operators for pencils of differential operators of high order is relevant. In this direction, we can note the works of M.G. Gasymov and A.M. Magerramov, the works of E.G. Orudzhev and others. In the last mentioned works, the multiplicities of the roots of characteristic polynomial of a pencil of differential operators are the same. Additional difficulties arise when the multiplicities of the roots of the characteristic polynomial are different.

The works of M.G. Gasymov, F.G. Magsudov and others are devoted to obtaining formulas for expansion in eigenfunctions in the case of the presence of a continuous spectrum of a non-self-adjoint operator. In the case where the multiplicities of the roots of the characteristic polynomial of a pencil of differential operators of the 4th order on the half-axis are the same, the expansion formulas in terms of eigenfunctions were obtained in the works of M.G. Gasymov, A.M. Magerramov and others. A similar problem in the case of a pencil of differential operators of the 4th order on the entire axis was considered in the works of S.S. Mirzoev, E.G. Orudzhev and A.Kh. Khanmamedov and others. Particular difficulties arise when deriving formulas for the expansion in eigenfunctions of a pencil of fourth-order differential operators when the multiplicities of the roots of the characteristic polynomial of this pencil are different. This is connected, first of all, with the inclusion of the third derivative in the bundle of differential operators. Therefore, obtaining formulas for expansion in eigenfunctions of a pencil of fourth-order differential operators with triple characteristics is of scientific interest.

Object and subject of research. The proof of the existence of transformation operators for a pencil of higher-order differential operators and their application to spectral theory are topical issues. For a pencil of fourth-order differential operators with triple characteristics, the differential equation includes a third-order derivative. This circumstance leads to the fact that the Wronskian of the system of fundamental solutions of the perturbed and unperturbed equations depends on the variable. The latter factor will require significant modification of existing reasoning. Therefore, the construction of transformation operators for a pencil of fourth-order differential operators and the spectral analysis of this pencil are of particular interest in those cases where the characteristic polynomial of the pencil has roots with different multiplicities.

The goal and objectives of the study. The main objective of the dissertation is to construct transformation operators for a pencil of fourth-order differential operators with triple characteristics, to study the spectrum and resolvent of this pencil, and to obtain formulas for the expansion in eigenfunctions of a pencil of fourth-order differential operators with triple characteristics.

Research methods. In substantiating the results obtained in the dissertation, methods of the spectral theory of differential operators, the theory of functions of real and complex variables, and functional analysis were used.

Basic theses for defence.

1. Construction of transformation operators with conditions at infinity for a pencil of fourth-order differential operators with triple characteristics;

2. Construction of transformation operators with conditions at the point $x = 0$ for a pencil of fourth-order differential operators with triple characteristics;

3. Study of the spectrum of a pencil of differential operators of the 4th order with triple characteristics;

4. Find an integral representation of the resolvent of a pencil of differential operators of the 4th order with triple characteristics;

5. Obtaining formulas for expansion in eigenfunctions of continuous and discrete spectra of a pencil of differential operators of the 4th order with triple characteristics.

Scientific novelty of the research. The following main results were obtained in the dissertation work:

- Transformation operators with conditions at infinity for a pencil of fourth-order differential operators with triple characteristics are constructed. The properties of kernels included in the representations of transformation operators are investigated;

- Transformation operators with conditions at the point $x = 0$ for a pencil of fourth-order differential operators with triple characteristics are constructed. The properties of kernels included in the representations of transformation operators are investigated;

- The distribution of eigenvalues of a pencil of differential operators of the 4th order with triple characteristics was studied;

- An integral representation of the resolvent of a pencil of differential operators of the 4th order with triple characteristics was found;

- Using various methods, formulas for the expansion in eigenfunctions of the continuous and discrete spectra of a pencil of fourth-order differential operators with triple characteristics were obtained.

Theoretical and practical significance of the research.

The results obtained in the dissertation are theoretical in nature. The obtained results can be used in spectral theory for pencils of differential operators of the 4th order, as well as in the study of inverse spectral problems.

Approbation and implementation. The main results of the dissertation were presented at the scientific seminars of Baku State University department of "Mathematical Economics" (head professor prof. E.G.Orudjev), scientific seminar of department of "Non-harmonic Analysis" (head corresponding member of the NAS of Azerbaijan, professor B.T. Bilalov) of the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan), Azerbaijan University department of " Mathematics and Computer Science" (head professor prof. I.M.Guseynov), Baku Engineering University department of "Mathematics" (head professor R.F.Efendiyev), at the national scientific conference "Modern Problems of Mathematics and Mechanics" dedicated to the 60th anniversary of the IMM NAS of Azerbaijan(Baku, 2019), at the international conference "Problems of Modern Mathematics" dedicated to the 70-th anniversary of prof. A.A.Borubaeva (Bishkek, Isyk-Kul, 2021).

