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Arzu G.ALIYEVA

INVESTIGATION OF GENERALIZED SOLUTION OF ONE-DIMENSIONAL MIXED PROBLEM FOR A CLASS OF FOURTH ORDER SEMI-LINEAR EQUATIONS OF SOBOLEV TYPE. I.

Abstract

The paper deals with the existence and uniqueness of the generalized solution of one-dimensional mixed problem with Rickier type conditions for fourth order semi-linear equations of Sobolev type. The notion of the generalized solution of the mixed problem under consideration is introduced. After applying the Fourier method, the solution of the input problem is reduced to the solution of some denumerable system of nonlinear integral equations with respect to Fourier unknown coefficients of the desired solution. Then the global uniqueness, small existence and global existence theorems of the generalized solution of the mixed problem under consideration are proved.

In the paper we study the existence and uniqueness of the generalized solution of the following one-dimensional mixed problem:

$$u_{txx}(t,x)-\alpha u_{xxxx}(t,x)=F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x)) \quad (0\le t\le T,\ 0\le x\le\pi), \tag{1}$$

$$u(0,x)=\varphi(x) \quad (0\le x\le\pi), \tag{2}$$

$$u(t,0)=u(t,\pi)=u_{xx}(t,0)=u_{xx}(t,\pi)=0 \quad (0\le t\le T), \tag{3}$$

where $\alpha>0$ is a fixed number; $0<T<+\infty$; F,φ are the given functions, $u(t,x)$ is a desired function, and under the generalized solution of problem (1)-(3) we understand the following:

Definition. *Under the generalized solution of problem (1)-(3) we understand the function $u(t,x)$ having the following properties:*

a) $u(t,x),u_x(t,x),u_{xx}(t,x),u_t(t,x)\in C\left([0,T]\times[0,\pi]\right);$
$u_{xxx}(t,x),u_{tx}(t,x)\in C\left([0,T];L_2(0,\pi)\right);$

b) all the conditions of (2) and (3) are satisfied in the ordinary sense;

c) the integral identity

$$\int_0^T\int_0^\pi\left\{u_{tx}(t,x)V_x(t,x)-\alpha u_{xxx}(t,x)V_x(t,x)+F(u(t,x))V(t,x)\right\}dxdt=0 \tag{4}$$

is fulfilled for any function $V(t,x)$ having the properties

$$V(t,x)\in C\left([0,T]\times[0,\pi]\right),V(t,0)=V(t,\pi)=0 \quad (0\le t\le T),$$

$$V_x(t,x)\in L([0,T];L_2(0,\pi)), \tag{5}$$

where

$$F(u(t,x))\equiv F(t,x,(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x)). \tag{6}$$

1. Auxiliary facts

For investigating the generalized solution of problem (1)-(3), we cite some known facts and set up a number of new auxiliary facts.

1. Since the system $\{\sin nx\}_{n=1}^{\infty}$ forms a basis in the space $L_2(0,\pi)$, then it is obvious that each generalized solution $u(t,x)$ of problem (1)-(3) has the form:

$$u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx, \tag{7}$$

where

$$u_n(t)=\frac{2}{\pi}\int_0^{\pi}u(t,x)\sin nxdx \quad (n=1,2,...;\ t\in[0,T]). \tag{8}$$

Then after applying the formal scheme of the Fourier method, the finding of the functions $u_n(t)$ $(n=1,2,...)$ is reduced the solution of the following denumerable system of nonlinear integral equations:

$$u_n(t)=\varphi_n e^{-\alpha n^2 t}-\frac{2}{\pi n^2}\int_0^t\int_0^{\pi}F(u(\tau,x))\sin nxe^{-\alpha n^2(t-\tau)}dxd\tau \ (n=1,2,...;\ t\in[0,T]). \tag{9}$$

where

$$\varphi_n\equiv\frac{2}{\pi}\int_0^{\pi}\varphi(x)\sin nxdx \quad (n=1,2,...), \tag{10}$$

$$F(u(t,x))\equiv F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x)). \tag{11}$$

