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EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A PROBLEM ON OPTIMAL CONTROL OF ECOLOGICAL SYSTEMS DESCRIBED BY ORDINARY DIFFERENTIAL EQUATIONS

Abstract

In the paper we study the existence and uniqueness of the solution of an optimal control problem for second order ordinary differential equation with Lions functional type quality criterion when the controls appear in the coefficients and in the right side of the equation. The theorems of the existence of even one and unique solution of the problem under consideration are proved.

Introduction

In the paper we consider the existence and uniqueness of the solution of an optimal control problem for second order ordinary differential equation with Lions functional type quality criterion, that oftenly arises while modeling and control of ecological processes [1]. It is known that wind velocity, metabolism factors that appear in the coefficient of the equation, and also ecological active sources density contained in the right side of this equation may play as a control. It should be noted that such optimal control problems with a special quality criterion, especially with Lions quality have not been studied enough. In this direction we can note the papers [2,3] wherein optimal control problems for a first order ordinary differential equations are studied. However, the optimal control problem considered in this paper differs from the previous ones by its statement and functional spaces used in the paper.

1. Problem Statement

Let the controlled process be described by the following equation:

$$\rho\left(x\right)\frac{d^2u}{dx^2}+\frac{1}{x}\nu_1\left(x\right)\frac{du}{dx}-\nu_0\left(x\right)u=w\left(x\right), \tag{1}$$

where $a\le x\le b$, $a>0$, $b>0$ are the given numbers, $\rho\left(x\right)$ is the mass, $\nu_0\left(x\right)$ is the mass transfer factor, $\nu_1\left(x\right)$ is the wind velocity, $w\left(x\right)$ is the ecological active sources density. It is known that the equation describes gas or liquid flow and oftenly arises while modeling stationary ecological processes. It is clear that by changing $\nu_0\left(x\right)$, $\nu_1\left(x\right)$, $w\left(x\right)$ we can affect on an ecological object described by equation (1), i.e. we can control this object. In place of the control we'll choose the vector functions $\nu=\nu\left(x\right)=\left(\nu_0\left(x\right),\nu_1\left(x\right),w\left(x\right)\right)$. We define the set of admissible controls in the form

$$V\equiv\{\nu=\nu\left(x\right)=\left(\nu_0\left(x\right),\nu_1\left(x\right),w\left(x\right)\right),\quad \nu_m\in L_2\left(a,b\right),$$

$$m=0,1,\quad b_0\le\nu_0\left(x\right)\le\widetilde{b}_0,\quad 0\le\nu_1\left(x\right)\le b_1,\ \overset{0}{\forall}x\in\left(a,b\right),\quad \left\|w\right\|_{L_2(a,b)}\le b_2\},$$

where $\widetilde{b}_0 > 0, \quad b_m > 0, \quad m = \overline{0,2}$ are the given numbers.

For each $\nu \in V$, denote by $u_1 = u_1(x)$ the solution of equation (1) under the boundary condition

$$u(a) = u(b) = 0, \tag{2}$$

and by $u_2 = u_2(x)$ the solution of equation (1) under the boundary condition:

$$\frac{du(a)}{dx} = \frac{du(b)}{dx} = 0. \tag{3}$$

It is clear that $u_1 = u_1(x)$ is the solution of the first boundary value problem, $u_2 = u_2(x)$ of the second boundary value problem for a second order ordinary differential equation of the form (1).