Applicant's personal contribution. All results obtained in the dissertation belong to the applicant.

Author's publications. 7 articles of the author's 11 scientific works, have been published in scientific publications recommended by the AAK under the President of the Republic of Azerbaijan (2 in journals indexed by Web of Science), and 4 in various international conference proceedings (2 of which in abroad).

The name of the organization where the dissertation was performed. The dissertation work was completed at the "General Mathematics" department of Nakhchivan State University.

The volume of the dissertation's structural sections separately and the general volume. The total volume of the dissertation is ~ 193022 signs (title – 397 signs, table of contents ~ 1771 signs, introduction ~ 29278 signs, chapter I ~ 80000 signs, chapter II ~ 80000 signs, conclusions –1576 signs). The bibliography consists of 126 names.

THE CONTENT OF THE DISSERTATION

The dissertation consists of an introduction, two chapters, a conclusion and a bibliography. The introduction substantiates the relevance of the topic, provides an overview of works related to the topic of the dissertation, and presents a brief summary of the work.

In the first chapter, integral representations of solutions of a pencil of differential equations of the 4th order with triple characteristics are found through transformation operators. The properties of the kernels of transformation operators are studied.

The first paragraph of the first chapter provides auxiliary information about the concept of a transformation operator in linear topological spaces and mentions some properties of the transformation operator.

Section 1.2 is devoted to the construction of transformation operators with a condition at infinity for fourth-order differential equations that depend polynomially on a parameter. In this section, the equation

$$\ell\left(x,\frac{d}{dx},\lambda\right)y=\left(\frac{d}{dx}+i\lambda\right)^3\left(\frac{d}{dx}-i\lambda\right)y+ \\ +r(x)\frac{dy}{dx}+\left(\lambda p(x)+q(x)\right)y=0, \tag{1}$$

is considered on the semiaxis $0<x<+\infty$, where complex-valued functions $r(x)$, $q(x)$ and $p(x)$ are defined in the interval $[0,+\infty)$, have continuous derivatives of the 3rd, 4th and 5th orders, respectively, and satisfy the conditions:

$$\int_0^\infty x^4\left|r^{(s)}(x)\right|dx<\infty,\qquad s=0,1,2,3,$$

$$\int_0^\infty x^4\left|p^{(s)}(x)\right|dx<\infty,\qquad s=0,1,2,3,4,5,$$

$$\int_0^\infty x^4\left|q^{(s)}(x)\right|dx<\infty,\qquad s=0,1,2,3,4. \tag{2}$$

We will look for solutions $F_j(x,\lambda)$, $j=0,1,2,3$ of equation (1) such that

$$\lim_{x\to+\infty}\frac{d^s}{dx^s}\left\{F_j(x,\lambda)-x^je^{-i\lambda x}\right\}=0,\ \ j=0,1,2,\ \ s=0,1,2,3$$

$$\lim_{x\to+\infty}\frac{d^s}{dx^s}\left\{F_3(x,\lambda)-e^{i\lambda x}\right\}=0,\quad s=0,1,2,3. \tag{3}$$

In this section it is proved that if the coefficients of the equation (1) satisfy conditions (2), then solutions with asymptotics (3) exist and are unique. Representations of these solutions by means of transformation operators are found.

Let's put

$$\sigma_j(x) = \frac{1}{4}\int\limits_x^{\infty} s^{j+1}|p(s)|ds + \frac{1}{8}\int\limits_x^{\infty} s^{j+1}|\, q(s) - r'(s)|\, ds +$$

$$+\frac{1}{4}\int\limits_x^{\infty} s^{j+1}|r\,(s)|ds.$$

Theorem 1. *If the functions* $r(x)$, $q(x)$ *and* $p(x)$ *are defined in the interval* $[0, +\infty)$, *have continuous derivatives of the 3rd, 4th and 5th orders respectively and satisfy conditions (2), then equation (1) for all values of* λ *from the closed lower half-plane* $\mathrm{Im}\lambda \le 0$ *has solutions in the form*

$$F_j(x,\lambda) = x^{j-1}e^{-i\lambda x} + \int\limits_x^{\infty} K_j(x,t)e^{-i\lambda t}dt,\ j = 0,1,2\ , \tag{4}$$

and for all values of λ *from the closed upper half-plane* $\mathrm{Im}\lambda \ge 0$ *has a solution of the form*

$$F_3(x,\lambda) = e^{i\lambda x} + \int\limits_x^{\infty} K_3(x,t)e^{i\lambda t}dt\ . \tag{5}$$

Moreover, for each $x \ge 0$, *the kernels* $K_j(x,t)$, $j = 0,1,2,3$ *satisfy the conditions*

$$\int\limits_x^{\infty} |K_j(x,t)|^2 dt < \infty\ . \tag{6}$$

In addition, the following estimates are valide are estimates

$$|K_j(x,t)| \le C_j \sigma_j\left(\frac{x+t}{2}\right), \tag{7}$$

where C_j, $j = 0,1,2,3$ *are constants*.

