Magomed F. MEKHTIYEV, Nihan A. ALIYEV, MEHRAN R. FATEMI

ON FREDHOLM PROPERTY OF A BOUNDARY VALUE PROBLEM FOR A FIRST ORDER EQUATION WITH GENERAL BOUNDARY CONDITIONS

Abstract

In the paper, we consider a boundary value problem for a loaded integro-differential equation. The differential part of the equation is an elliptic type equation of first order. In addition to integro-differential terms, the considered equation contains boundary values in the form of loading. Considering that the differential part of the equation is a Cauchy-Riemann equation (a first order equation), it is not possible to give local boundary conditions so that, the boundary were a support for a boundary condition (may be there is no solution). Therefore, non-local boundary conditions are considered. Without loss of generality, we add global terms to boundary conditions (in the form of integral).

Introduction. As is known, [1], [6] in theory of partial equations, Laplace equation (second order equation) is taken as a model for elliptic type equation. As it is possible to consider local boundary conditions (the number of conditions equals the half of order of equation) for Laplace equation (being a support on the boundary), the Dirichlet and Neumann problems are the simplest problems.

Here, as the principal part of the equation of the considered problem is Cauchy-Riemann equation, it is not convenient to consider the local boundary condition. Since the boundary is not a support for the condition one obtains a generalized Cauchy problem instead of a boundary value problem and this is not correct [3], [4]. Here we used non-local boundary conditions for the boundary to be a support in the boundary condition. For the problem to cover all possible general cases, in addition to integro-differential terms we added loaded terms to the equation, and in addition to non-local terms we added global terms (integral) to the boundary conditions.

Notice that if in integro-differential equation, the derivatives of the unknown function participate with differentiable coefficients in the integrals, then by the piecewise integration method we remove the derivative in these terms, there will arise loaded terms.

Problem statement. Let's consider the following problem for a first order linear partial integro-differential loaded equation under the general linear boundary condition:

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} + \int\limits_D K(x,\xi)u(\xi)d\xi +$$

$$+\int_{a_1}^{b_1}\left[K_1(x,\xi_1)u(\xi_1,\gamma_1(\xi_1))+K_2(x,\xi_1)u(\xi_1,\gamma_2(\xi_1))\right]d\xi_1+$$

$$+K_{01}(x)u(x_1,\gamma_1(x_1))+K_{02}(x)u(x_1,\gamma_2(x_1))=f(x),\ x=(x_1,x_2)\in D\subset R^2 \quad (1)$$

$$\alpha_1(x_1)u(x_1,\gamma_1(x_1))+\alpha_2(x_1)u(x_1,\gamma_2(x_1))+$$

$$+\int_D A(x_1,\xi)u(\xi)d\xi+\int_{a_1}^{b_1}\left[A_1(x_1,\xi_1)u(\xi_1,\gamma_1(\xi_1))+\right.$$

$$\left.+A_2(x_1,\xi_1)u(\xi_1,\gamma_2(\xi_1))\right]d\xi_1=g(x_1),\ x_1\in[a_1,b_1], \quad (2)$$

here $i=\sqrt{-1}$, $K(x,\xi),K_j(x,\xi_1),K_{oj}(x),j=1,2;\ f(x),\ \alpha_j(x_1),\ A_j(x_1,\xi_1),\ j=1,2; A(x_1,\xi)$ and $g(x_1)$ are the given continuous functions, $u(x)$ is a desired function. Domain D is a convex plane domain in the direction x_2, its Γ boundary is Lyapinov line [7].

It we project this domain parallel to x_2 to the axis x_1, it is assumed that its boundary are the equations $x_2=\gamma_1(x_1)$ and $x_2=\gamma_2(x_1)$ of parts τ_1 and τ_2 of Γ.

Fundamental solution. Here we'll use the expression

$$U(x-\xi)=\frac{1}{2\pi}\frac{1}{x_2-\xi_2+i(x_1-\xi_1)} \quad (3)$$

known [2] for the fundamental solution of Cauchy-Riemann equation. As the expression (3) is a fundamental solution for Cauchy-Riemann equation, the relation

$$\frac{\partial U(x-\xi)}{\partial x_2}+i\frac{\partial U(x-\xi)}{\partial x_1}=\delta(x-\xi) \quad (4)$$

is satisfied. Here $\delta(x)$ is Dirac's $\delta-$function [7], [8].