Allowing for the above-mentioned remarks, we can state the following optimal control problem on minimization of the functional:

$$J_\alpha(\nu) = \|u_1 - u_2\|^2_{L_2(a,b)} + \alpha \|\nu - \omega\|^2_H \tag{4}$$

on the set V under the conditions

$$\rho(x)\frac{d^2u_k}{dx^2} + \frac{1}{x}\nu_1(x)\frac{du_k}{dx} - \nu_0(x)u_k = w(x), \; x \in (a,b), \quad k = 1,2, \tag{5}$$

$$u_1(a) = u_1(b) = 0, \tag{6}$$

$$\frac{du_2(a)}{dx} = \frac{du_2(b)}{dx} = 0, \tag{7}$$

where $\alpha \geq 0$ is a given number, $H = (L_2(a,b))^3$, $\omega \in H$ is a given element, $\rho = \rho(x)$ is a given measurable bounded function satisfying the condition:

$$\rho_0 \leq \rho(x) \leq \rho_1, \quad \overset{0}{\forall} x \in (a,b), \quad \rho_0, \rho_1 = const > 0. \tag{8}$$

For each $\nu \in V$ a problem on definition of the functions $u_k = u_k(x) \equiv u_k(x;\nu)$, $k = 1,2$ from conditions (5)-(7) will be called a reduced problem.

Definition 1. *Under the solution of the reduced problem, for each $\nu \in V$ we understand the functions $u_k = u_k(x) \equiv u_k(x;\nu)$, $k = 1,2$ from the spaces $W^2_{2,0}(a,b)$ and $W^2_2(a,b)$, respectively, satisfying equations of (5) for almost all $x \in (a,b)$ and boundary conditions (6), (7).*

It is clear that the reduced problem consists of two boundary value problems (5), (6) and (5), (7). Problem (5), (6) is a first, (5), (7) is a second boundary value problem for a second order ordinary differential equation with measurable bounded coefficients and quadratically summable right side. It is known that under the accepted conditions, these boundary value problems are special cases of boundary value problems for elliptic equations studied in the papers [4-6]. Indeed, if we multiply the both sides of equation (1) by the function:

$$\mu(x) = e^{\int\limits_a^x \frac{\nu_1(\tau)}{\tau\rho(\tau)}d\tau}, \quad \forall x \in [a,b], \tag{9}$$

we get the following equation:

$$\frac{d}{dx}\left(a_0(x)\frac{du}{dx}\right) - a_1(x)u = f(x), \quad x \in (a,b), \tag{10}$$

where

$$a_0(x) = \mu(x), \quad a_1(x) = \frac{\nu_1(x)}{\rho(x)}\mu(x), \quad f(x) = \frac{w(x)}{\rho(x)}\mu(x). \tag{11}$$

Under the accept conditions and for each $\nu \in V$, the coefficients $a_0(x)$, $a_1(x)$ and the right side $f(x)$ of equation (10) satisfy the conditions:

$$1 \leq a_0(x) \leq e^{\frac{b_1}{a\rho_0}}, \quad \left|\frac{da_0(x)}{dx}\right| \leq \frac{b_1}{a\rho_0} e^{\frac{b_1(b-a)}{a\rho_0}}, \ \forall x \in (a,b), \tag{12}$$

$$\frac{b_0}{\rho_1} \leq a_1(x) \leq \frac{\widetilde{b}_0}{\rho_0} e^{\frac{b_1(b-a)}{a\rho_0}}, \ \forall x \in (a,b), \tag{13}$$

$$f \in L_2(a,b). \tag{14}$$

Now if we consider boundary value problems on definition of $u_1 = u_1(x)$ from conditions (10), (2), and $u_2 = u_2(x)$ from conditions (10), (3), then by means of the results on solvability of boundary value problems for elliptic equations, known from the papers [4,5], we can formulate the following statement.

Theorem 1. *Let $a > 0$ and condition (8) be fulfilled. Then for each $\nu \in V$ the reduced problem (5)-(7) has a unique solution, $u_1 \in W_{2,0}^2(a,b)$, $u_1 \in W_2^2(a,b)$, and the following estimations are valid:*

$$\|u_1\|_{W_{2,0}^2(a,b)} \leq c_1 \|w\|_{L_2(a,b)}, \tag{15}$$

$$\|u_2\|_{W_2^2(a,b)} \leq c_2 \|w\|_{L_2(a,b)}, \tag{16}$$

where $c_1 > 0$, $c_2 > 0$ are the constants independent of w.

