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ON THE SPECTRUM AND REGULARIZED TRACE OF STURM-LIOUVILLE OPERATOR EQUATION GIVEN ON A FINITE INTERVAL

Abstract

In the paper in a Hilbert space we consider a differential equation with a trace class operator potential on a finite interval. The structure spectrum and regularized trace of the given operator is studied.

1. Introduction. Let H be a separable Hilbert space. In the Hilbert space $H_1 = L_2([0,\pi]; H)$ consider a self-adjoint operator L generated by the differential expression

$$l(y) = -y''(x) + Q(x)\, y(x) \tag{1}$$

and boundary conditions

$$y(0) = 0, \quad y'(\pi) = 0. \tag{2}$$

Suppose that the operator function $Q(x)$ satisfies the following conditions:

1^0. For each $x \in [0,\pi]$, $Q(x) : H \to H$ is a trace class self-adjoint operator. Furthermore, $Q(x)$ has a fourth derivative in the norm of the space $\sigma_1(H)$ on the interval $[0,\pi]$ and for each $x \in [0,\pi]$, $Q^{(i)}_{(x)} : H \to H$ are self-adjoint operators $(i = 1,2,3,4)$.

2^0. $\sup\limits_{0\le x\le \pi} \|Q(x)\|_H < 1$;

3^0. In the space H there exists an orthonormed basis $\{\varphi_n\}_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} \|Q(x)\varphi_n\|_{H_1} < \infty$.

4^0. $\int\limits_0^{\pi} Q(x)\, dx = 0$.

5^0. $Q^{(2i-1)}(0) = Q^{(2i-1)}(\pi) = 0, \ (i = 1,2)$.

Here $\sigma_1(H)$ denotes a space of trace operators acting in the space H.

Let L_0 be an operator generated by the differential expression $l_0(y) = -y''(x)$ and boundary conditions (2).

It is easy to show that $\lambda_m^{(0)} = \left(m - \frac{1}{2}\right)^2$, $m = 1,2,...$ are infinite-to-one eigen values of the operator L_0. The appropriate orthonomed eigen vector-functions are the functions:

$$\psi_{mn} = \sqrt{\frac{2}{\pi}} \sin\left(m - \frac{1}{2}\right) x \cdot \varphi_n = 1,2,.... \tag{3}$$

Denote the resolvents of the operators L_0 and L by R_λ^0 and R_λ, respectively.

2. On the spectrum of problem (1)-(2)

The followings hold for the spectrum of the operator L [see 1].

Lemma. *If the operator-function* $Q(x)$ *satisfies condition* 3^0 *and* $\lambda \overline{\in} \left\{ \left(m + \frac{1}{2}\right)^2 \right\}_{m=1}^{\infty}$, *then* $Q(x) R_\lambda^0 : H_1 \to H_1$ *is a trace formula operator.*

Theorem 1. *If the operator-function $Q(x)$ satisfies conditions 2^0 and 3^0, then the spectrum of the operator L is a subset of the union of the following intervals:*

$$\Omega_m = \left[\left(m-\frac{1}{2}\right)^2 - \|Q\|_{H_1}, \left(m-\frac{1}{2}\right)^2 + \|Q\|_{H_1}\right], \quad m = 1,2,...$$

Therewith

a) each point different from $\left(m-\frac{1}{2}\right)^2$ of the spectrum of the operator L from the interval Ω_m is an isolated eigen value of finite multiplicity.

b) the points $\left(m-\frac{1}{2}\right)^2$, $m=1,2,...$ may be an eigen value of finite or infinite multiplicity.

c) if $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigen value of the operator L from the interval Ω_m, then $\lim\limits_{n\to\infty}\lambda_{mn} = \left(m-\frac{1}{2}\right)^2$.

