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# ON THE BASIS IN THE SPACE $L_p(0,1)$, $1<p<+\infty$ OF THE SYSTEM OF EIGEN FUNCTIONS OF STURM-LIOUVILLE PROBLEM WITH A SPECTRAL PARAMETER IN BOUNDARY CONDITIONS

**Abstract**

*We consider the following spectral problem*

$$-y''(x)=\lambda y(x), \quad x\in(0,1),$$

$$(a_0\lambda+b_0)\,y(0)=(c_0\lambda+d_0)\,y'(0),$$

$$(a_1\lambda+b_1)\,y(1)=(c_1\lambda+d_1)\,y'(1),$$

*where* $\lambda$ *is a spectral parameter,* $a_i,\ b_i,\ c_i,\ d_i,\ i=\overline{0,1}$ *are real constants, moreover*

$$\sigma_0=a_0d_0-b_0c_0<0,\quad \sigma_1=a_1d_1-b_1c_1>0.$$

*Necessary and sufficient basicity conditions in the space* $L_p(0,1)$, $1<p<\infty$ *of the system of eigen functions of this problem with two removed functions are found.*

Consider the following boundary value problem

$$-y''(x)+q(x)\,y(x)=\lambda y(x), \quad x\in(0,1), \tag{1}$$

$$(a_0\lambda+b_0)\,y(0)=(c_0\lambda+d_0)\,y'(0), \tag{2}$$

$$(a_1\lambda+b_1)\,y(1)=(c_1\lambda+d_1)\,y'(1), \tag{3}$$

where $\lambda$ is a spectral parameter, $q(x)$ is a real continuous function on $[0,1]$, $a_i$, $b_i$, $c_i$, $d_i$, $i=\overline{0,1}$ are real constants, moreover

$$\sigma_0=a_0d_0-b_0c_0<0,\quad \sigma_1=a_1d_1-b_1c_1>0. \tag{4}$$

The problem of the form (1)-(3) arises, for example, by separating variables in a dynamic boundary value problem describing small torsional vibrations of a bar with both pulley stiffened ends. A more complete information on the physical sense of the problems of type (1)-(3) may be found in [1] and [2].

In the paper [3] the complete description of general characteristics of arrangement of eigenvalues on a real axis is given, vibrational properties of eigenfunctions are studied, asymptotic formulae for eigenvalues and eigenfunctions of problem (1)-(3) are obtained. The basis properties of eigenfunctions where it is established that the system of eigenfunctions of this problem after removing two arbitrary functions having different parity ordinal number forms a basis in the space $L_p$, $1<p<\infty$ is also investigated.

In the paper [4], problem (1)-(3) is reduced to an eigen value problem for a linear operator acting in the Hilbert space $H = L_2(0,1) \oplus C^2$, necessary and sufficient condition of basicity in $L_p(0,1)$, $1<p<\infty$, of the subsystem of eigen functions of this problem is established. More exactly, it is proved the following theorem.

**Theorem A.** *Let $r$ and $l$ be arbitrary fixed entire non-negative numbers. If*

$$\Delta(r,l) = \begin{vmatrix} 1 & 1 \\ c_1 y_r'(1) - a_1 y_r(1) & c_1 y_l'(1) - a_1 y_l(1) \end{vmatrix} \neq 0. \tag{5}$$

*then the system of eigen functions $\{y_k\}_{k=0,k\neq r,l}^{\infty}$ of problem (1)-(3) forms a basis in the space $L_p(0,1)$, $1<p<\infty$, for $p=2$ the Riesz basis, if $\Delta(r,l)=0$, this system is incomplete and not minimal in the space $L_p(0,1)$, $1<p<\infty$.*