2. Existence of the solution of an optimal control problem

In this section we'll study the existence of the solution of optimal control problem (4)-(7) for any $\omega \in H$ at $\alpha \geq 0$.

Theorem 2. *Let the conditions of theorem 1 be fulfilled, and $\alpha \geq 0$ be a given number. Then optimal control problem (4)-(7) has even if one solution for any $\omega \in H$.*

Proof. Let $\{\nu^m\} \subset V$ be any minimizing sequence in problem (4)-(7), i.e.

$$\lim_{m\to\infty} J_\alpha(\nu^m) = J_{\alpha^*} = \inf_{\nu\in V} J_\alpha(\nu).$$

From the structure of the set V it is clear that it is a bounded set from the space $B \equiv (L_\infty(a,b))^2 \times L_2(a,b)$. Then from this sequence $\{\nu^m\}$ we can distinguish a subsequence that for simplicity of presentation we'll again denote by $\{\nu^m\}$, i.e.

$$\nu_p^m \to \nu_p\,(*) \quad weakly \ \ in \ \ L_\infty(a,b), \quad p = 0,1, \tag{17}$$

$$w^m \to w \ \ weakly \ \ in \ \ L_2(a,b) \tag{18}$$

as $m \to \infty$. From the convexity and closeness of the set V, in the space B we can get that it is a $(*)$ weakly closed set in B. Therefore, $\nu \in V$ and the following limit relations hold:

$$\int_a^b \nu_p^m(x)\, q(x)\, dx \to \int_a^b \nu_p(x)\, q(x)\, dx, \quad p = 0,1, \tag{19}$$

$$\int\limits_a^b w^m(x)\, g(x)\, dx \to \int\limits_a^b w(x)\, g(x)\, dx, \tag{20}$$

for any functions $q \in L_1(a,b)$, $g \in L_2(a,b)$ as $m \to \infty$.

Let $u_{km}(x) \equiv u_k(x, \nu^m)$, $k = 1,2$, $m = 1,2,...$ be the solution of reduced problem (5)-(7) for each $\nu^m \in V$, $m = 1,2,....$ By theorem 1, reduced problem (5)-(7) for each $\nu^m \in V$, $m = 1,2,...$ has a unique solution $u_{1m} \in W_{2,0}^2(a,b)$, $u_{2m} \in W_2^2(a,b)$ and the following estimations are valid:

$$\|u_{1m}\|_{W_{2,0}^2(a,b)} \le c_1 \|w^m\|_{L_2(a,b)} \le c_3, \quad m = 1,2,..., \tag{21}$$

$$\|u_{2m}\|_{W_2^2(a,b)} \le c_2 \|w^m\|_{L_2(a,b)} \le c_4, \quad m = 1,2,..., \tag{22}$$

where $c_3 = c_1 b_2$, $c_4 = c_2 b_2$ are independent of m. The uniform boundedness of sequences $\{u_{km}(x)\}$, $k = 1,2$ in the space $W_2^2(a,b)$ follows from these estimations. Therefore, from this sequences we can distinguish subsequences that for convenience of presentation will be again denoted by $\{u_{km}(x)\}$, $k = 1,2$ that

$$u_{km} \to u_k, \quad \frac{du_{km}}{dx} \to \frac{du_k}{dx}, \quad \frac{d^2u_{km}}{dx^2} \to \frac{d^2u}{dx^2} \tag{23}$$

$k = 1,2$ weakly in $L_2(a,b)$ as $m \to \infty$. As the space $W_2^2(a,b)$ is compactly imbedded into $W_2^1(a,b)$, we have

$$u_{km} \to u_k, \quad \frac{du_{km}}{dx} \to \frac{du_k}{dx}, \quad k = 1,2 \tag{24}$$

strongly in $L_2(a,b)$ as $m \to \infty$.