3. On regularized trace of the operator L

Theorem 2. *If the operator function $Q(x)$ satisfies conditions 1^0-3^0, then for the regularized trace of the operator L it holds the following formula*

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left[\lambda_{nm} - \left(m-\frac{1}{2}\right)^2\right] - \frac{1}{\pi}\int_0^{\pi} trQ(x)\,dx\right] = \frac{1}{4}\left[trQ(\pi) - trQ(0)\right]. \tag{4}$$

In [2], the following formula for the second regularized trace of the operator L is proved:

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left[\lambda_{nm}^2 - \left(m-\frac{1}{2}\right)^4\right] - \frac{(2m+1)^2}{2\pi}\int_0^{\pi} trQ(x)\,dx - C\right] =$$

$$= \frac{1}{8}tr\left[Q''(0) - Q''(\pi) - 2Q^2(0) + 2Q^2(\pi)\right] \tag{5}$$

where

$$C = \frac{1}{2\pi}\int_0^{\pi} trQ^2(x)\,dx + \frac{1}{2\pi^2}tr\left[\int_0^{\pi} Q(x)\,dx\right]^2 + \frac{1}{2\pi}\left[trQ'(0) + trQ'(\pi)\right].$$

The goal of the paper is to calculate the third regularized trace of the operator L generalized by expression (1) and boundary conditions (2).

Note that the first work on calculation of a regularized trace for a differential operator belongs to I.M. Gelphand and B.M. Levitan [3]. In this paper a formula for the sum of two differences of two Sturm-Liouville regular operators on the interval $[0,\pi]$ is obtained. L. A. Dikiy [4] calculated regularized traces for some differential operators. At the same time he obtained the regularization of the sum $\sum\limits_{n=1}^{\infty}\lambda_n^k$ (λ_n are eigenvalues, k is a natural number).

In [5], M.G. Gasymov and B.M. Levitan obtained a formula for the sum of differences of eigen values of two singular self-adjoint Sturm-Liouville operators that differ one from another by a finite potential. In [7], by means of a zeta-function and teta-functions $\theta(t) = \sum\limits_{n=1}^{\infty} e^{-\lambda_n t}$ V.A. Sadovnichiy [6], V.A. Sadovnichiy and V.V. Dubrovsky [7] obtained a regularized trace formula for higher order differential operatpors.

In [8] R.Z. Khalilova obtained an analogy of I.M. Gelphand and B.M. Levitan formula for Sturm-Liouville operator with an operator coefficient.

Regularized traces for differential operators with operator coefficients were investigated also in the papers of E. Abdukadyrov [9], M.Bayramoglu [10], A.A. Adygezalov [11], F.G. Maksudov, M.Bayramoglu, A.A. Adygezalov [12], N.M. Aslanova [13] and others.

4. Some formulae related to a resolvent

Let $\{\varphi_{nm}(x)\}$ be orthonormed eigen functions of the operator L, corresponding to eigen values $\{\lambda_{nm}\}_{n,m=1}^{\infty}$. Introduce the following denotation:

$$\Gamma_p = \left\{ \lambda : |\lambda| = \left(p - \frac{1}{2}\right)^2 + p \right\}, \quad B_{mn}^0 = \left(\circ, \psi_{mn}^0\right)_{H_1} \psi_{mn}^0,$$

$$B_{mn} = \left(\circ, \psi_{mn}\right)_{H_1} \psi_{mn}, \quad L_{om}^{(r)} = \sum_{n=1}^{\infty} \left(m - \frac{1}{2}\right)^2 \cdot B_{mn}^0, \quad L_m^{(r)} = \sum_{n=1}^{\infty} \lambda_{mn}^r \cdot B_{mn}.$$

Since for the resolvent R_λ^0 and R_λ the following expansions hold:

$$R_\lambda^0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{B_{mn}^0}{\left(m - \frac{1}{2}\right)^2 - \lambda}, \quad R_\lambda = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{B_{mn}}{\lambda_{mn} - \lambda}$$

then for the difference $R_\lambda - R_\lambda^0$ it holds the formula

$$R_\lambda - R_\lambda^0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{B_{mn}}{\lambda_{mn} - \lambda} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{B_{mn}^0}{\left(m - \frac{1}{2}\right)^2 - \lambda}. \tag{6}$$

It is proved that if the operator function $Q(x)$ satisfies conditions 2^0 and 3^0, then the series $\sum\limits_{n=1}^{\infty} \left[\lambda_{pn} - \left(p - \frac{1}{2}\right)^2\right]$, $p = 1, 2, ...$ absolutely converge.