2. Proceeding from the definition of the generalized solution of problem (1)-(3), we easily prove the following lemma.

Lemma. *If* $u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx$ *is any generalized solution of problem (1)-(3), then the functions* $u_n(t)$ $(n=1,2,...)$ *satisfy system (9).*

3. Denote by $B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ the totality of all the functions $u(t,x)$ of the form (7), considered in $[0,T]\times[0,\pi]$, for which all the functions $u_n(t)\in C^{(l)}([0,T])$ and

$$J_T(u)\equiv\sum_{i=0}^{l}\left\{\sum_{n=1}^{\infty}\left(n^{\alpha_i}\max_{0\le t\le T}\left|u_n^{(i)}(t)\right|\right)^{\beta_i}\right\}^{\frac{1}{\beta_i}}<+\infty,$$

where $l\ge 0$ is an integer, $\alpha_i\ge 0$ $(i=\overline{0,l}\ )$, $1\le\beta_i\le 2$ $(i=\overline{0,l})$. We define the norm in this set as follows: $\|u\|=J_T(u)$. It is known (see [1]) that all these spaces are Banach.

In the sequel, for the functions $u(t,x)\in B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ we'll use the denotation:

$$\|u\|_{B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,t}}\equiv\sum_{i=0}^{l}\left\{\sum_{n=1}^{\infty}\left(n^{\alpha_i}\max_{0\le\tau\le t}\left|u_n^{(i)}(\tau)\right|\right)^{\beta_i}\right\}^{\frac{1}{\beta_i}} \quad (0\le t\le T). \tag{12}$$

4. For the function $u(t,x) \equiv \sum\limits_{n=1}^{\infty} u_n(t)\sin nx \in B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$, call the function $u_n(t)$ its n-th component. Let $\mathcal{M}$ be any non-empty set from the space $B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$. The totality of n-th components of all the functions from $\mathcal{M}$ denote by $\mathcal{M}_n$. The following theorem (see[1]) is valid.

Theorem 1. *For compactness of the set* $M \subset B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ *in* $B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ *it is necessary and sufficient that the following two conditions to be fulfilled:*

a) for each fixed $n (n = 1, 2, ...)$ *the set* M_n *is compact in* $C^{(l)}([0,T])$;

b) for any $\varepsilon > 0$ *there exists the number* n_ε *one and the same for all* $u(t,x) = \sum\limits_{n=1}^{\infty} u_n(t)\sin nx \in M$ *such that*

$$\sum_{i=0}^{l}\left\{\sum_{n=n_\varepsilon}^{\infty}\left(n^{\alpha_i}\max_{0\le t\le T}\left|u_n^{(i)}(t)\right|\right)^{\beta_i}\right\}^{\frac{1}{\beta_i}} < \varepsilon \ \ \forall u \in \mathcal{M}.$$

5. It is obvious that if $u(t,x) = \sum\limits_{n=1}^{\infty} u_n(t)\sin nx \in B^k_{2,T}$ ($k \ge 1$ is an integer), then $\forall t \in [0,T]$:

$$\|u\|_{B^{k-1}_{1,t}} \equiv \sum_{n=1}^{\infty} n^{k-1}\max_{0\le\tau\le t}|u_n(\tau)| \le \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}} \times$$

$$\times\left\{\sum_{n=1}^{\infty}\left(n^k\max_{0\le\tau\le t}|u_n(\tau)|\right)^2\right\}^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}}\|u\|_{B^k_{2,t}}. \qquad (13)$$

6. Let $u(t,x) \equiv \sum\limits_{n=1}^{\infty} u_n(t)\sin nx \in B^3_{2,T}$. Then using estimation (13), for $k = 3$, $\forall t \in [0,T]$ and $x \in [0,\pi]$ we have:

$$\left|\frac{\partial^i u(t,x)}{\partial x^i}\right| \le \sum_{n=1}^{\infty} n^i\,|u_n(t)| \le \sum_{n=1}^{\infty} n^i\max_{0\le\tau\le t}|u_n(\tau)| \le$$

$$\le \sum_{n=1}^{\infty} n^2\max_{0\le\tau\le t}|u_n(\tau)| = \|u\|_{B^2_{1,t}} \le \frac{\pi}{\sqrt{6}}\|u\|_{B^3_{2,t}} \quad (i=\overline{0,2}). \qquad (14)$$