It is clear that the elements of the sequences $\{u_{km}(x)\}$, $k = 1,2$ for each $m = 1,2,...$ satisfy the following integral identities:

$$\int\limits_a^b \left(\rho(x)\frac{d^2u_{km}}{dx^2} + \frac{1}{x}\nu_1^m(x)\frac{du_{km}}{dx} - \nu_0^m(x)\, u_{km} - w^m(x)\right) \times$$

$$\eta_k(x)\, dx = 0, \ k = 1,2 \tag{25}$$

for any functions $\eta_k = \eta_k(m)$ from $L_2(a,b)$, $k = 1,2$, and the conditions

$$u_{1m}(a) = u_{1m}(b) = 0, \quad m = 1,2,..., \tag{26}$$

$$\frac{du_{2m}(a)}{dx} = \frac{du_{2m}(b)}{dx} = 0, \quad m = 1,2,.... \tag{27}$$

Allowing for limit relations (19), (20), (23), (24), if we pass to limit with respect to $m \to \infty$ in integral identities (25), we get validity of the identities:

$$\int\limits_a^b \left(\rho(x)\frac{d^2u_k}{dx^2} + \frac{1}{x}\nu_1(x)\frac{du_k}{dx} - \nu_0(x)\, u_k - w(x)\right)\eta_k(x)\, dx = 0, \ k = 1,2 \tag{28}$$

for any functions $\eta_k \in L_2(a,b)$, $k = 1,2$. Hence we get that the limit functions $u_k = u(x)$, $k = 1,2$ satisfy equations (5) for almost all $x \in (a,b)$. Prove that these functions satisfy boundary conditions (6), (7). To this end, at first prove that the limit function $u_1 = u_1(x)$ satisfies condition (6). From the compactness of the imbedding of $W_2^2(a,b)$ into the space $C[a,b]$ we have:

$$u_{1m}(x) \to u_1(x) \quad as \quad m \to \infty \tag{29}$$

uniformly with respect to $x \in [a,b]$. Then it is clear that the following limit relations hold:

$$u_{1m}(a) \to u_1(a), \quad u_{1m}(b) \to u_1(b) \tag{30}$$

as $m \to \infty$. Allowing for these limit relations and boundary conditions (26), from the inequalities

$$|u_1(s)| \le |u_1(s) - u_{1m}(s)| + |u_1(s)|, \quad s = a, b$$

with passing to limit with respect to $m \to \infty$ in both sides we get the validity of the boundary conditions $u_1(a) = u_1(b) = 0$. Finally, prove that the limit function $u_2 = u_2(x)$ satisfies conditions (7). It is known that the space $W_2^2(a,b)$ is compactly imbedded into $C^1[a,b]$. Then it is clear that the following limit relation holds:

$$\frac{du_{2m}(x)}{dx} \to \frac{du_2(x)}{dx} \quad as \quad m \to \infty \tag{31}$$

uniformly with respect to $x \in [a,b]$. Therefore, allowing for limit relations:

$$\frac{du_{2m}(s)}{dx} \to \frac{du_2(s)}{dx}, \quad s = a, b, \quad as \quad m \to \infty. \tag{32}$$

and boundary conditions (27), from the inequalities

$$\left|\frac{du_2(s)}{dx}\right| \le \left|\frac{du_2(s)}{dx} - \frac{du_{2m}(s)}{dx}\right| + \left|\frac{du_{2m}(s)}{dx}\right|,$$

$s = a, b$ with passing to limit in both sides we get the validity of the conditions: $\frac{du_2(a)}{dx} = \frac{du_2(b)}{dx} = 0$. Thus, we proved that the limit functions $u_1 \in W_{2,0}^2(a,b)$, $u_1 \in W_2^2(a,b)$ is the solution of reduced problem (5)-(7) for $\nu \in V$. Consequently, $u_k = u(x) \equiv u_k(x;\nu)$, $k = 1,2$. Estimations (15), (16) that follow from estimations (21), (22) with passing to the lower limit in subsequences $\{u_{km}(x)\}$, $k = 1,2$, $\{w^m(x)\}$ weakly converging to the functions $u(x)$, $k = 1,2$, $w(x)$ respectively, are valid for these functions.