In the section 1.2 the properties of solutions $F_j(x,\lambda)$, $j = 0,1,2,3$ are studied.

In addition, partial differential equations are obtained for the kernels $K_j(x,t)$ included in the representations (4), (5).

Theorem 2. *If the functions* $r(x)$, $q(x)$ *and* $p(x)$ *are defined in the interval* $[0, +\infty)$, *have continuous derivatives of the 3rd, 4th and 5th orders respectively and satisfy conditions (2), then the kernels* $K_j(x,t)$ *have continuous derivatives up to the fourth order and the relations*

$$\ell\left(x, \frac{\partial}{\partial x}, i\frac{\partial}{\partial t}\right)K_j(x,t) = 0, \tag{8}$$

$$\lim_{x+t\to\infty} \frac{\partial^{\alpha+\beta} K_j(x,t)}{\partial x^{\alpha} \partial t^{\beta}} = 0,\ \alpha + \beta \le 4. \tag{9}$$

are valid. In addition, on the characteristic $t = x$ *the functions* $K_j(x,t)$ *satisfy the Goursat type conditions.*

The last paragraph of the first chapter is devoted to the construction of transformation operators with a condition at the point $x=0$ for fourth-order differential equations that depend polynomially on a parameter.

Theorem 3. *If complex-valued functions* $r(x)$, $q(x)$ *and* $p(x)$ *are defined on the entire number axis and have continuous derivatives of the 3rd, 4th and 5th orders, respectively, then equation (1) for all values of* λ *has solutions in the form*

$$\begin{gathered} F_j(x,\lambda) = x^j e^{i\lambda x} + \int_{-x}^{x} K_j(x,t) e^{-i\lambda t} dt, \quad j = 0,1,2, \\ F_3(x,\lambda) = e^{-i\lambda x} + \int_{-x}^{x} K_3(x,t) e^{i\lambda t} dt, \end{gathered} \tag{10}$$

where the functions $K_j(x,t)$, $j = 0,1,2,3$ *have partial derivatives up to the fourth order.*

The second chapter is devoted to the spectral analysis of the boun-dary value problem generated by the equation

$$\ell_+\left(x, \frac{d}{dx}, \lambda\right)y = \left(\frac{d}{dx} - i\lambda\right)^3 \left(\frac{d}{dx} + i\lambda\right)y +$$

$$r(x)y' + (\lambda p(x) + q(x))y = 0, \ \ 0 < x < \infty \tag{11}$$

and the boundary conditions

$$\begin{aligned} U_v(y) = \alpha_{v_0} y(0) + \alpha_{v_1} y'(0) + \alpha_{v_2} y''(0) + \\ + \alpha_{v_3} y'''(0) = 0, \quad v = 1,2,3, \end{aligned} \tag{12}$$

where the complex-valued functions $r(x)$, $q(x)$ and $p(x)$ are defined in the interval $[0, +\infty)$, have continuous derivatives of the 3rd, 4th and 5th orders, respectively, and satisfy conditions (2). It is assumed that the linear forms $U_v(y)$, $v = 1,2,3$, included in (12) are linearly independent, i.e. the rank of the matrix

$$\begin{pmatrix} \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{30} & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

is equal to three. Based on the results of the second paragraph of the first chapter, it is established that under conditions (2), equation (11) for all $\operatorname{Im}\lambda \geq 0$ has solutions

$$y_j(x,\lambda) = x^{j-1} e^{i\lambda x} + \int_x^\infty K_j^+(x,t) e^{i\lambda t} dt, \ j = 1,2,3 \ , \tag{13}$$

and for $\operatorname{Im}\lambda \leq 0$ has a solution

$$y_4(x,\lambda) = e^{-i\lambda x} + \int_x^\infty K_4^+(x,t) e^{-i\lambda t} dt \ . \tag{14}$$