The basis properties of the system of root functions in the space $L_p(0,1)$, $1 < p < \infty$ of problem (1)-(3) (in special cases) were investigated also in the papers [5] and [6]. In the case $q\equiv 0$, $b_j=c_j=0, (-1)^{j+1}a_j>0$, $d_j=1$, $j=\overline{0,1}$, (therewith $\sigma_0<0$, $\sigma_1>0$) in [5] it was proved that if $a_0\neq a_1$ then the system of eigenfunctions of problem (1)-(3) with two removed arbitrary functions forms a basis in the space $L_p(0,1)$, $1<p<\infty$; if $a_0=-a_1$, then the system of eigenfunctions with two removed arbitrary functions having different parity numbers forms a basis in the space $L_p(0,1)$, $1<p<\infty$, if $a_0=-a_1$, then the system of eigenfunction with two arbitrary removed functions having the same order parity, is neither complete nor minimal in $L_p(0,1)$, $1<p<\infty$. In the case $q\equiv 0$, $b_j=c_j=0$, $a_j<0, d_j=1$, $j=\overline{0,1}$ (therewith $\sigma_0<0$, $\sigma_1<0$) necessary and sufficient basicity condition in $L_p(0,1)$, $1<p<\infty$, of the system of the root functions of problem (1)-(3) with two removed root functions is found in [6].

The present paper is devoted to the investigation of basis properties of the subsystem of eigenfunctions of problem (1)-(3) for $q\equiv 0$.

Note that the solution of equation (1) satisfying the initial conditions

$$y(0,\lambda) = c_0\lambda + d_0, \quad y'(0,\lambda) = a_0\lambda + b_0 \tag{6}$$

is of the form

$$y(x,\lambda) = (c_0\lambda + d_0)\cos\sqrt{\lambda}x + (a_0\lambda + b_0)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}. \tag{7}$$

Taking into account boundary condition (3), we get

$$\cot\sqrt{\lambda}\left\{(a_0\lambda+b_0)(c_1\lambda+d_1)-(a_1\lambda+b_1)(c_0\lambda+d_0)\right\} =$$

$$= \frac{1}{\sqrt{\lambda}}(a_0\lambda+b_0)(a_1\lambda+b_1)+(c_0\lambda+d_0)(c_1\lambda+d_1)\sqrt{\lambda}.$$

Thus, the eigenvalues $\lambda_0<\lambda_1<...<\lambda_k<...$ of problem (1)-(3) are the roots of the equation

$$\cot\sqrt{\lambda} = \frac{(a_0\lambda+b_0)(a_1\lambda+b_1)+(c_0\lambda+d_0)(c_1\lambda+d_1)\lambda}{\left\{(a_0\lambda+b_0)(c_1\lambda+d_1)-(a_1\lambda+b_1)(c_0\lambda+d_0)\right\}\sqrt{\lambda}} \tag{8}$$

and by (7) has only the eigenfunctions

$$y_k(x) = (c_0\lambda + d_0)\cos\sqrt{\lambda_k}x + (a_0\lambda + b_0)\frac{\sin\sqrt{\lambda_k}x}{\sqrt{\lambda_k}}, \quad k = 0, 1, ....$$

Then we have

$$y_k(1) = (c_0\lambda_k + d_0)\cos\sqrt{\lambda_k} + (a_0\lambda_k + b_0)\frac{\sin\sqrt{\lambda_k}}{\sqrt{\lambda_k}} =$$

$$= \sin\sqrt{\lambda_k}\left\{(c_0\lambda_k + d_0)\cot\sqrt{\lambda_k} + \frac{1}{\sqrt{\lambda_k}}(a_0\lambda_k + b_0)\right\} =$$

$$= \sin\sqrt{\lambda_k}\left\{\frac{(c_0\lambda + d_0)\{(a_0\lambda_k + b_0)(a_1\lambda_k + b_1) + (c_0\lambda_k + d_0)(c_1\lambda_k + d_1)\lambda_k\}}{\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda + d_0)\}\sqrt{\lambda}}+\right.$$

$$\left.+\frac{(a_0\lambda_k + b_0)\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda_k + d_0)\}}{\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda + d_0)\}\sqrt{\lambda}}\right\} =$$

$$= \sin\sqrt{\lambda_k}\frac{(c_1\lambda_k + d_1)\left\{(a_0\lambda_k + b_0)^2 + (c_0\lambda_k + d_0)^2\lambda_k\right\}}{\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda + d_0)\}\sqrt{\lambda_k}}. \tag{9}$$