From estimation (14) and the structure of the space $B^3_{2,T}$ it follows that

$$u(t,x), u_x(t,x), u_{xx}(t,x) \in C([0,T]\times[0,\pi]). \qquad (15)$$

Besides, obviously, $\forall t \in [0,T]$:

$$\int_0^{\pi} u^2_{xxx}(t,x)dx = \frac{\pi}{2}\sum_{n=1}^{\infty}(n^3u_n(t))^2 \le \frac{\pi}{2}\sum_{n=1}^{\infty}(n^3\max_{0\le\tau\le t}u_n(\tau))^2 = \frac{\pi}{2}\|u\|^2_{B^3_{2,t}}. \qquad (16)$$

Hence, from the structure of the space $B^3_{2,T}$ it follows that

$$u_{xxx}(t,x) \in C([0,T]; L_2(0,\pi)). \qquad (17)$$

Further, using relations (14)-(17) and the known properties of Nemytsky operator, we prove the following

Theorem 2. *Let*

1. $F(t,x,u_1,...,u_4) \in C([0,T] \times [0,\pi] \times (-\infty,\infty)^4)$.

2) $\forall R > 0$ *in* $[0,T] \times [0,\pi] \times [-R,R]^3 \times (-\infty,\infty)$

$$|F(t,x,u_1,...,u_4)| \le C_R(1+|u_4|), \tag{18}$$

where $C_R > 0$ *is a constant.*

Then:

a)

$$\begin{aligned}\forall u \in B^3_{2,T} F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),u_{xxx}(t,x)) \equiv \\ \equiv F(u(t,x)) \in C([0,T];L_2(0,\pi));\end{aligned} \tag{19}$$

b)

$$\begin{aligned}\forall u \in B^3_{1,T}, V \in B^3_{2,T} \quad F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),V_{xxx}(t,x)) \equiv \\ \equiv F_u(V(t,x)) \in C([0,T];L_2(0,\pi)).\end{aligned} \tag{20}$$

7. Let for a natural number k:

$$\varphi(x) \in C^{(k-1)}\left([0,\pi]\right), \varphi^k(x) \in L_2(0,\pi), \varphi^{(2s)}(0) =$$

$$= \varphi^{(2s)}(\pi) = 0 \quad \left(s = \overline{0,\left[\frac{k-1}{2}\right]}\right). \tag{21}$$

Then with the help of integration by parts, using the Bessel inequality (for an off k) and the Parseval equality (for an odd k), we easily prove that

$$\sum_{n=1}^{\infty}(n^k\varphi_n)^2 \le \frac{2}{\pi}\left\|\varphi^{(k)}(x)\right\|^2_{L_2(0,\pi)}, \tag{22}$$

where the numbers φ_n $(n=1,2,...)$ are defined by relation (10), moreover it is obvious that estimation (22) is valid for $k=0$ as well, if $\varphi(x) \in L_2(0,\pi)$.

8. In the end of the section, let as agree all the quantities to be real, the functions real-valued, and everywhere to understand the integrals in Lebesgues's sense.

2. Investigation of uniqueness of generalized solution of problem (1)-(3)

With the help of Bellman's inequality we prove the following global uniqueness theorem of the generalized solution of problem (1)-(3).

Theorem 3. *Let*

1. $F(t,x,u_1,...,u_4) \in C([0,T] \times [0,\pi] \times (-\infty,\infty)^4)$.