In this case, the kernels $K_j^+(x,t)$ have continuous partial derivatives up to the fourth order and satisfy the relations

$$\ell_t\left(x, \frac{\partial}{\partial x}, i\frac{\partial}{\partial t} \right) K_j^+(x,t) = 0, \tag{15}$$

$$\begin{aligned} \lim_{x+t\to\infty} \frac{\partial^{\alpha+\beta} K_j^+(x,t)}{\partial x^\alpha \partial t^\beta} = 0, \ \alpha + \beta \leq 0, \\ \int_x^\infty \left| K_j^+(x,t) \right| dt < \infty. \end{aligned} \tag{16}$$

By D we denote the set of functions $y = y(x) \in L_2(0, \infty)$ that satisfy the following conditions:

1) functions $y^{(\nu)}(x), \nu = \overline{0,3}$ are absolutely continuous on each finite segment $[0, b]$ and

$y^{(\nu)}(x) \in L_2(0, \infty), \nu = 0,...,3, \ y^{(3)}(x) \to 0, \ x \to \infty$;

2) $\ell_+\left(x, \frac{d}{dx}, \lambda\right) y \in L_2(0, \infty)$.

Let D_α be the set of functions from D satisfying the boundary conditions (12).

By the operator pencil L_λ^α we will understand a family of operators with domain of definition D_α and acting according to the rule

$$L_\lambda^\alpha y = \ell_+\left(x, \frac{d}{dx}, \lambda\right) y, y = y(x) \in D$$

where λ is the spectral parameter..

In section 2.1, a description of the domain of definition of the adjoint operator $\left(L_\lambda^\alpha\right)^*$ is given and the form of the adjoint operator is found.

In the second section of the second chapter, under additional conditions, i.e. if the coefficients of equation (11) satisfy the conditions

$$\begin{gathered} e^{\eta x} r^{(\nu)}(x) \in L_2(0, \infty), \\ e^{\eta x} p^{(\nu)}(x) \in L_2(0, \infty), \nu = 0,1, e^{\eta x} q(x) \in L_2(0, \infty) \end{gathered} \tag{17}$$

for some $\eta > 0$, the discrete spectrum of the operator pencil L_λ^α is studied. When we introduce the determinant A according to the formula When $\operatorname{Im}\lambda > 0$ we introduce the determinant $A(\lambda)$ using the formula

$$A(\lambda)=\begin{vmatrix} U_1(y_1(x,\lambda)) & U_1(y_2(x,\lambda)) & U_1(y_3(x,\lambda)) \\ U_2(y_1(x,\lambda)) & U_2(y_2(x,\lambda)) & U_2(y_3(x,\lambda)) \\ U_3(y_1(x,\lambda)) & U_3(y_2(x,\lambda)) & U_3(y_3(x,\lambda)) \end{vmatrix}. \qquad (18)$$

Theorem 4. *A complex number* λ_0, *where* $\operatorname{Im}\lambda_0 > 0$, *is an eigenvalue of the operator pencil* L_λ^α *if and only if*

$$A(\lambda_0)=0 \ . \qquad (19)$$

Theorem 5. *A complex number* λ_0, *where* $\operatorname{Im}\lambda_0 < 0$, *is an eigenvalue of the operator pencil* L_λ^α *if and only if the solution* $y_4(x,\lambda_0)$ *of equation (12) satisfies the equalities*

$$u_v(y_4(x,\lambda_0))=0, \ v=1,2,3 \qquad (20)$$

In this section we establish that an operator pencil L_λ^α can have at most a countable number of eigenvalues.

Theorem 6. *The eigenvalues of the operator pencil* L_λ^α *in the half-plane* $\operatorname{Im}\lambda > 0$ *form no more than a countable set, the limit points of which can only be located on the real axis.*

Theorem 7. *The eigenvalues of the operator pencil* L_λ^α *in the half-plane* $\operatorname{Im}\lambda < 0$ *form no more than a countable set, the limit points of which can only be located on the real axis.*

Theorem 8. *The operator pencil* L_λ^α *has no real eigenvalues.*

In the third section of Chapter II, the resolvent and continuous spectrum of the operator pencil L_λ^α are investigated. Let $\operatorname{Im}\lambda > 0$ and λ is not an eigenvalue of the operator pencil L_λ^α. Let $R_\lambda^{+\alpha}$ denote the resolvent of this pencil. Let $R_\lambda^{+\alpha} f = Y$, i.e. for a function $f(x) \in L_2(0,\infty)$ with bounded support the function $Y(x,\lambda)$ serves as a solution to the equation

$$\ell_+\left(x,\frac{d}{dx},\lambda\right)Y = f \ . \qquad (22)$$

This solution belongs to $L_2(0,\infty)$ and satisfies the boundary conditions (12). Then equation (11) has a fundamental system of solutions $Y_k(x,\lambda),\ k=\overline{1,4}$, where $Y_4(x,\lambda)\notin L_2(0,\infty)$. Moreover, without loss of generality, we can assume that the functions $Y_k(x,\lambda),\ k=1,2,3$ coincide with the solutions $y_k(x,\lambda),\ k=\overline{1,4}$, defined by formulas (13).