Using weak lower semi-continuity of the norm in the spaces $L_2(a,b)$ and H, and also the condition $\alpha \ge 0$ for any $\omega \in H$ we have:

$$J_{\alpha^*} \le J_\alpha(\nu) \le \underline{\lim}_{m\to\infty} J_\alpha(\nu^m) = J_{\alpha^*}.$$

Hence it follows that $J_{\alpha^*} = J_\alpha(\nu)$ i.e. $\nu = \nu(x)$ from V is the solution of optimal control problem (4)-(7). The theorem is proved.

3. Existence of a unique solution of an optimal-control problem

In this section we'll study the existence of a unique optimal control in problem (4)-(7) for $\alpha > 0$. To this end we city one known theorem on the existence and uniqueness of the solution of non-convex optimization known from the paper [7].

Theorem 3. (Goebel M. [7]). *Let $\widetilde{X}$ be a uniformly convex space, U be a closed bounded set from $\widetilde{X}$, the functional $I(\nu)$ on U be lower semi-continuous and lower bounded, $\alpha > 0$, $\beta \geq 1$ be the given numbers. Then there exists a dense subset G of the space $\widetilde{X}$ such that for any $\omega \in G$ the functional*

$$J_\alpha(v) = I(\nu) + \alpha \|\nu - \omega\|^\beta_{\widetilde{X}}$$

attains its least value on U. If $\beta > 1$, the minimal value of the functional is attained on a unique element.

Theorem 4. *Let the conditions of theorem 1 be fulfilled. Then there exists a dense subset G of the space $H \equiv (L_2(a,b))^3$ and such that for any $\omega \in G$ and $\alpha > 0$ optimal control problem (4)-(7) has a unique solution.*

Proof. At first prove the continuity of the functional

$$J_0(\nu) = \|u_1 - u_2\|^2_{L_2(a,b)} \tag{33}$$

on the set V. Take any element $\nu \in V$ and give it an increment $\Delta\nu \in B \equiv (L_\infty(a,b))^2 \times L_2(a,b)$ such that $\nu + \Delta\nu \in V$. Let $u_k(x) \equiv u_k(x;\nu)$, $u_{k\Delta}(x) \equiv u_k(x;\nu+\Delta\nu)$, $k=1,2$ be the solutions of reduced problem (5)-(7) corresponding to the controls $\nu \in V$ and $\nu + \Delta\nu \in V$. Then it is clear that the functions $\Delta u_k(x) = u_{k\Delta}(x) - u_k(x) \equiv u\,(x;\nu+\Delta\nu) - u_k(x;\nu)$, $k=1,2$ will be the solutions of the following boundary value problem:

$$\rho(x)\frac{d^2\Delta u_k}{dx^2} + \frac{1}{x}(\nu_1(x) + \Delta\nu_1(x))\frac{d\Delta u_k}{dx} - (\nu_0(x) + \Delta\nu_0(x))\Delta u_k =$$

$$= \Delta w(x) - \frac{1}{x}\Delta\nu_1(x)\frac{du_k}{dx} + \Delta\nu_0(x)u_k, \quad x \in (a,b), \quad k=1,2 \tag{34}$$

$$\Delta u_1(a) = \Delta u_1(b) = 0, \quad \frac{d\Delta u_2(a)}{dx} = \frac{d\Delta u_2(b)}{dx} = 0. \tag{35}$$