2. $\forall R > 0$ *in* $[0,T] \times [0,\pi] \times [-R,R]^3 \times (-\infty,\infty)$

$$|F(t,x,u_1,...,u_4) - F(t,x,\widetilde{u}_1,...,\widetilde{u}_4)| \le C_R\sum_{i=1}^{4}|u_i - \widetilde{u}_i|, \tag{23}$$

where $C_R > 0$ *is a constant.*

Then problem (1)-(3) may have at most one generalized solution.

3. Investigation of the small existence of the generalized solution of problem (1)-(3).

In this section, by combining the generalized principle of contracted mappings with Schauder principle on a fixed point, we prove the following small existence theorem (i.e. valid for rather small values of T) of the generalized solution of problem (1)-(3).

Theorem 4. *Let*

1. $\varphi(x) \in C^{(2)}([0,\pi]), \varphi'''(x) \in L_2(0,\pi)$ *and* $\varphi(0) = \varphi(\pi) = \varphi''(0) = \varphi''(\pi) = 0$.

2. $F(t,x,u_1,...,u_4) \in C([0,T] \times [0,\pi] \times (-\infty,\infty)^4)$.

3. $\forall R > 0$ *in* $[0,T] \times [0,\pi] \times [-R,R]^3 \times (-\infty,\infty)$

$$|F(t,x,u_1,u_2,u_3,u_4) - F(t,x,u_1,u_2,u_3,\widetilde{u}_4)| \leq C_R\,|u_4 - \widetilde{u}_4|\,, \tag{24}$$

where $C_R > 0-$*is a constant.*

Then there exists a small generalized solution of problem (1)-(3).

Proof. For each fixed $u \in B^2_{1,T}$ define in $B^3_{2,T}$ the operator (with respect to V) $\mathfrak{P}_u$:

$$\mathfrak{P}_u(V(t,x)) = \widetilde{V}(t,x) \equiv \sum_{n=1}^{\infty} \widetilde{V}_n(t)\sin nx, \tag{25}$$

where

$$\widetilde{V}_n(t) = \varphi_n e^{-\alpha n^2 t} - \frac{2}{\pi n^2}\int_0^t\int_0^\pi F_u(V(\tau,x))\sin nx e^{-\alpha n^2(t-\tau)}dxd\tau$$

$$(n = 1,2,...; t \in [0,T]), \tag{26}$$

the numbers of φ_n $(n = 1,2,...)$ were determined by relation (10) and

$$F_u(V(t,x)) \equiv F(t,x,u(t,x),u_x(t,x),u_{xx}(t,x),V_{xxx}(t,x)). \tag{27}$$

Obviously,

$$\forall u \in B^3_{2,T} \qquad F_u(u(t,x)) = F(u(t,x)), \tag{28}$$

where the operator F was determined by relation (6).

From (26) we get that for each fixed $u \in B^2_{1,T}$ $\forall V \in B^3_{2,T}$:

$$\|\mathfrak{P}_u(V)\|^2_{B^3_{2,T}} = \left\|\widetilde{V}\right\|^2_{B^3_{2,T}} \leq a_0 + \frac{2}{\alpha\pi}\int_0^T\int_0^\pi \{F_u(V(\tau,x))\}^2\,dxd\tau, \tag{29}$$

where

$$a_0 \equiv 2\sum_{n=1}^{\infty}(n^3\varphi_n)^2, \tag{30}$$

and the finiteness of a_0 follows from (22) for $k = 3$.

Since by theorem 2, $F_u(V(t,x)) \in C([0,T];L_2(0,\pi))$, then from (29) it follows that for any fixed $u \in B^2_{1,T}$, the operators $\mathfrak{P}_u$ acts in the space $B^3_{2,T}$.