We introduce the Wronskian $W(x,\lambda)$ using the formula

$$W(x,\lambda)=\begin{vmatrix} Y_1(x,\lambda) & Y_2(x,\lambda) & Y_3(x,\lambda) & Y_4(x,\lambda) \\ Y_1'(x,\lambda) & Y_2'(x,\lambda) & Y_3'(x,\lambda) & Y_4'(x,\lambda) \\ Y_1''(x,\lambda) & Y_2''(x,\lambda) & Y_3''(x,\lambda) & Y_4''(x,\lambda) \\ Y_1'''(x,\lambda) & Y_2'''(x,\lambda) & Y_3''(x,\lambda) & Y_4''(x,\lambda) \end{vmatrix} \tag{21}$$

Let $W_i(x,\lambda)$ denote the algebraic complement of the element $Y_i'''(x,\lambda)$ of the determinant $W(x,\lambda)$. Let

$$Z_{5-k}(x,\lambda)=\frac{W_k(x,\lambda)}{W(x,\lambda)},\ k=1,2,3,4.$$

$$A(\lambda)=\det\left|U_v(Y_k)\right|_{v,k=1}^{3}\neq 0,$$

and the determinant of $A_k(\lambda)$ is obtained from $A(\lambda)$ by replacing $U_v(Y_k)$ with $U_v(Y_4)$. Let us introduce the following notations:

$$h_k(x,\lambda)=\frac{A_k(\lambda)}{A(\lambda)}Z_1(x,\lambda),\ k=1,2,3.$$

Note that the functions $Z_k(x,\lambda),\ k=1,2,3,4$ serve as solutions to the adjoint equation $\ell_+^*\left(x,\frac{d}{dx},\lambda\right)Z=0$. Let

$$Z_{5-i}(\xi,\lambda)=\omega_i^+(\xi,\lambda),\quad i=1,\dots,4.$$

$$h_i^+(\xi,\lambda)=\frac{A_i(\lambda)}{A(\lambda)}\omega_4^+(\xi,\lambda)..$$

$$K^+(x,\xi,\lambda)=\begin{cases}\sum_{i=1}^{3}\left[h_i^+(\xi,\lambda)+\omega_i^+(\xi,\lambda)\right]Y_i(x,\lambda), & \xi<x,\\ \sum_{i=1}^{3}h_i^+(\xi,\lambda)Y_i(x,\lambda)-\omega_4^+(\xi,\lambda)Y_4(x,\lambda), & \xi>x.\end{cases}$$

Theorem 9. *The following integral representation holds for the resolvent* $R_\lambda^{+\alpha}$ *:*

$$Y(x,\lambda)=\left(R_\lambda^{+\alpha}f\right)(x)=\int_0^\infty K^+(x,\xi,\lambda)f(\xi)d\xi \qquad (23)$$

Further, in this section the resolvent is also investigated for values of λ lying in the lower half-plane. According to the general theory, equation (11) for $\mathrm{Im}\lambda<0$ has linearly independent solutions $Y_1^-(x,\lambda), Y_2^-(x,\lambda), Y_3^-(x,\lambda), Y_4^-(x,\lambda)$, where $Y_1^-(x,\lambda)\in L_2(0,\infty)$, $Y_i^-(x,\lambda)\notin L_2(0,\infty)$, $i=2,3,4$. The corresponding adjoint equation to (11) has linearly independent solutions $Z_i^-(x,\lambda)$ such that $Z_i^-(x,\lambda)\in L_2(0,\infty)$, $i=1,2,3$ and $Z_4^-(x,\lambda)\notin L_2(0,\infty)$. In this case, to find the solution $Y(x,\lambda)=R_\lambda^{-\alpha}f$ of the equation $\ell_+\left(x,\frac{d}{dx},\lambda\right)Y=f$ the method of variation of constants is used and the same reasoning is repeated as for $R_\lambda^{+\alpha}$. Let

$$Z_{5-i}^-(\xi,\lambda)=\omega_i^-(\xi,\lambda), \quad i=1,\ldots,4,$$

$$h^-(x,\lambda)=\frac{1}{U_v\left(Y_1^-\right)}\sum_{i=2}^{4}U_v\left(Y_i^-\right)\omega_i^-(\xi,\lambda),$$

$$K^-(x,\xi,\lambda)=\begin{cases}\left[h^-(\xi,\lambda)+\omega_i^-(\xi,\lambda)\right]Y_1^-(x,\lambda), & \xi<x,\\ h^-(\xi,\lambda)Y_1^-(x,\lambda)-\sum_{i=2}^{4}\omega_i^-(\xi,\lambda)Y_1^-(x,\lambda), & \xi>x.\end{cases}$$