At first establish an estimation for solving this problem. To this end write equation (34) in the following form:

$$\frac{d}{dx}\left(\widetilde{a}_0(x)\frac{d\Delta u_k}{dx}\right) - \widetilde{a}_0(x)\Delta u_k = \widetilde{f}_k(x), \quad x \in (a,b), \quad k=1,2, \tag{36}$$

where

$$\widetilde{a}_0(x) = e^{\int_a^z \frac{\nu_1(\tau)+\Delta\nu_1(\tau)}{\tau\rho(\tau)}}, \tag{37}$$

$$\widetilde{a}_1(x) = \frac{\nu_0(x) + \Delta\nu_0(x)}{\rho(x)}\widetilde{a}_0(x), \tag{38}$$

$$\widetilde{f}_k(x) = \widetilde{a}_0(x)\left(\Delta w(x) - \frac{1}{x}\Delta\nu_1(x)\frac{du_k}{dx} + \Delta\nu_0(x)u_k\right)/\rho(x), \quad k=1,2. \tag{39}$$

Multiply the both sides of equation (36) by the function $\Delta u_k(x)$, $k=1,2$ and integrate with respect to the interval (a,b). Then in the obtained equality we use the integration by parts formula, boundary conditions (35) and have:

$$\int_a^b \widetilde{a}_0(x)\left(\frac{d\Delta u_k}{dx}\right)^2 dx + \int_a^b \widetilde{a}_1(x)(\Delta u_k)^2 dx = -\int_a^b \widetilde{f}_k(x)\Delta u_k(x)\,dx, \quad k=1,2. \tag{40}$$

For $\nu \in V$, $\nu + \Delta\nu \in V$ by means of formulae (37), (38) and condition (8) we have:

$$1 \leq \widetilde{a}_0(x) \leq e^{\frac{b_1(b-a)}{a\rho_0}}, \quad \forall x \overset{0}{\in} (a,b), \tag{41}$$

$$\frac{b_0}{\rho_1} \leq \widetilde{a}_1(x) \leq \frac{\widetilde{b}_0}{\rho_0} e^{\frac{b_1(b-a)}{a\rho_0}}, \quad \forall x \overset{0}{\in} (a,b). \tag{42}$$

Using these conditions and the Cauchy inequality with ε for $\varepsilon = \frac{b_0}{\rho_1}$ we get the validity of the estimations:

$$\left\| \frac{d\Delta u_k}{dx} \right\|^L_{L_2(a,b)} + \frac{b_0}{2\rho_1} \|\Delta u_k\|^L_{L_2(a,b)} \leq \frac{\rho_1}{2b_0} \left\| \widetilde{f}_k \right\|^L_{L_2(a,b)}, \quad k = 1,2. \tag{43}$$

By means of formula (39) and estimation (15), (16) from these estimations we can establish the validity of the estimations:

$$\|\Delta u_1\|^L_{W^1_{2,0}(a,b)} \leq c_5 \|\Delta\nu\|^2_B, \tag{44}$$

$$\|\Delta u_2\|^L_{W^1_2(a,b)} \leq c_6 \|\Delta\nu\|^2_B, \tag{45}$$

where $c_5 > 0$, $c_6 > 0$ are the constants independent of $\Delta\nu$. Now consider the increment of the functional $J_0(\nu)$ on any element $\nu \in V$. From formula (33) we have

$$\Delta J_0(\nu) = J_0(\nu + \Delta\nu) - J_0(\nu) = 2\int_a^b (u_1(x) - u_2(x))(\Delta u_1(x) - \Delta u_2(x))\,dx +$$

$$+ \|\Delta u_1\|^2_{L_2(a,b)} + \|\Delta u_k\|^2_{L_2(a,b)} - 2\int_a^b \Delta u_1(x)\,\Delta u_2(x)\,dx, \tag{46}$$

where $\Delta u_k = \Delta u_k(x)$, $k = 1,2$ is the solution of boundary value problem (34), (35). Hence from estimations (15), (16) and (44), (45) and the Cauchy-Bunyakovsky inequality we get:

$$|\Delta I_0(\nu)| \leq c_7 \left(\|\Delta\nu\|_B + \|\Delta\nu\|^2_B \right), \quad \forall \nu \in V. \tag{47}$$