It is easy to show that all the eigen values of the operators L_0 and L except $\left(m - \frac{1}{2}\right)^2$ and $\{\lambda_{nm}\}_{n=1}^{\infty}$ are arranged outside of the circle Γ_p.

5. On the third regularized trace of the operator L

Since $R_\lambda - R_\lambda^0 \leftarrow \sigma_1(H_1)$, for $\lambda \in \rho(L)$, then it follows formula (6) that

$$tr\left(R_\lambda - R_\lambda^0\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\left(m - \frac{1}{2}\right)^2 - \lambda} \right). \tag{7}$$

Having multiplied the both hand sides of this equality by $\frac{\lambda^3}{2\pi i}$ and integrating with respect to the circle $|\lambda| = b_p = \left(p - \frac{1}{2}\right)^2 + p\,(p \geq 1)$, we get

$$\frac{1}{2\pi i} \int\limits_{|\lambda| = b_p} \lambda^3 tr\left(R_\lambda - R_\lambda^0\right) d\lambda = \frac{1}{2\pi i} \int\limits_{|\lambda| = b_p} \lambda^3 \sum_{m=1}^{p}\sum_{n=1}^{\infty} \left[\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\left(m - \frac{1}{2}\right)^2 - \lambda} \right] d\lambda +$$

$$+\frac{1}{2}\int\limits_{|\lambda|=b_p}\lambda^3\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_{mn}-\lambda}-\frac{1}{\left(m-\frac{1}{2}\right)^2-\lambda}\right]d\lambda. \qquad (8)$$

Since $\left(m-\frac{1}{2}\right)^2-\|Q\|_{H_1}\leq\lambda_{mn}\leq\left(m-\frac{1}{2}\right)^2+\|Q\|_{H_1}$, then for $m<p\,(p\geq 1)$ we get

$$\lambda_{mn}\leq\left(m-\frac{1}{2}\right)^2+\|Q\|_{H_1}\leq\left(p-\frac{1}{2}\right)^2+\|Q\|_{H_1}<\left(p-\frac{1}{2}\right)^2+p=b_p,$$

i.e. for $m<p$ and $p>1$

$$|\lambda_{mn}|\leq b_p. \qquad (9)$$

Furthermore, if $m>p$, then

$$\lambda_{mn}\geq\left(m-\frac{1}{2}\right)^2-\|Q\|_{H_1}\geq\left(p+1-\frac{1}{2}\right)-\|Q\|_{H_1}>\left(p-\frac{1}{2}\right)^2+p=b_p,$$

$$\lambda_{mn}>b_p,\quad m>p,\quad n=1,2,.... \qquad (10)$$

Using (9) and (10), from (8) we get:

$$\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^3 tr\left(R_\lambda-R_\lambda^0\right)dx=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\left(m-\frac{1}{2}\right)^6-\lambda_{mn}^3\right]. \qquad (11)$$

From formula $R_\lambda=R_\lambda^0-R_\lambda QR_\lambda^0$ it follows that

$$R_\lambda-R_\lambda^0=\sum_{j=1}^{N}(-1)^j R_\lambda^0\left(QR_\lambda^0\right)^j+(-1)^{j+1}R_\lambda\left(QR_\lambda^0\right)^{N+1}, \qquad (12)$$

where N is an arbitrary natural number.