Theorem 10. *The following integral representation holds for the resolvent* $R_\lambda^{-\alpha}$ *:*

$$Y(x,\lambda)=\left(R_{\lambda}^{-\alpha} f\right)(x)=\int_{0}^{\infty} K^{-}(x,\xi,\lambda) f(\xi) d\xi \ .$$

At the end of section 2.3 the following theorem is proved for the resolvent.

Theorem 11. *If a complex number* λ *belongs to the resolvent set of an operator pencil* L_{λ}^{α}, *then its resolvent is a bounded integral operator in* $L_2(0,\infty)$. *The kernel of the integral operator satisfies Carleman-type conditions. As* λ *tends to the real axis, the norm of the resolvent increases infinitely and all points of the real axis belong to the continuous spectrum of the operator pencil* L_{λ}^{α}.

Section 2.4 is devoted to the derivation of the formula for the expansion in eigenfunctions of the operator pencil L_{λ}^{α}. In this section it is assumed that the coefficients of equation (11), in addition to conditions (2), also satisfy conditions (17). In this case, the operator pencil L_{λ}^{α} can have only a finite number of eigenvalues.

Theorem 12. *Let a pencil* L_{λ}^{α} *have a finite number of non-real eigenvalues* $\lambda_1,\lambda_2,\ldots,\lambda_p$ *and have no spectral singularities. Then for a sufficiently smooth and compactly supported function* $f(x)$ *the spectral decomposition formula holds*

$$f(x)=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\lambda^{3}\left[R_{\lambda+i0}^{+\alpha}-R_{\lambda-i0}^{-\alpha}\right] f d\lambda+\sum_{j=1}^{p_1}\underset{\lambda=\lambda_j}{res}\left[\lambda^{3}R_{\lambda}^{\alpha} f\right], \tag{24}$$

where

$$\left[R_{\lambda+i0}^{+\alpha}-R_{\lambda-i0}^{-\alpha}\right] f=\int_{0}^{\infty}\left[K^{+}(x,\xi,\lambda+i0)-K^{-}(x,\xi,\lambda-i0)\right] f(\xi) d\xi \ .$$

In Section 2.5, the spectral properties of the operator pencil L_{λ}^{α} with compactly supported coefficients are studied. Let the coefficients $r(x)$, $q(x)$, $p(x)$ of the operator L_{λ}^{α} vanish outside the finite segment $[0,a]$. Let us assume that $f(x)$ is a sufficiently smooth function whose support does not contain the point $x=0$. Let

$$y_a^{\pm}(x,\lambda)=\int_0^{\infty} K_a^{\pm}(x,\xi,\lambda)f(\xi)d\xi,$$

$$y_a(x,\lambda)=\begin{cases} y_a^{+}(x,\lambda), & \operatorname{Im}\lambda\ge 0,\\ y_a^{-}(x,\lambda), & \operatorname{Im}\lambda<0.\end{cases}$$

Theorem 13. *Let the coefficients of an operator pencil* L_λ^α *vanish outside a finite interval* $[0,a]$ *and let this pencil have a finite number of non-real eigenvalues* $\lambda_j^+(a),\ j=1,...,l,\ \lambda_j^-(a),\ j=1,...,s$ *and real spectral singularities* $\mu_j^+(a),\ j=1,...,n,\ \mu_j^-(a),\ j=1,...,n$. *Then for a sufficiently smooth and compactly supported function* $f(x)$ *the following spectral expansion formulas hold:*

$$0=\sum_{j=1}^{\ell}\underset{\lambda=\lambda_j^+(a)}{res}\lambda^k y_a^+(x,\lambda)+\sum_{j=1}^{s}\underset{\lambda=\lambda_j^+(a)}{res}\lambda^k y_a^-(x,\lambda)+$$

$$+\sum_{j=1}^{m}\underset{\lambda=\mu_j^+(a)}{res}\lambda^k y_a(x,\lambda)+\sum_{j=1}^{n}\underset{\lambda=\mu_j^-(a)}{res}\lambda^k y_a(x,\lambda)-$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda^k\left[y_a^+(x,\lambda)-y_a^-(x,\lambda)\right]d\lambda,\qquad k=0,1,2,$$