Further, from estimations (14), $\forall u \in B^2_{1,T}$ there exists a number $R_u > 0$ such that $\forall t \in [0,T]$ and $x \in [0,\pi]$:

$$-R_u \le u(t,x), u_x(t,x), u_{xx}(t,x) \le R_u. \tag{31}$$

Now, using relations (25)-(27) allowing for (31), using inequalities (24) for $R = R_u$ and estimation (16) for $u = V_1 - V_2$, similar to (29) we get that for any fixed $u \in B^2_{1,T}$ $\forall V_1, V_2 \in B^3_{2,T}$ and $t \in [0,T]$:

$$\|\mathfrak{P}_u(V_1) - \mathfrak{P}_u(V_2)\|^2_{B^3_{2,t}} \le \frac{2}{\alpha\pi} \int\limits_0^t \int\limits_0^\pi \{F_u(V(\tau,x)) - F_u(V_2(\tau,x))\}^2 \, dx d\tau \le$$

$$\le \frac{2}{\alpha\pi} C^2_{R_u} \int\limits_0^t \int\limits_0^\pi \{V_{1,xxx}(\tau,x) - V_{2,xxx}(\tau,x)\}^2 \, dx d\tau \le$$

$$\le \frac{2}{\alpha\pi} C^2_{R_u} \frac{\pi}{2} \int\limits_0^t \|V_1 - V_2\|^2_{B^3_{2,\tau}} \, d\tau \le \frac{1}{\alpha} C^2_{R_u} \|V_1 - V_2\|^2_{B^3_{2,T}} \cdot t,$$

....

$$\left\| \mathfrak{P}^k_u(V_1) - \mathfrak{P}^k_u(V_2) \right\|_{B^3_{2,T}} \le \left(\frac{1}{\alpha} C^2_{R_u} \right)^k \|V_1 - V_2\|^2_{B^3_{2,T}} \frac{t^k}{k!}, \tag{32}$$

where k is any natural number.

Thus, for any fixed $u \in B^2_{1,T}$ $\quad \forall V_1, V_2 \in B^3_{2,T}$:

$$\left\| \mathfrak{P}^k_u(V_1) - \mathfrak{P}^k_u(V_2) \right\|_{B^3_{2,T}} \le q_k(u) \|V_1 - V_2\|_{B^3_{2,T}}, \tag{33}$$

where

$$q_k(u) \equiv \frac{1}{\sqrt{k!}} \left(\frac{1}{\alpha} C^2_{R_u} T \right)^{\frac{k}{2}}. \tag{34}$$

Obviously, for rather large $k = k_u : q_k(u) < 1$. For such k the operator $\mathfrak{P}^k_u$ turns out to be contractive in the space $B^3_{2,T}$. Then, by the generalized principle of contracted mappings, a fixed point V of the operator $\mathfrak{P}^k_u$ unique in $B^3_{2,T}$, is also a unique fixed point of the operator $\mathfrak{P}_u$:

$$V = \mathfrak{P}_u(V), \quad V \in B^3_{2,T}. \tag{35}$$

Take to each $u \in B^2_{1,T}$ a fixed point $B^3_{2,T}$ of the operator V unique in $\mathfrak{P}_u$, and generate an operator H:

$$H(u) = V = \mathfrak{P}_u(V), \tag{36}$$

acting from $B^2_{1,T}$ to $B^3_{2,T}$.

Then we show that the operator H acts from $B^2_{1,T}$ to $B^3_{2,T}$ continuously, and all the more in $B^2_{1,T}$ it acts continuously.

Now show the compactness of the operator H in $B^2_{1,T}$. Let $\mathfrak{K}=\mathfrak{K}_R$ be any closed ball of the space $B^2_{1,T}$ of radius R and centered at zero. Then from (14) it is obvious that for any $u\in\mathfrak{K}_R$ $\forall t\in[0,T]$ and $x\in[0,\pi]$:

$$-R\le u(t,x),u_x(t,x),u_{xx}(t,x)\le R. \tag{37}$$

Then using inequality (24) and estimation (16) for $u=V$, similar to (29) we get that for any $u\in\mathfrak{K}_R$ $\forall t\in[0,T]$:

$$\|H(u)\|^2_{B^3_{2,t}}\equiv\|V\|^2_{B^3_{2,t}}\equiv\|\mathfrak{P}_u(V)\|^2_{B^3_{2,t}}\le a_0+\frac{2}{\alpha\pi}\int\limits_0^t\int\limits_0^\pi\{F_u(V(\tau,x))\}^2\,dxd\tau\le$$

