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ON BEHAVIOR OF SOLUTIONS DEGENERATE PARABOLIC EQUATIONS

Abstract

In this paper behavior of solutions of the initial-boundary problem for denigrate quasi-linear parabolic equations in unbounded domains with noncompact boundary is study.

1. Introduction

The goal of the paper is to study behavior of solutions of the initial-boundary problem for degenerate quasi-linear parabolic equations in unbounded domains with noncompact boundary.

For linear elliptic and parabolic equations on behavior of solution were studied in the paper of O.A. Oleinik [1], [2]. For quasilinear equations, similar results were obtained in the papers of A.F. Tedeev, A.E. Shishkov [3], T.S. Gadjiev [4]. S. Bonafade [5] is studied quality properties of solutions for degenerate equations. Also we mention papers [6], [7].

We obtained some estimations that analogies of Saint-Venant's principle known in theory of elasticity. By means of these estimations we obtained estimation on behavior of solution of type Fragmen-Lindelyof.

In unbounded domain Q which contains in layer

$$H_T = \{(x,t) : 0 < 1 < T < \infty\}$$

of Euclid space $R^{n+1}_{x,t}$ consider initial-boundary problem

$$\frac{\partial u}{\partial t} - \sum_{|\alpha|\le m} (-1)^{|\lambda|} D^{\alpha} A_{\alpha}\left(x,t,u,Du,...,D^m_u\right) = 0 \tag{1}$$

$$u\left|_{t=0}\right. = 0 \tag{2}$$

$$D^{\alpha}_x u\left|_{\Gamma=0}\right., \quad |\alpha| \le m-1, \tag{3}$$

where $D^{\alpha} = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_m^{\alpha_m}}, \quad |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_m, m \ge 1.$

The domain Q has noncompact boundary $\partial Q = \Gamma_0 \cup \Gamma_T \cup \Gamma$, where

$$\Gamma_0 = \partial Q \cap \{(x,t) : t = 0\}\,,\ \Gamma_T = \partial Q \cap \{(x,t) : t = T\}\,.$$

Assume that the coefficients $A_{\alpha}(x,t,\xi)$ are measurable with respect to $(x,t) \in Q$, continuous with respect to $\xi \in R^M$ (M is the number of different multi-indices of length no more than m) and satisfy the conditions

$$\sum_{|\alpha|\le m} A_{\alpha}(x,t,\xi)\, \xi^m_{\alpha} > \omega(x) \left|\xi^m\right|^P - c_1 \omega(x) \sum_{i=1}^{m-1} \left|\xi_i\right|^P - f_1(x,t) \tag{4}$$

$$|A_\alpha(x,t,\xi)| \leq C_2\omega(x)\sum_{i=0}^{m}|\xi_i|^P + f_2(x,t), \tag{5}$$

where $\xi = \left(\xi^0,...,\xi^m\right),\ \xi^i = \left(\xi^i_\alpha\right),\ |\alpha| = i,\ P > 1,\ f_1(x,t) \in L_p(0,T\ L_{p,loc}(\Omega_t)),$

$$f_2(x,t) \in L_{1,loc}(Q)\Omega_\tau = Q \cap \{(x,t) : t = \tau\}.$$

The space $L_p\left(0,T,W^m_{q,\omega}(\Omega_t)\right)$ defined as $\left\{u(x,t) : \int\limits_0^T \left(\|u\|_{W^m_{q,\omega}(\Omega_t)}\right)^p dt < \infty\right\}$, where Q-bounded subdomain Q. $\Omega_t = Q' \cap \{(x,t) : t = \tau\}$ $W^m_{q,\omega}(\Omega_t)$ is a closure of Ω_t the functions from $C^m(\overline{\Omega})$ with respect to the norm

$$\|u\|_{W^m_{q,\omega}(\Omega'_1)} = \left(\int\limits_{\Omega'_t}\omega(x)\sum_{|\alpha|\leq m}|D^\alpha_x|^q\, dxdt.\right)^{\frac{1}{q}}.$$

Assume that $\omega(x),\ x \in Q$ is a measurable non negative function satisfying the conditions:

$$\omega(x) \in L_{1,loc}(Q), \tag{6}$$

where $\Omega_s = \Omega_t \cap B_s,\quad B_s = \{x : |x| < s\},\ C_i$-are positive constants dependent only the problems data. In particular, it follows from condition (6) that $\omega \in A_\tau$ (see [8]), i.e. for any $\rho > 0$

$$\int\limits_{\Omega_\rho}\omega dx\left[\int\limits_{\Omega_\rho}\omega(x)^{-\frac{1}{\sigma-1}}dx\right]^{\sigma-1} \leq C_4\rho^{n\sigma}. \tag{7}$$

