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INTEGRAL LIMIT THEOREM FOR THE FIRST PASSAGE TIME FOR THE LEVEL OF RANDOM WALK, DESCRIBED WITH $AR(1)$ SEQUENCES

Abstract

In the paper the integral limit theorem is proved for the first passage for a level of random walk described by an autoregression sequences $AR(1)$.

1.Introduction. Let $\xi_n;\ n \geq 1$ be a sequence of independent identically distributed random variables determined on some probability space (Ω, F, P).

As is known, the autoregressive sequence of first order $AR(1)$ is determined as the solution of the equation

$$X_n = \beta X_{n-1} + \xi_n,\ n \geq 1,\ X_0 = x, \tag{1}$$

where x and β are non-random constants, and we'll suppose $x \geq 0$ and $|\beta| < 1$.

Assume

$$T_n = \sum_{k=1}^{n} X_{k-1} X_k,\ n \geq 1$$

and consider the first passage time

$$\tau_a = \inf\{n \geq 1 : T > a\} \tag{2}$$

of the process $T_n,\ n \geq 1$ for the level $a \geq 0$.

The first passage time of type (1) was an investigation object in the papers [1-5], where different boundary problems for $AR(1)$ sequences were studied.

Sufficient conditions for exponential boundedness of the first passage time for the level of $AR(1)$ sequences are found and an identity for the mean time of the first passage is obtained in the paper [1].

In [5], the limit distribution of the first overshoot for the level of the $AR(1)$ sequence is found.

In the present paper we prove an integral limit theorem for the first passage time τ_a of the form (2) under which we understand any assertion on the convergence in distribution

$$\frac{\tau_a - A(a)}{B(a)} \xrightarrow{d} \eta,$$

where η is some non-degenerate random variable, $A(a)$ and $B(a) > 0$ are normalized non-random constants dependent on a. Integral limit theorems play an important part in theoretical and applied problems of theory of random walks. The role and value of these theorems are explained in [2], [3] (see also [9]).

2. Conditions and formulation of the main result

At first we give the following definition that plays a fundamental role in investigation of weak convergence of the sum of the random number of random variables ([6], [9]).

Definition. *A sequence of random variables* η_n, $n \geq 1$ *is said to be uniformly continuous in probability if for any* $\varepsilon > 0$

$$\lim_{\delta\to 0}\sup_{n\geq 1} P\left\{\max_{0\leq k\leq n\delta}\left|\eta_{n+k}-\eta_n\right| > \varepsilon\right\} = 0 \tag{3}$$

Remark 1. Note that any sequence of random variables converging almost surely to finite limit, is uniformly continuous in probability.

Note that the sum of two random variables uniformly continuous in probability is uniformly continuous in probability (see [9]).

We'll assume that

$$0 < \beta < 1, \quad E\xi_n = 0 \text{ and } D\xi_n = 1.$$

Enumerate some properties of $AR(1)$-sequence. It is easy to see that X_n has the following representation

$$X_n = \xi_n + \beta\xi_{n-1} + \beta^2 X_{n-1} = ... = \sum_{i=0}^{n-1}\beta^k\xi_{n-i} + \beta^n x.$$

Hence it follows that

$$EX_n = x\beta^n \text{ and } DX_n = \frac{1-\beta^{2n}}{1-\beta^2}$$

Taking into account that the random variables X_{n-1} and ξ_n are independent, we have

$$EX_nX_{n-1} = \beta EX_n + E\xi_n EX_{n-1} = \beta EX_{n-1}^2$$

Then from (4) we find

$$EX_nX_{n-1} \to \frac{\beta}{1-\beta^2} = \lambda, \tag{5}$$

$EX_n^2 \to \dfrac{1}{1-\beta^2}$ as $n\to\infty$.

Remark 2. Note that $AR(1)$ sequence with the initial value $X_0 = x$ is non-stationary since EX_n and DX_n obviously depend on n. By $|\beta| < 1$, the limit values of its mean value and variance coincide with appropriate characteristics of the stationary $AR(1)$-sequence satisfying (1) for all $n = 0, \ \pm 1, \pm 2...$ (see [10]).

Assume

$$\sigma = \frac{1}{\lambda}\sqrt{\frac{1-\beta^2}{\lambda}} \text{ and } \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{x} e^{-y^{2/2}}dy.$$

It holds

Theorem. *Let* $E\xi_n = 0$, $D\xi_n = 1$ *and* $0 < \beta < 1$.

Then

$$\lim_{a\to\infty} P\left(\frac{\tau_a - \frac{a}{\lambda}}{\sigma\sqrt{a}} \leq x\right) = \Phi(x), \ x \in R.$$

3. Proof of the theorem. For proving theorem, we need a number of known statements formulated in the form of the following lemmas 1-5.

Lemma 1. *For* $|\beta| < 1$ *it holds*

$$\frac{T_n}{n} \xrightarrow{a.s.} \lambda \text{ as } n \to \infty.$$

The statement of this lemma was proved in [7] (see also [6]).

Lemma 2. *For* $|\beta| < 1$ *it holds*

$$\lim_{n\to\infty} P(T_n^* \le x) = \Phi(x)$$

where $T_n^* = \dfrac{T_n - \lambda_n}{\sqrt{n(1-\beta^2)}}$.

This statement was proved in [7], (see also [6]).

Lemma 3. T*he sequence* T_n^*, $n \ge 1$ *is uniformly continuous in probability. This lemma was proved in [6].*

Lemma 4. *Let* t_c, $c > 0$ *be a family of integer random variables such that* $\frac{t_c}{c} \xrightarrow{P} \theta > 0$ *as* $c \to \infty$, *and let the sequence of random variables* Y_n, $n \ge 1$ *satisfy condition (2) and converge in distribution* $Y_n \xrightarrow{d} y$.. *Then* $Y_{t_c} \xrightarrow{d} y$ as $c \to \infty$.

