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MARCINKIEWICZ INTEGRAL ON GENERALIZED WEIGHTED MORREY SPACES

Abstract

In this paper, we study the boundedness of the Marcinkiewicz operators μ_Ω and their commutators $\mu_{b,\Omega}$ on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair (φ_1, φ_2) with $w \in A_p(\mathbb{R}^n)$ which ensures the boundedness of the operators μ_Ω from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. In all cases the conditions for the boundedness of the operator μ_Ω is given in terms of Zygmund-type integral inequalities on (φ_1, φ_2) and w, which do not assume any assumption on monotonicity of $\varphi_1(x, r)$, $\varphi_2(x, r)$ in r.

1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [27] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [12, 13, 17, 27]. Recently, Komori and Shirai [24] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev [18] introduced the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ and studied the boundedness of the classical operators and its commutators in this spaces $M_{p,\varphi}(w)$, see also Guliyev et al [18, 21, 23].

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

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where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the com- mutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b} f(x) = \left(\int_0^\infty |F^b_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F^b_{\Omega,t}(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)] f(y)dy.$$

On the other hand, the study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [29] considered Lp estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R^L_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. Then, Dziubanński and Zienkiewicz [11] introduced the Hardy type space $H^1_L(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$.

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions $\mu_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu^L_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K^L_j(x,y) f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $K^L_j(x,y) = \widetilde{K^L_j}(x,y)|x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In particular, when $V = 0$, $K^\Delta_j(x,y) = \widetilde{K^\Delta_j}(x,y)|x-y| = \frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K^\Delta_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} \Delta^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In this paper, we write $K_j(x,y) = K^\Delta_j(x,y)$ and

$$\mu_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K_j(x,y) f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Let $f \in L^{\rm loc}_1(\mathbb{R}^n)$. The maximal operator with rough kernel M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)|\, |f(y)|dy.$$

It is obvious that when $\Omega \equiv 1$, M_Ω is the Hardy-Littlewood maximal operator M. Obviously, $\mu_{j,\Omega}$ are classical Marcinkiewicz functions with rough kernel. Therefore, it will be an interesting thing to study the property of $\mu^L_{j,\Omega}$. The main purpose of this paper is to show that Marcinkiewicz operators with rough kernel associated with

Schrödinger operators $\mu_{j,\Omega}^L$, $j = 1, \ldots, n$ are bounded from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $\Omega \in L_\infty(S^{n-1})$ and $1 < p < \infty$.

2.Preliminaries

We say that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$ if there exists a constant $C > 0$ such that $|\Omega(x') - \Omega(y')| \le C|x' - y'|^\alpha$ for all $x', y' \in S^{n-1}$.

The operator μ_Ω was first defined by Stein [31]. And Stein proved that if is continuous and satisfies a $\mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \le 1)$ condition, then μ_Ω is an operator of type (p,p) $(1 < p \le 2)$ and of weak type $(1,1)$. In [4], Benedek, Calderón and Panzone proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The L_p boundedness of μ_Ω has been studied extensively. See [4, 22, 31, 32], among others. A survey of past studies can be found in [6]. Ding, Fan and Pan [7] proved the weighted $L_p(\mathbb{R}^n)$ boundedness with A_p weighs for a class of rough Marcinkiewicz integrals. Recently, Ding, Fan and Pan [8] improved the results mentioned above and showed that if $\Omega \in H^1(S^{n-1})$, the Hardy space on the unit sphere, then μ_Ω is still a bounded operator on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. In [33], Xu, Chen and Ying proved the same result as [8] using a different method.

Theorem 2.1. *([10]) Suppose that Ω be satisfies the conditions* (1) *and $\Omega \in L_\infty(S^{n-1})$. Then for every $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, there is a constant C independent of f such that*

$$\|M_\Omega(f)\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

Theorem 2.2. *([7]) Suppose that Ω be satisfies the conditions* (1), (2) *and $\Omega \in L_\infty(S^{n-1})$. Then for every $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, there is a constant C independent of f such that*

$$\|\mu_\Omega(f)\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

Note that a nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \le C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \qquad (3)$$

holds for every ball $x \in \mathbb{R}^n$ and $r > 0$, where $B(x,r)$ denotes the open ball centered at x with radius r; see [29]. It is worth pointing out that the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only n and the constant C in (3), such that $V \in B_{q+\varepsilon}$. Throughout this paper, we always assume that $0 \ne V \in B_n$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem in the case $w = 1$ was proved in [5].

Theorem 2.3. *Let v_1, v_2 and w be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w^* g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{4}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{5}$$

Moreover, the value $C = B$ is the best constant for (4).

Remark 2.4. In (4) and (5) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

3. Generalized weighted Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [27] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [15, 25].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We recall that a weight function w is in the Muckenhoupt's class $A_p(\mathbb{R}^n)$ [28], $1 < p < \infty$, if

$$\begin{aligned}[w]_{A_p} :&= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1}\end{aligned} \tag{6}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B by Hölder's inequality

$$[w]_{A_p(B)}^{1/p} = |B|^{-1} \|w\|_{L_1(B)}^{1/p} \|w^{-1/p}\|_{L_{p'}(B)} \ge 1. \tag{7}$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \le Cw(x)$ with $[w]_{A_1} = \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ $A_\infty(\mathbb{R}^n) = \bigcup_{1\le p<\infty} A_p(\mathbb{R}^n)$ and $[w]_{A_\infty} = \inf_{1\le p<\infty} [w]_{A_p}$.