Well describe geometry ∂Q with weight nonlinear basis frequency $\lambda_p(r,\tau)$ of section $\sigma(r,\tau) = S(r)\cap\Omega_\tau$, where $S(r) = Q \cap \partial Q(r),\quad Q(r) = Q \cap \{B_S \times (0,T)\}$

$$\lambda^p_p(r,\tau) = \inf\left(\int\limits_{\sigma(r,\tau)}\omega(x)\nabla_s v\,|^p\, d\tau\right)\left(\int\limits_{\tau(r,\tau)}\omega(x)\,|\nabla v|^p\, d\tau\right)^{-1},$$

where the lower bound is taken by all continuously differentiable functions in the vicinity of $\sigma(r,\tau)$ that vanish on ∂Q; $\nabla_s v(x)$ is a projection of the vector $\nabla_s v(x)$ on a tangential plane to $\sigma(r,\tau)$ at the point x.

The function $u(x,t) \in L_p\left(0,T,\overset{\circ}{W}{}^m_{p,\omega,loc}(\Omega_t)\right) \cap W^1_2\left(0,T,L_{2,loc}(\Omega_t)\right)$ is said to be a generalized solution of the problem (1)-(3) of the integral identity

$$\int\limits_Q\frac{\partial u}{\partial t}\varphi dxdt + \int\limits_Q\sum_{|\alpha|\leq m}A_\alpha(x,t,u,Du,D^m_u)D^\alpha\varphi dxdt = 0 \tag{8}$$

is fulfilled for the arbitrary function $\varphi(x,t) \in L_p\left(0,T,\overset{\circ}{W}{}^m_{p,\omega}(\Omega'_t)\cap h_2(Q')\right)$.

We will consider classes of domains, for which hold estimate

$$\int\limits_{S_r} \omega(x) |u|^p \, dxdt \le \lambda_p^{-p}(r) \int\limits_{S_r} \omega(x) |\nabla u|^p \, dxdt. \tag{9}$$

The necessary and sufficiently conditions on domains for holds estimate (9) is given for example [9].

2. Behavior of solutions

Let $k(x) \in C^m_{loc}(\Omega)$ positive function, $k(0) = 0$ and that at $x \in \Omega$ hold estimates

$$|D_s k(x)| \ge h_1 > 0,$$
$$\left| D^j_x k(x) \right| \le h_2 (k(x))^{-j+1}, h_2 > 0, j = 1, 2, ..., m.$$

Is defined $\lambda^2_{\mu(s)}(r, \tau) = \lambda^2_2(r, \tau) + \mu^{\frac{2}{m}}_{(s)}, \quad \forall s, \tau > 0.$

$$J_{\mu(s),p}(r, \tau) \equiv \int\limits_{\Omega_\tau(s)} \left(|D^m u|^p + \mu^2(s) u^2 \right) dx,$$

$J_{\mu(S),2}(r, \tau), \Omega_\tau(s_1) \backslash \Omega_\tau(s_2) = M_\tau(s_1, s_2)$ where $\mu(v)$ function which define later.

Lemma 1. *Let $u(x,t) \in h_p(0, T; W^m_{p,\omega}(\Omega'_t) \cap L_2(Q'))$ and $\mu(k(x))$ be a measurable non-negative function locally bounded in Ω. Then the inequality*

$$\int\limits_{M_r(s_1,s_2)} \left| D^j_x u \right|^2 \lambda_{\mu(s)}(k(x), \tau) f(k(x)) dx \le \frac{h_2}{h_1} \times$$

$$\times \int\limits_{M_r(s_1,s_2)} \left(|D^m_x u|^2 + \mu^2(s) \left| D^j u \right|^2 \right) f(k(x)) dx, \tag{10}$$

is valid, where $j \le m$.