$$\le a_0+\frac{4}{\alpha\pi}\int\limits_0^t\int\limits_0^\pi\left\{[F_u(V(\tau,x))-F_u(0)]^2+[F_u(0)]^2\right\}dxd\tau\le$$

$$\le a_0+\frac{4}{\alpha\pi}C_R^2\int\limits_0^t\int\limits_0^\pi V^2_{xxx}(\tau,x)dxd\tau+\frac{4}{\alpha\pi}\|F_u(0)\|^2_{C(Q_T)}\cdot\pi t\le$$

$$\le a_0+\frac{4T}{\alpha}\mathcal{A}_R^2+\frac{4}{\alpha\pi}C_R^2\frac{\pi}{2}\int\limits_0^t\|V\|^2_{B^3_{2,\tau}}\,d\tau, \tag{38}$$

where the number a_0 is determined by relation (30), $Q_T\equiv[0,T]\times[0,\pi]$, $\mathcal{A}_R$ is the maximum of the function $|F(t,x,u_1,u_2,u_3,0|$ in the closed domain $0\le t\le T,\ 0\le x\le\pi,\ -R\le u_1,u_2,u_3\le R$.

Having applied the Belleman inequality, from (38) we get $\forall u\in\mathfrak{K}_R$:

$$\|H(u)\|^2_{B^3_{2,T}}\equiv\|V\|^2_{B^3_{2,T}}\le\left(a_0+\frac{4T}{\alpha}\mathcal{A}_R^2\right)\exp\left(\frac{2}{\alpha}C_R^2\cdot T\right)\equiv a_R^2. \tag{39}$$

Consequently, the set $H(\mathfrak{K}_R)$ is bounded in $B^3_{2,T}$.

Further, we show that $\forall u\in\mathfrak{K}_R$:

$$\max_{0\le t\le T}\left|\int\limits_0^\pi F_u(V(t,x))\sin dx\right|\le\pi\left(C_R^2a_R^2+2\mathcal{A}_R^2\right)^{\frac{1}{2}}\equiv b_R\quad(n=1,2,...), \tag{40}$$

$$\int\limits_0^T\int\limits_0^\pi\{F_u(V(\tau,x))\}^2\,dxd\tau\le\pi T\left(C_R^2a_R^2+2\mathcal{A}_R^2\right)^{\frac{1}{2}}\equiv c_R^2, \tag{41}$$

$$\|(H(u))_t\|^2_{B^1_{2,T}}\equiv\|V_t\|^2_{B^1_{2,T}}\equiv\left\|\sum_{n=1}^\infty V_n'(t)\sin nx\right\|^2_{B^1_{2,T}}\le\sum_{n=1}^\infty(n\max_{0\le t\le T}|V_n'(t)|)^2\le$$

$$\le 3\alpha^2\sum_{n=1}^\infty\left(n^5\varphi_n\right)^2+2b_R^2+\frac{3\alpha}{\pi}c_R^2\equiv d_R^2. \tag{42}$$

Thus, it follows from (39) and (42) that

$$\forall u\in \mathfrak{K}_R \; \|H(u)\|_{B^{3,1}_{2,2,T}} \equiv \|V\|_{B^{3,1}_{2,2,T}} = \|V\|_{B^{3}_{2,T}} + \|V_t\|_{B^{1}_{2,T}} \le a_R + d_R \equiv c_R. \quad (43)$$