Note that

$$\sin\sqrt{\lambda_k} = (-1)^k\frac{1}{\left(1 + \cot^2\sqrt{\lambda_k}\right)^{1/2}}.$$

We have

$$1 + \cot^2\sqrt{\lambda_k} = 1 + \frac{\{(a_0\lambda_k + b_0)(a_1\lambda_k + b_1) + (c_0\lambda_k + d_0)(c_1\lambda + d_1)\lambda_k\}^2}{\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda + d_0)\}^2\lambda_k} =$$

$$= \{\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda_k + d_0)\}^2\lambda_k +$$

$$+\{(a_0\lambda_k + b_0)(a_1\lambda_k + b_1) + (c_0\lambda_k + d_0)(c_1\lambda_k + d_1)\lambda_k\}^2\times$$

$$\times\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda_k + d_0)\}^{-2}\lambda_k^{-1} =$$

$$= \left\{(a_0\lambda_k + b_0)^2(c_1\lambda_k + d_1)^2 + (a_1\lambda_k + b_1)^2(c_0\lambda_k + d_0)^2\right\}\lambda_k +$$

$$+\left\{(a_0\lambda_k + b_0)^2(a_1\lambda_k + b_1)^2 + (c_0\lambda_k + d_0)^2(a_1\lambda_k + b_1)^2\lambda 2_k\right\}\times$$

$$\times\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda_k + d_0)\}^{-2}\lambda_k^{-1} =$$

$$= \frac{\left\{(a_0\lambda_k + b_0)^2(c_0\lambda_k + d_0)^2\lambda_k(a_1\lambda_k + b_1)^2 + (c_1\lambda + d_1)^2\lambda_k\right\}}{\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda + d_0)\}^2\lambda_k}.$$

Taking into account the last two equalities, from (9) we find

$$y_k(1) = (-1)^k\times$$

$$\times\frac{\{(a_0\lambda_k + b_0)(c_1\lambda_k + d_1) - (a_1\lambda_k + b_1)(c_0\lambda + d_0)\}\sqrt{\lambda}}{\left\{(a_0\lambda_k + b_0)^2 + (c_0\lambda_k + d_0)^2\lambda_k\right\}^{1/2}\left\{(a_1\lambda_k + b_1)^2 + (c_1\lambda_k + d_1)^2\lambda_k\right\}^{1/2}}\times$$

$$\times\frac{(c_1\lambda_k+d_1)\left\{(a_0\lambda_k+b_0)^2+(c_0\lambda_k+d_0)^2\lambda_k\right\}}{\{(a_0\lambda_k+b_0)(c_1\lambda_k+d_1)-(a_1\lambda_k+b_1)(c_0\lambda_k+d_0)\}\sqrt{\lambda_k}}=$$

$$=(-1)^k(c_1\lambda_k+d_1)\left(\frac{(a_0\lambda_k+b_0)^2+(c_0\lambda_k+d_0)^2\lambda_k}{(a_1\lambda_k+b_1)^2+(c_1\lambda_k+d_1)^2\lambda_k}\right)^{1/2}.$$

Thus, we have

$$y_k(1)=(-1)^k(c_1\lambda_k+d_1)\left(\frac{(a_0\lambda_k+b_0)^2+(c_0\lambda_k+d_0)^2\lambda_k}{(a_1\lambda_k+b_1)^2+(c_1\lambda_k+d_1)^2\lambda_k}\right)^{1/2}. \tag{10}$$