This inequality yields the continuity of the functional $J_0(\nu)$ on any element $\nu \in V$, i.e. on the set V. Consequently,

$$\Delta I_0(\nu) \to 0 \quad as \quad \|\Delta\nu\|_B \to 0, \quad \forall \nu \in V. \tag{48}$$

Besides, $J_0(\nu) \geq 0$, $\forall \nu \in V$ i.e. the functional $J_0(\nu)$ is lower bounded on the set V. Along with these, it is easy to prove that the set V is a closed and bounded set in the space H. From the uniform convexity of the space $H \equiv (L_2(a,b))^3$ [8] we get that all the conditions of theorem 3 are fulfilled for $I(\nu) = J_0(\nu)$ and $U \equiv V$. Therefore, from the statement of theorem 3 we deduce that for any $\omega \in G$ and $\alpha > 0$ optimal control problem (4)-(7) has a unique solution. Theorem 4 is proved.

By means of this theorem, the existence and uniqueness of the solution of optimal control problem (4)-(7) for $\alpha > 0$ was proved not for each $\omega \in H$. The following theorem establishes the existence and uniqueness of the solution of an optimal control problem for each $\omega \in H$ and any $\alpha > \alpha_0$, where $\alpha_0 > 0$ is some number dependent only on the problem data.

Theorem 5. *Let the conditions of theorem 1 be fulfilled. Then there exists some number $\alpha_0 > 0$ dependent only on the data of problem (4)-(7), and optimal control problem (4)-(7) has a unique solution for any $\omega \in H$ and any $\alpha > \alpha_0 > 0$.*

The proof of this theorem is carried out by means of the statement of theorem 2 that says that optimal control problem (4)-(7) for any $\alpha > 0$ and for any $\omega \in H$ has even if one solution, and by means of the statement that there exists some number $\alpha_0 = \widetilde{c} > 0$, that the functional $J_\alpha(\nu)$ is a strongly convex functional on a convex set V with the strong convexity index $\chi = \alpha - \alpha_0 > 0$, i.e. for any ν^0, $\nu^1 \in V$ and $\beta \in [0,1]$ the following inequality is valid:

$$J_\alpha\left(\beta\nu^1 + (1-\beta)\,\nu^0\right) \leq \beta J_\alpha\left(\nu^1\right) + (1-\beta)\,J_\alpha\left(\nu^0\right) - \chi\beta\,(1-\beta)\left\|\nu^1 - \nu^0\right\|_H^2 .$$

References

[1]. Marchuk G. I. *Mathematical simulation in environment problem.* M. Nauka, 1982, 320 p. (Russian).

[2]. Salmanov V.I. *On an optimal control problem for concentrated systems with special quality criterion.* Vestnik Lenkoranskogo Gosudarstvennogo Universiteta. Ser. estestvennykh i ekonomicheskikh Nauk, 2005, pp. 83-90 (Russian).

[3].Mahmudov N.M., Salmanov V.I. *Existence of the solution of an optimal control problem for a linear system of ordinary differential equations of first order.* Proc. of the VII All-Russian scientific conference "Mathematical simulation and boundary value problems"Ṡamara, 2010, June 3-6, pp. 187-192 (Russian).

[4]. Ladyzhenskaya O.A., Uraltseva N.N. *Linear and quasilinear equations of elliptic type.* M. Nauka, 1973, 576 p. (Russian).

[5]. Ladyzhenskaya O.A., Uraltseva N.N. *Boundary value problems of mathematical physics.* M. Nauka, 1973, 408 p. (Russian).

[6]. Mikhailov V.P. *Partial differential equations.* M. Nauka, 1983, 424 p. (Russian).

[7]. Goebel M. *On existence of optimal control.* Math. Nachr., 1979, vol. 93, pp. 67-73.

[8]. Iosida K. *Functional analysis.* M. Mir, 1967, 624 p. (Russian).

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