Consequently, the set $H(\mathfrak{K}_R)$ is bounded in $B^{3,1}_{2,2,T}$. Hence, from theorem 1 it follows that the set $H(\mathfrak{K}_R)$ considered as a subset of the space $B^{2}_{1,T}$ is compact in $B^{2}_{1,T}$. Thus, the operator H acts in $B^{2}_{1,T}$ compactly. Since the operator H acts in $B^{2}_{1,T}$ continuously as well, then it acts in $B^{2}_{1,T}$ completely continuously. From (13) for $k=3$ and (39), $\forall u\in \mathfrak{K}_R$ we have:

$$\|H(u)\|_{B^{2}_{1,T}} \le \frac{\pi}{\sqrt{6}}\|H(u)\|_{B^{3}_{2,T}} \le \frac{\pi}{\sqrt{6}}a_6 = \frac{\pi}{\sqrt{6}}\left(a_0 + \frac{4}{\alpha}\mathcal{A}_R^2 T\right)^{\frac{1}{2}} \exp\left\{\frac{1}{\alpha}C_R^2 T\right\}. \quad (44)$$

It is seen from (44) that if the number

$$R > \frac{\pi}{\sqrt{6}}\sqrt{a_0} \quad (45)$$

is fixed, then for rather small values of T

$$\forall u\in \mathfrak{K}_R \qquad \|H(u)\|_{B^{2}_{1,T}} \le R,$$

i.e. $H(\mathfrak{K}_R)\subset \mathfrak{K}_R$.

Thus, for any fixed R satisfying condition (45), for rather small values of T, the operator H transforms the ball $\mathfrak{K}_R$ into itself completely continuously. Consequently, from the Schauder's principle on a fixed point, for rather small values of T the operator H has in $\mathfrak{K}_R$ at least one fixed point $u : u = H(u)$. Since $u = H(u) = V = \mathfrak{P}_u(V)$, then $u = V$, and consequently, $u = H(u) = \mathfrak{P}_u(u)$, moreover by (43), $u(t,x)\in B^{3,1}_{2,2,T}$.

Further, by $u = V$ and (28), for the found fixed point $u(t,x) = \sum_{n=1}^{\infty} u_n(t)\sin nx$, the functions $u_n(t)$ $(n = 1, 2, ...)$ satisfy system (9).

Using this fact, we show that the function

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t)\sin nx \in B^{3,1}_{2,2,T} \quad (46)$$

is the generalized solution of problem (1)-(3). The theorem is proved.

Remark 1. It should be noted that condition 1 of theorem 4 imposed on the input function $\varphi(x)$, is not only sufficient but also necessary for the existence of the generalized solution of problem (1)-(3).

4. Investigation of global existence of generalized solution of problem (1)-(3)

In this section, by means of Schauder's strong principle on a fixed point, we prove a theorem on global existence of a generalized solution of problem (1)-(3).

Theorem 5. *Let*

1) all the conditions of theorem 4 be fulfilled.

2) In $[0,T]\times[0,\pi]\times(-\infty,\infty)^4$

$$|F(t,x,u_1,...,u_4)|\le C(1+|u_1|+...+|u_4|), \tag{47}$$

where $C>0$ is a constant.

Then there exists a generalized solution of problem (1)-(3).

Proof. For proving the given theorem, it suffices to make some changes and additions in the proof of theorem 4. More exactly, let H be an operator introduced in the process of proof of theorem 4. As it was shown in the process of proof of theorem 4, the operator H acts in the space $B^2_{1,T}$ completely continuously, it even moves from $B^2_{1,T}$ to $B^3_{2,T}$.

By definition of the operator H:

$$\forall u\in B^2_{1,T}\quad H(u)=V=\mathfrak{P}_u(V),$$

where the operator $\mathfrak{P}_u$ was determined by relations (25)-(27).

Now, let's consider in $B^2_{1,T}$ the equations

$$u=\lambda H(u),\quad 0\le\lambda\le 1, \tag{48}$$

and a priori estimate their all possible solutions in $B^2_{1,T}$. Since

$$u=\lambda H(u)=\lambda V=\lambda\mathfrak{P}_u(V),$$

then similar to (29) and (38), we get $\forall t\in[0,T]$:

$$\|u\|^2_{B^3_{2,t}}\equiv\|\lambda H(u)\|^2_{B^3_{2,t}}\equiv\|\lambda V\|^2_{B^3_{2,t}}\equiv\|\lambda\mathfrak{P}_u(V)\|^2_{B^3_{2,t}}\le\lambda^2a_0+$$

$$+\lambda^2\frac{2}{\alpha\pi}\int\limits_0^t\int\limits_0^\pi\{F_u(V(\tau,x))\}^2\,dxd\tau\le a_0+\frac{2}{\alpha\pi}\lambda^2\int\limits_0^t\int\limits_0^\pi\{F_u(V(\tau,x))\}^2\,dxd\tau, \tag{49}$$

where the number a_0 is determined from relation (30).