Let the following relation be fulfilled

$$a_1=a_0,\quad b_1=b_0,\quad c_1=-c_0,\quad d_1=-d_0. \tag{11}$$

Then from (10) we get

$$y_k(1)=(-1)^k(c_1\lambda_k+d_1). \tag{12}$$

If $(c_j\lambda_r+d_j)(c_j\lambda_l+d_j)\neq 0$, $j=0,1$ then by (14) from [4] (see also [3]), equalities (6) and (12) we have

$$\Delta(r,l)=\Delta_1(r,l)=\begin{vmatrix}1 & 1\\ c_1y_r'(1)-a_1y_r(1) & c_1y_l'(1)-a_1y_l(1)\end{vmatrix}=$$

$$=\begin{vmatrix}1 & 1\\ \frac{b_1c_1-a_1d_1}{c_1\lambda_r+d_1}y_r(1) & \frac{b_1c_1-a_1d_1}{c_1\lambda_l+d_1}y_r(1)\end{vmatrix}=$$

$$=\begin{vmatrix}1 & 1\\ \frac{y_r(1)}{c_1\lambda_r+d_1} & \frac{y_l(1)}{c_1\lambda_l+d_1}\end{vmatrix}=\begin{vmatrix}1 & 1\\ (-1)^r & (-1)^l\end{vmatrix}. \tag{13}$$

If $c_0\neq 0$, $\lambda_l=-d_0/c_0$, then $\lambda_l=-d_1/c_1$, $c_j\lambda_r+d_j\neq 0$, $j=0,1$. Consequently, by (12) we have

$$\Delta(r,l)=\Delta_5(r,l)=\begin{vmatrix}1 & 1\\ \frac{y_r(1)}{c_1\lambda_r+d_1} & -\frac{c_1y_l'(1)}{\sigma_1}\end{vmatrix}=\begin{vmatrix}1 & 1\\ (-1)^r & -\frac{c_1y_l'(1)}{\sigma_1}\end{vmatrix}. \tag{14}$$

From formula (7) we get

$$y_l(1)=y(1,\lambda_l)=(c_0\lambda_l+d_0)\cos\sqrt{\lambda_l}+(a_0\lambda_l+b_0)\frac{\sin\sqrt{\lambda_l}}{\sqrt{\lambda_l}}=$$

$$=(a_0\lambda_l+b_0)\frac{\sin\sqrt{\lambda_l}}{\sqrt{\lambda_l}}=0.$$

Taking into account this equality, from formula (7) we find

$$y_l'(1)=y'(1,\lambda_l)=-\sqrt{\lambda_l}(c_0\lambda_l+d_0)\sin\sqrt{\lambda_l}x+(a_0\lambda+b_0)\cos\sqrt{\lambda_l}=$$

$$=\left(a_0\left(-\frac{d_0}{c_0}\right)+b_0\right)\cos\sqrt{\lambda_l}=-\frac{a_0d_0-b_0c_0}{c_0}\cos\sqrt{\lambda_l}=$$

$$= -\frac{a_1 d_1 - b_1 c_1}{c_1} \cos\sqrt{\lambda_l} = -\frac{\sigma_1}{c_1}\sqrt{1-\sin^2\sqrt{\lambda_l}} = -\frac{(-1)^k \sigma_1}{c_1}.$$

Taking into account this relation in (14), we get

$$\Delta_5(r,l) = \begin{vmatrix} 1 & 1 \\ (-1)^r & (-1)^l \end{vmatrix}. \tag{15}$$

From (13) and (15) it follows that if $r$ and $l$ are entire non-negative numbers having the same parity, then $\Delta_{r,l} = 0$ and by theorem $A$, the system of eigenfunctions $\{y_k\}_{k=0,k\neq r,l}^{\infty}$ of problem (1)-(3) for $q \equiv 0$ is neither complete nor minimal in the space $L_p(0,1)$, $1 < p < +\infty$.