We introduce shear function $\xi(t)$ be m times continuously differentiable function, $0 < \xi(t) < 1, 2^{-1} < t < 1; \xi(t) = 1$ for $t < 2^{-1}, \xi(t) = 0$ for $t \ge 1$. Denote $\xi^{(t)}_h = \xi\left(\dfrac{t-h}{1-h}\right)$. The following estimations are true for this shear function

$$\left| D^j_x \xi_h \left(\frac{k(x)}{r} \right) \right| \le \frac{C_j}{[r(1-h)]^j}, \quad rh + \frac{r}{2}(1-h) < j(x) < r, \ j = 0, 1, ..., m.$$

$$D^j_x \xi_h \left(\frac{k(x)}{r} \right) = 0 \quad \text{for} \ \ g(x) \le rh + \frac{r}{2}(1-h), \quad \text{and} \ \ g(x) > r, \ \ j > 1.$$

Lemma 2. *Assume that the continuous non-decreasing on (t, ∞) function $I(t)$ satisfies inequality*

$$I(t) \le \theta I(t\psi(t)), \quad 0 < \theta < 1,$$
$$\psi(t) = 1 + \varphi(t), \quad \varphi(t) > 0 \tag{11}$$

and measurable function $\varphi(t)$ *satisfy*

$$(\varphi(t))^{-1}\inf\varphi(\tau)>\delta>0$$
$$t<\tau<t\psi(t). \tag{12}$$

Then the estimation

$$I(t)\geq\theta\exp\left(\delta In\theta^{-1}\int_{t_0}^{t}\frac{d\tau}{\tau\varphi(\tau)}\right)I(t_0)$$

is valid for $I(t)$.

Our main goal is to obtain estimations of behavior of the function $J_{\mu(\tau),p}(\tau)$ at $\tau\to\infty$.

We defined function $\psi(\tau)$ and $\mu(\tau)$:

$$\inf_{\substack{\tau<k(x)<\tau\psi(\tau)\\0<t<T}}\lambda_{\mu(\tau)}(k(x),t)\tau(\psi(\tau)-1)\geq h_0>0,\quad\forall\tau>\tau_0, \tag{13}$$

$$0<h\leq\mu(\tau\psi(\tau))(\mu(\tau))^{-1}\leq H<\infty,\quad\forall i>\tau. \tag{14}$$

We substitute to integral identity (8) of test function

$$\varphi(x,t)=u(x,t)\left[1-\xi\left(\frac{\varphi(\tau)-k(x)\tau^{-1}}{\psi(\tau)-1}\right)\right]\exp(-2\mu^2(\tau))t.$$

Then by virtue of condition (4), (5) having

$$J_{\mu(\tau,p)}(\tau)=\int_{\Omega_\tau}\left(\omega(x)\left|D_x^m\right|^p+\mu^2\left(\tau\right)u^2\right)\exp(-2\mu^2(\tau)t)dxdt\leq$$

$$\leq\int_{\Omega_{\tau\psi(\tau)}}\left[c_2\omega(x)\sum_{|\alpha|<m}\left|D_x^\alpha u\right|^p-c_3\omega(x)\left(\sum_{|\alpha|<m}\left|D^\alpha u\right|^{p-1}\right)\left(\sum_{|\alpha|<m}\left|D^\alpha u\right|+\right.\right.$$

$$\left.\left.+\sum_{|\alpha|<m}\left|f_2(x)\left|D^\alpha u\right|+\sum_{|\alpha|\leq m}\left|F_\alpha(x)D^\alpha u\right|\right|\right)\left[1-\xi\left(\frac{\varphi(\tau)-k(x)\tau^{-1}}{\psi(\tau)-1}\right)\right]\times$$

$$\times\exp(-2\mu^2(\tau)t)dxdt+\int_{\Omega_\tau\cap\Omega_{\tau\psi(\tau)}}\left[c_3k_2\omega(x)\left(\sum_{|\alpha|\leq m}\left|D^\alpha u\right|^{p-1}\right)\times\right.$$

$$\times\sum_{|\alpha|\leq m}\sum_{|\beta|\leq|\alpha|}\left|D^{\alpha-\beta}u\right|\left|D^\beta\xi\right|+\sum_{|\alpha|\leq m}\sum_{\beta\leq|\alpha|}\left|f_2(x)\right|D^{\alpha-\beta}u\left|D^\beta\xi\right|+$$

$$\left.+\sum_{|\alpha|\leq m}\sum_{|\beta|\leq\alpha}\left|f_\alpha(x)\right|D^{\alpha-\beta}u\left|D^\beta\xi\right|\right]\exp(-2\mu^2(t)t)dxdt. \tag{15}$$

For any $0 < s_1, s_2 < \infty,$ if we use Lemma 1 at $j \le m$, then

$$\int\limits_{\mu_\tau(s_1,s_2)} \left|D^j u\right|^2 \lambda_{\mu(s)}(k(x),\tau) j(k(x))dx \le$$

$$\le \frac{h_2}{h_1} \int\limits_{\mu_\tau(s_1,s_2)} \left(\left|D^{j+1}u\right|^2 + \mu^2(s)\left|D^j u\right|^2\right) j(k(x))dx. \tag{16}$$

Later we considering case $p = 2$. Under $p \neq 2$ calculus doing similarly.