Hence, using inequality (47), relation $\lambda V=u$ and estimations (14),(16) and (13) for $k=3$ we get $\forall t\in[0,T]$:

$$\|u\|^2_{B^3_{2,t}}\le a_0+\frac{2}{\alpha\pi}\lambda^2 5C^2\int\limits_0^t\left\{\pi+\int\limits_0^\pi u^2(\tau,x)dx+\int\limits_0^\pi u_x^2(\tau,x)dx+\right.$$

$$\left.+\int\limits_0^\pi u_{xx}^2(\tau,x)dx+\int\limits_0^\pi V_{xxx}^2(\tau,x)\,dx\right\}d\tau\le a_0+\frac{10}{\alpha}C^2T+$$

$$+\frac{10}{\alpha\pi}C^2\int\limits_0^t\left\{\int\limits_0^\pi u^2(\tau,x)dx+\int\limits_0^\pi u_x^2(\tau,x)dx+\int\limits_0^\pi u_{xx}^2(\tau,x)dx+\int\limits_0^\pi\lambda^2V_{xxx}^2(\tau,x)dx\right\}d\tau=$$

$$=a_0+\frac{10T}{\alpha}C^2+\frac{10}{\alpha\pi}C^2\int\limits_0^t\left\{\int\limits_0^\pi u^2(\tau,x)dx+\int\limits_0^\pi u_x^2(\tau,x)dx+\right.$$

$$+\int_0^\pi u_{xx}^2(\tau,x)dx+\int_0^\pi u_{xxx}^2(\tau,x)dx\Bigg\}d\tau\le a_0+\frac{10T}{\alpha}C^2+$$

$$+\frac{10}{\alpha\pi}C^2\int_0^t\left\{3\|u\|^2_{B^2_{1,\tau}}+\frac{\pi}{2}\|u\|^2_{B^3_{2,\tau}}\right\}d\tau\le a_0+\frac{10T}{\alpha}C^2+$$

$$+\frac{10}{\alpha\pi}C^2\int_0^t\left\{3\frac{\pi^2}{6}\|u\|^2_{B^3_{2,\tau}}+\frac{\pi}{2}\|u\|^2_{B^3_{2,\tau}}\right\}d\tau=$$

$$=a_0+\frac{10T}{\alpha}C^2+\frac{5(\pi+1)}{\alpha}C^2\int_0^t\|u\|^2_{B^3_{2,\tau}}d\tau. \tag{50}$$

Having applied the Bellman inequality, from (50) we get:

$$\|u\|^2_{B^3_{2,\tau}}\le\left(a_0+\frac{10T}{\alpha}C^2\right)\exp\left\{\frac{5(\pi+1)}{\alpha}C^2T\right\}\equiv C_0^2. \tag{51}$$

Thus, all possible solutions u of equations (48) in $B^2_{1,T}$ are a priori bounded in $B^3_{2,T}$, and all the more in $B^2_{1,T}$, since by (13) for $k=3$, $\|u\|_{B^2_{1,T}}\le\frac{\pi}{\sqrt{6}}\|u\|_{B^3_{2,T}}$. Then, by Schauder's strong principle on a fixed point in H the operator $B^2_{1,T}$ has a fixed point u that belongs to the space $B^{3,1}_{2,2,T}$ and is a generalized solution of problem (1)-(3).

The theorem is proved.

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Arzu G.Aliyeva
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, B.Vahabzade str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 539 47 20 (apt.).

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