$$f(x)=\sum_{j=1}^{\ell}\underset{\lambda=\lambda_j^+(a)}{res}\lambda^3 y_a^+(x,\lambda)+\sum_{j=1}^{s}\underset{\lambda=\lambda_j^+(a)}{res}\lambda^3 y_a^-(x,\lambda)+$$

$$+\sum_{j=1}^{m}\underset{\lambda=\mu_j^+(a)}{res}\lambda^3 y_a(x,\lambda)+\sum_{j=1}^{n}\underset{\lambda=\mu_j^-(a)}{res}\lambda^3 y_a(x,\lambda)-$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda^3\left[y_a^+(x,\lambda)-y_a^-(x,\lambda)\right]d\lambda.$$

In the last paragraph of the second chapter, the expansion formulas obtained in the previous paragraphs are also derived using the “cut coefficients” method.

Let's introduce the following function:

$$\eta_a(x) = \begin{cases} 1, & 0 \le x \le a, \\ 0, & x > 0. \end{cases}$$

Let us construct the following functions by means of the coefficients $r(x)$, $q(x)$, $p(x)$ of the operator pencil L_λ^α:

$$r_a(x) = r(x)\eta_a(x), \; p_a(x) = p(x)\eta_a(x),$$
$$q_a(x) = q(x)\eta_a(x).$$

It follows from conditions (2) that the relations

$$\lim_{a\to\infty} \int_0^\infty x^4 \left| r_a^s(x) - r^{(s)}(x) \right| dx = 0, \quad s = 0,..,3,$$
$$\lim_{a\to\infty} \int_0^\infty x^4 \left| p_a^s(x) - p^{(s)}(x) \right| dx = 0, \quad s = 0,...,5, \tag{26}$$
$$\lim_{a\to\infty} \int_0^\infty x^4 \left| q_a^s(x) - q^{(s)}(x) \right| dx = 0, \quad s = 0,...,4.$$

are also valid. Then since $f(x)$ is a rather smooth and finite function we consider the fololwing boundary value problem:

$$\ell_a\left(x, \frac{d}{dx}, \lambda\right) y_a(x,\lambda) = \left(\frac{d}{dx} - i\lambda\right)^3 \left(\frac{d}{dx} + i\lambda\right) y_a(x,\lambda) + \tag{27}$$
$$+ r_a'(x) y_a'(x,\lambda) + (\lambda p_a(x) + q_a(x)) y_a(x,\lambda) = f(x),$$

$$U_v(y_a(x,\lambda)) = 0, \; v = 1,2,3 \; . \tag{28}$$

Conditions (28) are the same with the boundary conditions $U_v(y\ (x,\lambda)) = 0, \; v = 1,2,3$. We are interested in the solution of problem (27), (28) that satisfies the condition $y_a(x,\lambda) \in L_2(0, \infty)$.

It is clear that if the function $y\ (x,\lambda)$ is a solution of the equation $L_\lambda^\alpha y = f$ satisfying the condition $y(x,\lambda) \in L_2(0, \infty)$ and the function $K(x,\xi,\lambda)$ is the kernel of the resolvent of the operator L_λ^α, then the formula

$$y(x,\lambda)=y_a(x,\lambda)-\int_0^\infty K(x,\xi,\lambda)[\eta_a(\xi)-1]r(\xi)y_a(\xi,\lambda)d\xi+$$

$$+\int_0^\infty K(x,\xi,\lambda)[\eta_a(\xi)-1](\lambda p(\xi)+q(\xi))y_a(\xi,\lambda)d\xi \qquad (29)$$

is valid. On the other hand, the indications

$$y(x,\lambda)=\int_0^\infty K(x,\xi,\lambda)f(\xi)d\xi,\ \ y_a(x,\lambda)=\int_0^\infty K_a(x,\xi,\lambda)f(\xi)d\xi$$

are also valid.

At first we construct the resolvent of the problem (27), (28) and derive expansion formulas by its means:

$$0=\sum_{j=1}^{\ell}\underset{\lambda=\lambda_j^+}{res}\lambda^k y(x,\lambda)+\sum_{j=1}^{\ell}\underset{\lambda=\lambda_j^-}{res}\lambda^k y(x,\lambda)-$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda^k\left[y^+(x,\lambda)-y^-(x,\lambda)\right]d\lambda, \qquad k=0,1,2,$$

$$f(x)=\sum_{j=1}^{\ell}\underset{\lambda=\lambda_j^+}{res}\lambda^3 y(x,\lambda)+\sum_{j=1}^{\ell}\underset{\lambda=\lambda_j^-}{res}\lambda^3 y(x,\lambda)-$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda^3\left[y^+(x,\lambda)-y^-(x,\lambda)\right]d\lambda.$$

So, when the coefficients of the L_λ^α operator pencil is from class (2) the spectral expansion formulas can be obtained by passing from the spectral expansion formulas of operator pencils corresponding to “continuous” coefficients to limit.