Using interpolation inequality of Nirenberg-Galiardo in following form

$$\int\limits_{\Omega_\tau(r)} \left|D^j u\right|^2 dx < \varepsilon \int\limits_{\Omega_\tau(r)} \left|D^m u\right|^2 dx + c_4 \varepsilon^{-\frac{j}{m-j}} \int\limits_{\Omega_\tau(r)} |u|^2 dx, \quad \forall \varepsilon > 0. \tag{17}$$

For inequality (17) it easy following inequality

$$\mu^{\frac{2(m-j)}{m}}(r) \int\limits_{\Omega} \left|D^j u\right|^2 dx \le \int\limits_{\Omega_\tau(r)} \left|D^m u\right|^2 dx + c_5 \mu^2(r) \int\limits_{\Omega_\tau(r)} |u|^2 dx \tag{18}$$

is obtained.

Now passing to estimates right hand (15), using (16), (18). Then

$$J_{\mu(\tau),p} \le \varepsilon \left[\int\limits_{\Omega_{\tau\psi(\tau)}} \omega(x)\left|D^m u\right|^2 dx + \frac{C_4(\varepsilon)}{\varepsilon} \int\limits_{\Omega_{\tau\psi(\tau)}} |u|^2 dx\right] +$$

$$+ \sum_{|\alpha|=m} \int\limits_{\Omega_{\tau\psi(\tau)}} |f_2(x)|\, dx + c_5 \sum_{|\alpha|<m} \left(\int\limits_{\Omega_{\tau\psi(\tau)}} |f_1(x)|^2 \lambda_{\mu(\tau)}^{-2(m-|\alpha|)}(k(x),t)dx\right)^{1/2} \times$$

$$\times \left(\int\limits_{\Omega_{\tau\psi(\tau)}} (\omega(x)\left|D^m u\right|^2 + \mu^2(\tau)u^2 dx)^{1/2}\right)^{1/2} +$$

$$+c_6 \sum_{|\alpha|\le m} \int\limits_{M_\tau(\tau,\tau\psi(\varepsilon))} |f_1(x)|^2 \lambda_{\mu(\tau)}^{-2(m-|\alpha|)}(k(x),t)dx \times$$

$$\times \int\limits_{M_\tau(\tau,\tau\psi(\varepsilon))} (\omega(x)\left|D^m u\right|^2 + \mu^2(\tau)u^2) \sum_{i=1}^{m} \frac{b_j^2 \lambda_{\mu(\tau)}^{-2i}(k(x),t)}{(\tau(\psi(\tau)-1)^{2i}} dx +$$

$$+c_7 \int\limits_{M_\tau(\tau,\tau\psi(\varepsilon))} (\omega(x)\left|D^m u\right|^2 + \mu^2(\tau)u^2)dx \int\limits_{M_\tau(\tau,\tau\psi(\varepsilon))} (\omega(x)\left|D^m u\right|^2 + \mu^2(\tau)u^2) \times$$

$$\times \sum_{k=1}^{m} \sum_{i=1}^{k} b_i^2 (\tau(\psi(\tau)-1)^{-2i} \lambda_{\mu(\tau)}^{-2(m-k+1)}(k(x),t)dx$$

By virtue choose function $\psi(\tau)$, $\mu(\tau)$ by (13), (14), and by inequality (19), from (15)

$$\int_0^\tau J_{\mu(\tau),p}(\tau,t)dt \le \varepsilon \int_0^\tau J_{\mu(\tau)}(\tau\psi(\tau),t)dx + (\varepsilon + c_8 d_1)\int_0^\tau (J(\tau\psi(\tau)) -$$

$$-J_{\mu(\tau)}dt + c_9(\varepsilon)\int_0^\tau g_{\mu_{(\tau)}}(\tau\psi(\tau),t)\exp(-2\mu^2(\tau)t)dt \tag{20}$$

is obtained. Near $\forall\varepsilon < 0$, $d_1 = \sum\limits_{v=1}^{m} \frac{b_i^2}{h_0^{2i}}$ and

$$g_{\mu_{(\tau)}}(\tau,t) = \int_{\Omega_t} \left[\sum_{|\alpha|=m} |f_2(x)| + \sum_{|\alpha|<m} |f_1(x)|^2 \lambda_{\mu(\tau)}^{-2(m-|\alpha|)}(k(x),t) \right] dx.$$