Main relations and necessary conditions. If we multiply the both hand sides of the given equation (1) by the fundamental solution $U(x-\xi)$, integrate with respect to the domain D, and apply the Ostragradsky Gauss formula, we get [7], [9]:

$$\int_D\left(\int_D K(x,\xi)u(\xi)d\xi\right)U(x-\eta)dx+\int_D\left(\int_{a_1}^{b_1}\left[K_1(x,\xi_1)u(\xi_1,\gamma_1(\xi_1))+\right.\right.$$

$$\left.\left.+K_2(x,\xi_1)u(\xi_1,\gamma_2(\xi_1)\right]d\xi_1\right)U(x-\eta)dx+\int_D K_{01}(x)u(x_1,\gamma_1(x_1))U(x-\eta)dx+$$

$$+\int_D K_{02}(x)u(x_1,\gamma_2(x_1))U(x-\eta)dx-\int_D f(x)U(x-\eta)dx+$$

$$\int_{\Gamma_1} u(x)U(x-\eta)\cos(\nu,x_2)dx+\int_{\Gamma_2} u(x)U(x-\eta)\cos(\nu,x_2)dx+$$

$$+i\int\limits_{\Gamma_1} u(x)U(x-\eta)\cos(\nu,x_1)dx+i\int\limits_{\Gamma_2} u(x)U(x-\eta)\cos(\nu,x_1)dx=$$

$$=\begin{cases} u(\eta), & \eta\in D,\\ \frac{1}{2}u(\eta), & \eta\in\Gamma, \end{cases} \tag{5}$$

here by ν we denote an external normal drawn to the boundary D of the domain Γ. So, we get the following statement:

Theorem 1. *If the plane domain D is convexly bounded in the direction x_2, its Γ boundary is Lyapinov line, the coefficients $K(x,\xi)$, $K_j(x,\xi_1), K_{0j}(x)$, $j=1,2$ and the right hand side $f(x)$ of equation (1) are continuous, then the relations (5) for arbitrary solution of this equation determined in D, are satisfied.*

The second expressions of the obtained (5) are necessary conditions. Let's write them separately

$$\frac{1}{2}u(\eta,\gamma_k(\eta_1))=\int\limits_D(\int\limits_D K(x,\xi)u(\xi)d\xi)U(x_1-\eta_1,x_2-\gamma_k(\eta_1))dx+$$

$$+\int\limits_D(\int\limits_{a_1}^{b_1}[K_1(x,\xi_1)u(\xi_1,\gamma_1(\xi_1))+K_2(x,\xi_1)u(\xi_1,\gamma_2(\xi_1))]\,d\xi_1)U(x_1-\eta_1,x_2-\gamma_k(\eta_1))dx+$$

$$+\int\limits_D K_{01}(x)u(x_1,\gamma_1(x_1))U(x_1-\eta_1,x_2-\gamma_k(\eta_1))dx+$$

$$+\int\limits_D K_{02}(x)u(x_1,\gamma_2(x_1))U(x_1-\eta_1,x_2-\gamma_k(\eta_1))dx-$$

$$-\int\limits_D f(x)U(x_1-\eta_1,x_2-\gamma_k(\eta_1))dx-$$

$$-\int\limits_{a_1}^{b_1} u(x_1,\gamma_1(x_1))U(x_1-\eta_1,\gamma_1(x_1)-\gamma_k(\eta_1))\left[1-i\gamma_1'(x_1)\right]dx_1+$$

$$+\int\limits_{a_1}^{b_1} u(x_1,\gamma_2(x_1))U(x_1-\eta_1,\gamma_2(x_1)-\gamma_k(\eta_1))\left[1-i\gamma_2'(x_1)\right]dx_1, \tag{6}$$

$$k=1,2;\ \eta_1\in(a_1,b_1)$$

It in the last two terms of the necessary condition (6), the variables x and η coincide in the same boundary Γ_s $(s=1,2)$, there arises singularity.