The statement of this lemma follows from the Anscombe theorem [8], [9].

Lemma 5. *Let the sequence* Y_n, $n \ge 1$ *of random variables converge almost surely to the random variable* $\left(Y_n \xrightarrow{a.s.} y\right)$, *and let for the family of integer random variables the convergence* $t_c \xrightarrow{a.s.} \infty$ *as* $c \to \infty$ *be fulfilled. Then* $Y_{t_c} \xrightarrow{d} y$ *as* $c \to \infty$. *This lemma was proved in [8].*

Prove the following lemma on asymptotic properties of the first passage time τ_a *of the form (1).*

Lemma 6. *Let* $0 < \infty < 1$. *Then the following statements are true:*

1) $P(\tau_a < \infty) = 1$ *for all* $a \ge 0$.

2) $\tau_a \xrightarrow{a.s.} \infty$ *as* $a \to \infty$

3) $\dfrac{\tau_a}{a} \xrightarrow{a.s.} \dfrac{1-\beta^2}{\beta}$ *as* $a \to \infty$.

Proof. From lemma 1 it follows that

$$P\left(\sup_n T_n = \infty\right) = 1.$$

Hence we have

$$P\left(\tau_a < \infty\right) = P\left(\sup_n T_n > a\right) = 1$$

for all $a \ge 0$.

Prove statement 2). The process τ_a, $a \ge 0$ as a function of a increases and therefore there exists the limit $\tau_\infty = \lim_{a\to\infty} \tau_a \le \infty$. On the other hand,

$$P\left(\tau_\infty \le n\right) = \lim_{a\to\infty}\left(\tau_a \le n\right) =$$

$$= \lim_{a\to\infty} P\left(\sup_{1\le k\le n} T_k > a\right) = 0$$

for all $n \geq 1$. Thus, $\lim\limits_{a\to\infty} P(\tau_a > n) = 1$ and $P(\tau_\infty = \infty) = 1$.

For proving statement 3) we note that by statement 2) of the proved lemma from lemma 1 and 5 we get that

$$\frac{T_{\tau_a}}{\tau_a} \overset{a.s.}{\to} \lambda \text{ as } a \to \infty \tag{6}$$

From definition of the first passage time τ_a we have

$$\frac{T_{\tau_a - 1}}{\tau_a} \leq \frac{a}{\tau_a} < \frac{T_{\tau_a}}{\tau_a} \tag{7}$$

Then statement 3) of lemma 6 follows from (6) and (7).

Theorem's proof. Having assumed $R_a = T_{\tau_a} - a$, we have

$$\frac{T_{\tau_a} - \lambda\tau_a}{\sqrt{\tau_a}} = \frac{a - \lambda\tau_a}{\sqrt{\tau_a}} + \frac{R_a}{\sqrt{\tau_a}}$$

or

$$T^*_{\tau_a} = \frac{T_{\tau_a} - \lambda\tau_a}{\sqrt{(1-\beta^2)\tau_a}} = -\frac{\tau_a - \dfrac{a}{\lambda}}{\dfrac{\sqrt{1-\beta^2}}{\lambda}\sqrt{\tau_a}} + \frac{R_a}{\sqrt{(1-\beta^2)\tau_a}} \tag{8}$$

By statement 3) of lemma 6, from lemmas 2,3, and 4 we have

$$\lim_{a\to\infty} p(T^*_{\tau_a} \leq x) = \Phi(x), \quad x \in R \tag{9}$$

For obtaining the statement of the theorem from equality (8), it suffices to show that

$$\frac{R_a}{\sqrt{\tau_a}} \overset{P}{\to} 0 \text{ as } a \to \infty \tag{10}$$

Indeed, taking into account $T_{\tau_a - 1} \leq a$, we have

$$R_a = T_{\tau_a} - a \leq T_{\tau_a} - T_{\tau_a - 1} = X_{\tau_a - 1} X_{\tau_a}$$

or

$$\frac{R_a}{\sqrt{\tau_a}} \leq \frac{X_{\tau_a - 1} X_{\tau_a}}{\sqrt{\tau_a}} \tag{11}$$

Applying the Cauchy-Bunyakovsky inequality and taking into account (4), (5), it is easy to show that

$$\frac{E\left|X_{n-1}X_n\right|}{\sqrt{n}} \to 0 \text{ for } n \to \infty.$$

Then from the last relation and the Chebyshev inequality it follows that

$$\frac{X_{n-1}X_n}{\sqrt{n}} \overset{P}{\to} 0 \text{ as } n \to \infty. \tag{12}$$

Further, we have

$$\frac{X_{n-1}X_n}{\sqrt{n}} = \frac{T_n - n\lambda}{\sqrt{n}} - \frac{T_n - (n-1)\lambda}{\sqrt{n}} + \frac{\lambda}{\sqrt{n}}.$$

Then by remark 1 and lemma 3, the sequence $\dfrac{X_{n-1}X_n}{\sqrt{\tau_a}}, n \geq 1$ is uniformly continuous in probability.

Now from (11) and lemma 4 we find

$$\frac{X_{\tau_a-1}X_{\tau_a}}{\sqrt{\tau_a}} \xrightarrow{P} 0 \text{ as } a \to \infty. \tag{13}$$

Consequently, (10) follows from (11) and (12).

From (8), (9) and (10) we have

$$\lim_{a\to\infty} P\left(\frac{\tau_a - \dfrac{a}{\lambda}}{\dfrac{\sqrt{1-\beta^2}}{\lambda}\sqrt{\tau_a}} \leq x\right) = \Phi(x).$$

By the statement of lemma 6, from the last relation we get the statement of the theorem.

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