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CONCLUSION

The dissertation is devoted to the construction of transformation operators for fourth-order differential equations with triple characteristics and the study of their spectral properties by means of transformation operators. The following main results were obtained in the work:

-transformation operators with a condition at infinity for a pencil of differential operators of the 4th order with triple characteristics are constructed. The properties of the kernels of integral representations of transformation operators are investigated;

- transformation operators with the condition at the point $x = 0$ for a pencil of differential operators of the 4th order with triple characteristics are constructed. The properties of the kernels of integral representations of transformation operators are investigated;

- the distribution of eigenvalues of a pencil of fourth-order differential operators with triple characteristics in the complex plane was studied;

- an integral representation of the resolvent of a pencil of differential operators of the 4th order with triple characteristics was found;

- formulas for expansion in eigenfunctions of discrete and continuous spectrum of a pencil of differential operators of the 4th order with triple characteristics are obtained.

The main results of the dissertation were published in the following works:

1. Aliev, S.A. On the existence of transformation operator for a fourth order differential equation with triple characteristics // -Baku: Proceedings of the Institute of Mathematics and Mechanics of ANAS, – 2013. v. 39, – p. 3-8.
2. Əliyev, S.A. Təkrar xarakteristikalı diferensial tənliklər üçün çevirmə operatorunun varlığı // "Regional inkişaf və böyük mədəniyyət: Mənşə, harmoniya və topologiya məsələləri" Beynəlxalq konfarns, -Naxçıvan: NDU – 2013. – s.58.
3. Алиев, С.А. О существовании оператора преобразования для дифференциалъного уравнения 4-го порядка с трехкратными характеристиками // Нахчыванский Гос. Унив. Научные труды, серия физ.-мат. и техн. наук, –Нахчыван, -2014, № 3 (59), – с.37-41.
4. Orudzhev, E.G., Aliyev, S.A. Construction of a kernel of the transformation operator for a fourth order differential bundle with multiple characteristics // -Baku: Proceedings of IMM of ANAS, Special issue in memory of M.G.Gasymov on his 75-th birthday, special issue -2014. v.40, №2, – p.351-358.
5. Алиев, С.А. Спектральное разложение по собственным функциям одного дифференциального пучка 4-го порядка с трех-кратным характеристическим корнем // Journal of Contemporary Applied Mathematics, – 2019, т.9, №2, – с. 46-56
6. Orudzheva, E.G., Aliev, S.A. Spectral analysis of a fourth-order differential bundle with a triple characteristic root // "Spectral Theory and its Applications" an Intern. Workshop dedicated to the 80 anniv. of an acad. Mirabbas Gasymov, –Baku: –7-8 June, – 2019, – p.133-136.
7. Оруджев, Э.Г., Алиев, С.А. Исследование спектра и резольвенты одного дифференциального пучка 4-то порядка с трехкратным характеристическим корнем // Научные Ведомости Белгородского Гос. Унив. Математика. Физика, – 2019, – т.51, №1, – с.52-63.

8. Алиев, С.А. 4-х кратное разложение по собственным функциям дифференциального пучка с 3-х кратным непрерывным спектром // “Scientific Collection Interconf” Proceedings of the 8-th Intern. Scientific and Practical confer. – Manchester, Great Britain: -26-28 December, – 2020, № 3(39), – с.1494-1506.
9. Оруджев Э. Г., Алиев С.А. Исследование спектра и резольвенты одного дифференциального пучка 4-то порядка с трехкратным характеристическим корнем // Вестник Воронежского Гос. Унив. Серия: Физика. Математика, 2020, №. 3, с.95-106.
10. Aliyev, S.A. On spectral properties of the family of fourth order differential operators with finite and exponentially decreasing coefficients // -Baku: Journal of Contemporary Applied Mathematics, –2020. v.10, №2, –p.101-109.
11. Aliyev, S.A. On transformation operators with conditions in zero for a fourth order differential equation with triple characteristics // “Problems of Modern Mathematics and its applications” 70-th anniversary of A.A.Borubaev, Kyrgyzstan, Bishkek - Issyk-Kul June 15-19,-2021, –p. 51-52.

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