Separation of singularity and regularization. Let's separate the singularity in relation (6). As

$$\int\limits_{a_1}^{b_1} u(x_1,\gamma_s(x_1))U(x_1-\eta_1,\gamma_s(x_1)-\gamma_s(\eta_1))(1-i\gamma_s'(x_1))dx_1=$$

$$= -\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_s(x_1))}{x_1-\eta_1}dx_1 + \frac{1}{2\pi}\int_{a_1}^{b_1}u(x_1,\gamma_s(x_1))\frac{i\left[\gamma_s'(\sigma_s)-\gamma_s'(x_1)\right]}{\left[\gamma_s'(\sigma_s)+i\right](x_1-\eta_1)}dx_1$$

from (6) we get

$$u(\eta_1,\gamma_1(\eta_1)) = \frac{i}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{x_1-\eta_1}dx_1 + \ldots,$$

$$u(\eta_1,\gamma_2(\eta_1)) = -\frac{i}{\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))}{x_1-\eta_1}dx_1 + \ldots, \tag{7}$$

$$\eta_1 \in (a_1,b_1).$$

here the sum of terms not containing singularity are denoted by dots.

Let's engage in regularization of singularities (7) contained in necessary conditions (6).

Notice that, there are two types of singularity. One of them is general state singularity. Such type singularities are regularized by the Poincare-Bertman formula [10] according to the scheme given in [11]. In the second case, parameter is on the spectrum. In this case the Poincare-Bertran formula is reduced to the first kind Fredholm integral equation and this singularity is regularized by a special way [9], [12]. For that we take into account (7) and consider the following linear combination:

$$\alpha_1(\eta_1)u(\eta_1,\gamma_1(\eta_1)) - \alpha_2(\eta_1)u(\eta_1,\gamma_2(\eta_1)) =$$

$$= \frac{i}{\pi}\int_{a_1}^{b_1}\frac{\alpha_1(\eta_1)u(\eta_1,\gamma_1(x_1)) + \alpha_2(\eta_1)u(\eta_1,\gamma_2(x_1))}{x_1-\eta_1}dx_1 + \ldots, \qquad \eta_1 \in (a_1,b_1).$$

Here if we take into account the boundary condition (2) in the right had side integrals, we get:

$$\alpha_1(\eta_1)u(\eta_1,\gamma_1(\eta_1)) - \alpha_2(\eta_1)u(\eta_1,\gamma_2(\eta_1)) =$$

$$= \frac{i}{\pi}\int_{a_1}^{b_1}\left\{g(x_1) - \int_D A(x_1,\xi)u(\xi)d\xi - \int_{a_1}^{b_1}\left[A_1(x_1,\xi_1)u(\xi_1,\gamma_1(\xi_1)) + \right.\right.$$

$$\left.\left. + A_2(x_1,\xi_1)u(\xi_1,\gamma_2(\xi_1))\right]d\xi_1\right\}\frac{dx_1}{x_1-\eta_1} + \ldots, \qquad \eta_1 \in (a_1,b_1). \tag{8}$$

We easily give meaning the first singular integral contained in the right hand side [13], i.e. it is regularized. In the remaining terms, these terms are regularized by changing the turn of integrals. Thus, we get:

Theorem 2. *Under the conditions of theorem 1, if variables $\alpha_j(x_1)$, $j = 1,2$ are from the Holder class, $A(x_1,\xi)$, $A_j(x_1,\xi_1)$, $j = 1,2$ is continuous, $g(x_1)$ is differentiable and satisfies the conditions $g(a_1) = g(b_1) = 0$, then regular relation (8)* *is satisfied.*

Fredholm property. The obtained regular expression (8) together with the given boundary condition (2), gives a system of Fredholm type normal integral equations of second kind with respect to the functions $u(\eta_1, \gamma_j(\eta_1))$, $j = 1, 2$. The kernel of this system is not regular. Its right hand side is linear dependent on the integral of $u(x)$ with respect to D. That is why, if this system will be solved with respect to boundary values $u(\eta_1, \gamma_j(\eta_1))$, $j = 1, 2$ these boundary values will linearly dependent on the integral of $u(x)$ with respect to D.

Finally, if we write the expression obtained for these boundary values in the left hand side of main relations (5), then the first expression of these main relation gives us Fredholm type second order integral equation for the function $u(x)$. The kernel of this integral equation is regular. So, we get the following statement.

Theorem 3. *If the conditions are satisfied within the conditions of theorem 2, then problem (1), (2) is of Fredholm property.*

Similar investigations were carried out for ordinary differential equations as well.

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Magomed F. Mekhtiyev, Nihan A. Aliyev, Mehran R. Fatemi
Baku State University.
23, Z.I. Khalilov str., AZ1148, Baku, Azerbaijan.
E-mail: fatemi.mehran@yahoo.com.

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