Thus, we proved

**Theorem 1.** *Let $r$ and $l$ be entire non-negative numbers having the same parity, and condition (11) be fulfilled. Then the system of eigenfunctions $\{y_k\}_{k=0,\ k\neq r,l}^{\infty}$ of problem (1)-(3) for $q \equiv 0$ is neither complete nor minimal in the space $L_p(0,1)$, $1 < p < +\infty$.*

From this theorem it is seen that the condition of theorem 4 from [3] about that the numbers $r$ and $l$ have different parities is essential.

Let $r$ and $l$ be entire non-negative numbers having the same parities. Then it holds the equality

$$\Delta(r,l) = \Delta_1(r,l) = \begin{vmatrix} 1 & 1 \\ \frac{y_r(1)}{c_1\lambda_r + d_1} & \frac{y_l(1)}{c_1\lambda_l + d_1} \end{vmatrix} =$$

$$= (-1)^r \begin{vmatrix} 1 & 1 \\ \left(\frac{(a_0\lambda_l+b_0)^2+(c_0\lambda_l+d_0)^2\lambda_l}{(a_1\lambda_l+b_1)^2+(c_1\lambda_l+d_1)^2\lambda_l}\right)^{1/2} & \left(\frac{(a_0\lambda_r+b_0)^2+(c_0\lambda_r+d_0)^2\lambda_r}{(a_1\lambda_r+b_1)^2+(c_1\lambda_r+d_1)^2\lambda_r}\right)^{1/2} \end{vmatrix}. \tag{16}$$

Now, let's consider the function

$$F(\lambda) = \left(\frac{(a_0\lambda+b_0)^2 + (c_0\lambda+d_0)^2\lambda}{(a_1\lambda+b_1)^2 + (c_1\lambda+d_1)^2\lambda}\right)^{1/2}. \tag{17}$$

Let $c_0^2 + c_1^2 > 0$ . We rewrite the function $F(\lambda)$ in the form

$$F(\lambda) = \left(\frac{(a_0\lambda+b_0)^2 + (c_0\lambda+d_0)^2\lambda}{(a_1\lambda+b_1)^2 + (c_1\lambda+d_1)^2\lambda}\right)^{1/2} =$$

$$= \left(\frac{c_0^2\lambda^3 + \left(a_0^2 + 2c_0d_0\right)\lambda^2 + \left(d_0^2 + 2a_0b_0\right)\lambda + b_0^2}{c_1^2\lambda^3 + \left(a_1^2 + 2c_1d_1\right)\lambda^2 + \left(d_1^2 + 2a_1b_1\right)\lambda + b_1^2}\right)^{1/2}.$$

Hence we have

$$F'(\lambda) = (1/2)\left(F(\lambda)\right)^{-1/2} \times$$

$$\times \left\{c_1^2\lambda^3 + \left(a_1^2 + 2c_1d_1\right)\lambda^2 + \left(d_1^2 + 2a_1b_1\right)\lambda + b_1^2\right\}^{-2} \times$$

$$\times\{\left(3c_0^2\lambda^2 + 2\left(a_0^2 + 2c_0d_0\right)\lambda + \left(d_0^2 + 2a_0b_0\right)\right) \times$$