From (20) we have

$$\int_0^\tau J_{\mu(\tau)}(\tau,t)dt < \frac{2\varepsilon + c_{10}d_1^{1/2}}{1+\varepsilon + c_{10}d_1^{12}} \int_0^\tau J_{\mu(\tau)}(\tau,t)dt +$$

$$+\frac{c_{12}(\varepsilon)}{1+\varepsilon+c_{10}d_1^{1/2}}\int_0^\tau g_{\mu(\tau)}(\tau\psi(\tau),t)\exp(-2\mu^2(\tau)t)dt \tag{21}$$

We defined $\theta = \frac{c_{10}d_1^{1/2}}{1+c_{10}d_1^{12}} < 1$.

By virtue condition (14) from (21) we obtained

$$\int_0^\tau J_{\mu(\tau)}(\tau,t)dt \le \frac{2\varepsilon + c_{10}d_1^{1/2}}{1+\varepsilon + c_{10}d_1^{1/2}} \int_0^\tau J_{\mu(\tau)}(\tau\psi(\tau),t)dx +$$

$$+c_{13}(\varepsilon,h,H)\int_0^\tau g_{\mu(\tau\psi(\tau))}(\tau\psi(\tau),t)\exp(-2\mu^2(\tau)t)dt \tag{22}$$

Now function $\mu(\tau)$ satisfying condition for

$$\theta \exp\left[\left(2\mu^2(\tau\psi(\tau),t) \right) - 2\mu^2(\tau))T \right] \le \beta < 1, \quad \text{for} \quad \forall\tau > \tau_0. \tag{23}$$

Other words condition (23) having mean

$$\mu^2(\tau\psi(\tau)) - \mu^2(\tau) \le 2T^{-1}(In\ \beta - In\ \theta). \tag{24}$$

Then for any $\varepsilon > 0$ from (21) and (22) we have

$$J_{\mu(\tau)}(\tau) \le (\beta+\varepsilon)\, J_{\mu(\tau)}(\tau\psi(\tau)) + \widetilde{c}_{13}(\tau)\, G_{\mu(\tau)}(\tau\psi(\tau)), \quad \forall\tau > \tau_0(\varepsilon) \tag{25}$$

where $G_\mu(\tau) = \int\limits_0^\tau g_{\mu(\tau)}(\tau, t)\exp(-2\mu^2(\tau)t)dt.$

From inequality (24) we obtained

$$J_{\mu(\tau)}(\tau) \leq (\beta+\varepsilon)\left(1+\widetilde{c}_{13}(\varepsilon)\frac{G_{\mu(\tau)}(\tau\psi(\tau))}{J_{\mu(\tau)}(\tau\psi(\tau))}\right)J_{\mu(\tau)}(\tau\psi(\tau)). \qquad (26)$$

From inequality (25) by Lemma 2 following basic theorem is obtained.

Theorem. *Let $u(x)$ be a generalized solution of problem (1)-(3) and measurable, locally bounded function $\mu(\tau), \psi(\tau) > 1$ satisfy conditions (13), (14). Moreover function $\varphi(\varphi) \equiv \psi(\tau) - 1$ satisfy condition (12) of lemma 2 with some $\partial > 0$. Then for integral of energy $J_{\mu(\tau)}(\tau)$ alternative*

1. *or* $\lim\limits_{\tau\to\infty} J_{\mu(\tau)}(\tau)\left(G_{\mu(\tau)}(\tau)\right)^{-1} < C < \infty;$

2. or $J_{\mu(\tau)} > (\beta+\varepsilon)\exp\left(\partial\ In(\beta+\varepsilon)^{-1}\int\limits_{\tau_0}^{-1}\frac{d\tau}{(\tau\psi(\tau)-1)}\right)J_{\mu(\tau)}(\tau).$

is valid, where $G_\mu(\tau) = \int\limits_0^\tau g_{\mu(\tau)}(\tau, t)\exp(-2\mu^2(\tau)t)dt.$

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