Taking into account that the operators $R_\lambda^0\left(QR_\lambda^0\right)^j\,(j=1,2,...,N),\ R_\lambda\left(QR_\lambda^0\right)^j$ are trace formula operators in H_1, from (11) and (12) we get

$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\lambda_{mn}^3-\left(m-\frac{1}{2}\right)^6\right]=\sum_{j=1}^{N}\frac{(-1)^j}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^3 tr\left[R_\lambda^0\left(QR_\lambda^0\right)^j\right]d\lambda+$$

$$+\frac{(-1)^{j+1}}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^3 tr\left[R_\lambda\left(QR_\lambda^0\right)\right]^{N+1}d\lambda. \qquad (13)$$

Denote

$$M_{pj}=\frac{(-1)^j}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^3 tr\left[R_\lambda^0\left(QR_\lambda\right)^j\right]d\lambda \qquad (14)$$

$$M_{PN}=\frac{(-1)^{N+1}}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^3 tr\left[R_\lambda\left(QR_\lambda^0\right)^{N+1}\right]d\lambda. \qquad (15)$$

Then

$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\lambda_{mn}^{3}-\left(m-\frac{1}{2}\right)^{6}\right]=\sum_{j=1}^{N}M_{pj}+M_{PN}. \tag{16}$$

Using conditions 1^0-5^0, it is proved that

$$M_{p1}=-\frac{3}{16\pi}\sum_{m=1}^{p}\int_{0}^{\pi}trQ^{(IV)}(x)\cos(2m-1)\,xdx+$$

$$+\frac{3}{\pi}\sum_{m=1}^{p}\left(m-\frac{1}{2}\right)^{2}\int_{0}^{\pi}trQ(x)\,dx \tag{17}$$

$$\lim_{p\to\infty}M_{pj}=0,\quad j\geq 2 \tag{18}$$

$$\lim_{p\to\infty}M_{PN}=0,\quad N\geq 8. \tag{19}$$

The following theorem is the main result of the paper:

Theorem 3. *It the operator function $Q(x)$ satisfies conditions 1^0-5^0, then it holds the following formula*

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}^{3}-\left(m-\frac{1}{2}\right)^{6}\right)-\frac{3\left(m-\frac{1}{2}\right)^{2}}{4\pi}\times\int_{0}^{\pi}trQ^{2}(x)\,dx-\right.$$

$$\left.-\frac{3}{16\pi}\int_{0}^{\pi}tr\left[Q'(x)\right]^{2}dx-\frac{1}{\pi}\int_{0}^{\pi}g(x)\,dx+h\right]=\frac{3}{64}\left[trQ^{(IV)}(\pi)-tr\,(Q)^{(IV)}(o)\right]+$$

$$+\frac{3}{8\pi}\left[trQ''(o)\,Q(o)-trQ''(\pi)\,Q(\pi)\right]+\frac{1}{4\pi}\left[g(\pi)-g(o)\right]-\frac{h}{2}, \tag{20}$$

where

$$h=\frac{15}{8}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\beta_{ij}\right|,$$

the number β_{ij} and the function $g(x)$ are determined as follows:

$$\beta_{ij}=\frac{1}{\pi^{3}}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\int_{0}^{\pi}\left(Q(x)\,\varphi_{n},\varphi_{q}\right)_{H}\cos ixdx\times$$

$$\times\int_{0}^{\pi}\left(Q(x)\,\varphi_{q},\varphi_{s}\right)_{H}\cos(i-j)\,dx\cdot\int_{0}^{\pi}\left(Q(x)\,\varphi_{s},\varphi_{n}\right)_{H}\cos jxdx \tag{21}$$

$$g(x)=\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\int_{0}^{\pi}\left(Q(x)\,\varphi_{n},\varphi_{q}\right)_{H}\left(Q(x)\,\varphi_{q},\varphi_{s}\right)_{H}\left(Q(x)\,\varphi_{s},\varphi_{n}\right). \tag{22}$$

The left hand side of equality (20) is said to be the third regularized trace of the operator L.

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