$$\times\left(c_1^2\lambda^3+\left(a_1^2+2c_1d_1\right)\lambda^2+\left(d_1^2+2a_1b_1\right)\lambda+b_1^2\right)-$$
$$\times\left(c_0^2\lambda^3+\left(a_0^2+2c_0d_0\right)\lambda^2+\left(d_0^2+2a_0b_0\right)\lambda+b_0^2\right)\times$$
$$\times\left(3c_1^2\lambda^2+2\left(a_1^2+2c_1d_1\right)\lambda+\left(d_1^2+2a_1b_1\right)\right)\}=$$
$$=(1/2)\left(F\left(\lambda\right)\right)^{-\frac{1}{2}}\left\{c_1^2\lambda^3+\left(a_1^2+2c_1d_1\right)\lambda^2+\left(d_1^2+2a_1b_1\right)\lambda+b_1^2\right\}^{-2}\times$$
$$\times\left\{c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)\right\}\lambda^4+2\{c_0^2\left(d_1^2+2a_1b_1\right)-$$
$$-c_1^2\left(d_0^2+2a_0b_0\right)\}\lambda^3+\{3\left(c_0^2b_1^2-c_1^2b_0^2\right)+\left(a_0^2+2c_0d_0\right)\left(d_1^2+2a_1b_1\right)-$$
$$-\left(a_1^2+2c_1d_1\right)\left(d_0^2+2c_1d_1\right)\left(d_0^2+2a_0b_0\right)\}\lambda^2+$$
$$+2\left\{b_1^2\left(a_0^2+2c_0d_0\right)-b_0^2\left(a_1^2+2c_1d_1\right)\right\}+$$
$$+b_1^2\left(d_0^2+2a_0b_0\right)-b_0^2\left(d_1^2+2a_1b_1\right)\}. \tag{18}$$

It follows from formula (18) that either
1) $c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)\neq 0$;
or
2) $c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)=0$ , $c_0^2\left(d_1^2+2a_1b_1\right)-c_1^2\left(d_0^2+2a_0b_0\right)\neq 0$;
or
3) $c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)=0$ , $c_0^2\left(d_1^2+2a_1b_1\right)-c_1^2\left(d_0^2+2a_0b_0\right)=0$, $c_0^2b_1^2-c_1^2b_0^2=0$, then there exists $\lambda^*\in R$ such that $F'\left(\lambda\right)\neq 0$ for $\lambda\geq\lambda^*$, i.e. the function $F\left(\lambda\right)$ is strongly monotone for $\lambda\geq\lambda^*$. Consequently, there exists an entire non-negative number $k^*$ such that for $r,l\geq k^*$ we'll have $\Delta\left(r,l\right)=\Delta_1\left(r,l\right)\neq 0$ from (16) and (17), i.e. condition (5) is fulfilled. Then on the base of theorem $A$, the system of eigen functions $\{y_k\left(x\right)\}_{k=0,k\neq r,l}^{\infty}$, $r,l\geq k^*$ of problem (1)-(3) for $q\equiv 0$ forms the Riesz basis in the space $L_p\left(0,1\right)$, $1<p<\infty$, for $p=2$ the Riesz basis.

And if
4) $c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)=0$, $c_0^2\left(d_1^2+2a_1b_1\right)-c_1^2\left(d_0^2+2a_0b_0\right)=0$, $c_0^2b_1^2-c_1^2b_0^2=0$, then $F'\left(\lambda\right)=0$ for all $\lambda\in R$, i.e. $F\left(\lambda\right)\equiv const$. Consequently, from (16), (17) we'll have $\Delta\left(r,l\right)=\Delta_1\left(r,l\right)=0$. Then on the base of theorem $A$ the system of eigenfunctions $\{y_k\left(x\right)\}_{k=0,k\neq r,l}^{\infty}$ of problem (1)-(3) for $q\equiv 0$ is neither complete nor minimal in the space $L_p\left(0,1\right)$, $1<p<\infty$.

Now let $c_0=c_1=0$. Then from formula (10) we get

$$y_k\left(1\right)=\left(-1\right)^k d_1\left(\frac{\left(a_0\lambda+b_0\right)^2+d_0^2\lambda_k}{\left(a_1\lambda+b_1\right)^2+d_1^2\lambda_k}\right)^{1/2}. \tag{19}$$

Consider the function

$$F\left(\lambda\right)=\left(\frac{\left(a_0\lambda+b_0\right)^2+d_0^2\lambda}{\left(a_1\lambda+b_1\right)^2+d_1^2\lambda}\right)^{1/2}. \tag{20}$$

We rewrite the function $F\left(\lambda\right)$ in the following form:

$$F\left(\lambda\right)=\left(\frac{\left(a_0\lambda+b_0\right)^2+d_0^2\lambda}{\left(a_1\lambda+b_1\right)^2+d_1^2\lambda}\right)^{1/2}=\left(\frac{a_0^2\lambda^2+\left(2a_0b_0+d_0^2\right)\lambda+b_0^2}{a_1^2\lambda^2+\left(2a_1b_1+d_1^2\right)\lambda+b_1^2}\right)^{1/2}. \tag{21}$$

From (21) we get

$$F'(\lambda)=\frac{1}{2}\left(F(\lambda)\right)^{-1/2}\left\{a_1^2\lambda+\left(d_1^2+2a_1b_1\right)\lambda+b_1^2\right\}^{-2}$$

$$\times\left\{\left(2a_0^2\lambda+\left(d_0^2+2a_0b_0\right)\right)\left(a_1^2\lambda^2+\left(d_1^2+2a_1b_1\right)\lambda+b_1^2\right)-\right.$$

$$\left.-\left(a_0^2\lambda^2+\left(d_0^2+2a_0b_0\right)\lambda+b_0^2\right)\left(2a_1^2\lambda+\left(d_1^2+2a_1b_1\right)\right)\right\}=$$

$$=(1/2)\left(F(\lambda)\right)^{-1/2}\times\left\{a_1^2\lambda+\left(d_1^2+2a_1b_1\right)\lambda+b_1^2\right\}^{-2}\times$$

$$\times\left\{\left\{a_0^2\left(d_1^2+2a_1b_1\right)-a_1^2\left(d_0^2+2a_0b_0\right)\right\}\lambda^2+\right.$$

$$\left.+2\left(a_0^2b_1^2-a_1^2b_0^2\right)\lambda+\left\{b_1^2\left(d_0^2+2a_0b_0\right)-b_0^2\left(d_1^2+2a_1b_1\right)\right\}\right\}. \qquad (22)$$

From (22) it follows that either

1) $a_0^2\left(d_1^2+2a_1b_1\right)-a_1^2\left(d_0^2+2a_0b_0\right)\neq 0$;

or

2) $a_0^2\left(d_1^2+2a_1b_1\right)-a_1^2\left(d_0^2+2a_0b_0\right)=0$, $a_0^2b_1^2-a_1^2b_0^2\neq 0$, then there exists $\lambda^{**}\in R$ such that $F'(\lambda)\neq 0$ for $\lambda\geq\lambda^{**}$, i.e. the function $F(\lambda)$ is a strictly monotone for $\lambda\geq\lambda^{**}$. Consequently, there exists an entire non-negative number $k^{**}$ such that for $r,l\geq k^{**}$ we'll have $\Delta(r,l)=\Delta_1(r,l)\neq 0$, from (16), (17). Then on the base of theorem $A$ the system of eigen functions $\{y_k(x)\}_{k=0,k\neq r,l}^{\infty}$, $r,l\geq k^{**}$ of problem (1)-(3) for $q\equiv 0$ forms a basis in the space $L_p(0,1)$, $1<p<\infty$, for $p=2$ the Riesz basis.

And if

3) $a_0^2\left(d_1^2+2a_1b_1\right)-a_1^2\left(d_0^2+2a_0b_0\right)=0$, $a_0^2b_1^2-a_1^2b_0^2=0$, then $F'(\lambda)=0$ for all $\lambda\in R$ i.e. $F'(\lambda)=const$. Consequently, from (16), (17) we'll have $\Delta(r,l)=\Delta_1(r,l)=0$. Then again on the base of theorem $A$, the system of eigenfunctions $\{y_k(x)\}_{k=0,k\neq r,l}^{\infty}$, of problem (1)-(3) for $q\equiv 0$ is neither complete nor minimal in the space $L_p(0,1)$, $1<p<\infty$.

So, we proved the following

**Theorem 2.** *Let $r$ and $l$ be entire non-negative numbers of the same parity. Then there exists such an entire non-negative number $\overline{k}$ $\left(\overline{k}=\max\{k^*,k^{**}\}\right)$, that for $r,l\geq\overline{k}$ the system of eigen functions $\{y_k(x)\}_{k=0,k\neq r,l}^{\infty}$ of problem (1)-(3) for $q\equiv 0$ in the cases*

*a) $c_0^2+c_1^2>0$ and either*

*1) $c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)\neq 0$, or*

*2) $c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)=0$, $c_0^2\left(d_1^2+2a_1b_1\right)-c_1^2\left(d_0^2+2a_0b_0\right)\neq 0$*

*or*

*3) $c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)=0$, $c_0^2\left(d_1^2+2a_1b_1\right)-c_1^2\left(d_0^2+2a_0b_0\right)=0$, $c_0^2b_1^2-c_1^2b_0^2\neq 0$,*

*b) $c_0=c_1=0$ and either*

*1) $a_0^2\left(d_1^2+2a_1b_1\right)-a_1^2\left(d_0^2+2a_0b_0\right)\neq 0$; or*

*2) $a_0^2\left(d_1^2+2a_1b_1\right)-a_1^2\left(d_0^2+2a_0b_0\right)=0$, $a_0^2b_1^2-a_1^2b_0^2\neq 0$ forms a basis in the spaces $L_p(0,1)$, $1<p<\infty$, for $p=2$ the Riesz basis. In the cases*

*c) $c_0^2+c_1^2>0$ and $c_0^2\left(a_1^2+2c_1d_1\right)-c_1^2\left(a_0^2+2c_0d_0\right)=0$,*
*$c_0^2\left(d_1^2+2a_1b_1\right)-c_1^2\left(d_0^2+2a_0b_0\right)=0$, $c_0^2b_1^2-c_1^2b_0^2=0$;*

*d)* $c_0 = c_1 = 0$ *and* $a_0^2 \left(d_1^2 + 2a_1 b_1\right) - a_1^2 \left(d_0^2 + 2a_0 b_0\right) = 0$, $a_0^2 b_1^2 - a_1^2 b_0^2 = 0$, *the system of eigenfunctions* $\{y_k(x)\}_{k=0, k\neq r,l}^{\infty}$ *is neither complete nor minimal in the space* $L_p(0,1)$, $1 < p < +\infty$.

Note that in the case $c_0 = c_1 = 0$ and $b_0 = b_1 = 0$

$$F(\lambda) = \left(\frac{(a_0^2\lambda + d_0^2)}{(a_1^2\lambda + d_1^2)}\right)^{1/2},$$

$$F'(\lambda) = \frac{1}{2}\left(\frac{(a_1^2\lambda + d_1^2)}{(a_0^2\lambda + d_0^2)}\right)^{1/2} \frac{\left(a_0^2 d_1^2 - a_1^2 d_0^2\right)\lambda^2}{\left(a_1^2\lambda + d_1^2\right)^2}.$$

Hence it follows that if $a_0^2 d_1^2 - a_1^2 d_0^2 \neq 0$, then $F'(\lambda) \neq 0$ for $\lambda \in R\backslash\{0\}$ and consequently the function $F(\lambda)$ is strongly monotone in $R$. From (16) and (17) we'll have $\Delta(r,l) = \Delta_5(r,l) \neq 0$. Then on the base of theorem $A$, the system of eigenfunctions $\{y_k(x)\}_{k=0,k\neq r,l}^{\infty}$ of problem (1)-(3) for $q \equiv 0$ forms a basis in the space $L_p(0,1)$, $1 < p < \infty$, for $p = 2$ the Riesz basis. But if $a_0^2 d_1^2 - a_1^2 d_0^2 = 0$, then this system is neither complete nor minimal in the space $L_p(0,1)$, $1 < p < +\infty$. Recall that in the case $d_0 = -1$, $d_1 = 1$ this result was obtained by N.Yu. Kapustin [5].

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