Definition 3.1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space consisting of all measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{_{B(x,r)}}\|_{WL_p(\mathbb{R}^n)} < \infty.$$

Also the spaces $L_p^{\rm loc}(\mathbb{R}^n)$ and $WL_p^{\rm loc}(\mathbb{R}^n)$ endowed with the natural topology are defined as the set of all functions f such that $f\chi_{_B} \in L_p(\mathbb{R}^n)$ and $f\chi_{_B} \in WL_p(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$, respectively.

According to this definition, we recover the space $M_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$

$$WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

We define the generalized weighed Morrey spaces as follows.

Definition 3.2. Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, w(B(x,r))^{-\frac{1}{p}} \, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{_{B(x,r)}}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y) dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_{p,w}^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, w(B(x,r))^{-\frac{1}{p}} \, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y)dy \right)^{\frac{1}{p}}.$$

Remark 3.3. (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin supp f$

$$|Tf(x)| \leq c_0 \int_{\mathbb{R}^n} |x-y|^{-n} \, |f(y)|dy, \tag{8}$$

where c_0 is independent of f and x.

We point out that the condition (8) was first introduced by Soria and Weiss in [30]. The condition (8) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operator, Hardy–Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [26], [30] for details).

The following statement, was proved in [23], see also [18, 21].

Theorem 3.4. *Let $1 \leq p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leq C\,\varphi_2(x,r), \tag{9}$$

where C does not depend on x and r. Let T be a sublinear operator satisfying condition (8) bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{1,w}(\mathbb{R}^n)$. Then the operator T is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.

The following statement, was proved in [21], see also [18].

Note that, in the case $w = 1$ Theorem 3.4 was proved in [19] and for the operators M and K in [1].

4. Marcinkiewicz operator μ_Ω in the spaces $M_{p,\varphi}(w)$

Lemma 4.1. *Suppose that Ω be satisfies the conditions (1), (2) and $\Omega \in L_\infty(S^{n-1})$. Then for every $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$ the inequality*

$$\|\mu_\Omega(f)\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, and for all $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$.

Proof. Let Ω be satisfies the conditions (1), (2) and $\Omega \in L_\infty(S^{n-1})$.

For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0 \tag{10}$$

and have

$$\|\mu_\Omega(f)\|_{L_{p,w}(B)} \le \|\mu_\Omega(f_1)\|_{L_{p,w}(B)} + \|\mu_\Omega(f_2)\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $\mu_\Omega(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of μ_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_p(\mathbb{R}^n)$ (see Theorem 2.2) it follows that

$$\|\mu_\Omega(f_1)\|_{L_{p,w}(B)} \le \|\mu_\Omega(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L_\infty(S^{n-1})} [w]_{A_p}^{\frac{1}{p}} \|f_1\|_{L_{p,w}(\mathbb{R}^n)}$$
$$\approx \|\Omega\|_{L_\infty(S^{n-1})} [w]_{A_p}^{\frac{1}{p}} \|f\|_{L_{p,w}(2B)}.$$

It's clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$. Then by the Minkowski inequality and conditions on Ω, we get

$$\mu_\Omega(f_2(x)) \lesssim \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy.$$

By Fubini's theorem we have

$$\int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy \approx \int_{\complement_{(2B)}} |\Omega(x-y)|\,|f(y)| \int_{|x-y|}^{\infty} \frac{dt}{t^{n+1}} dy$$
$$= \int_{2r}^{\infty} \int_{2r\le|x_0-y|<t} |\Omega(x-y)|\,|f(y)| dy \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}\ .$$

By applying Hölder's inequality for $w \in A_p(\mathbb{R}^n)$, we get

$$\int_{\complement_{(2B)}} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n} dy \lesssim \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_\infty(B(x_0,t))} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})} [w]_{A_p}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x_0,t))^{-\frac{1}{p}} |B(x_0,t)| \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})} [w]_{A_p}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}. \tag{11}$$

Moreover, for all $p \in (1, \infty)$ the inequality

$$\|\mu_\Omega (f_2)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_\infty(S^{n-1})} [w]_{A_p}^{\frac{1}{p}} w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

is valid. Thus

$$\|\mu_\Omega(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_\infty(S^{n-1})} [w]_{A_p}^{\frac{1}{p}} \Big(\|f\|_{L_{p,w}(2B)} + w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t} \Big).$$

On the other hand,

$$\begin{aligned} \|f\|_{L_{p,w}(2B)} &\approx |B| \, \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\ &\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}} \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim [w]_{A_p}^{\frac{1}{p}} w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}. \end{aligned}$$

Thus

$$\|\mu_\Omega(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_\infty(S^{n-1})} [w]_{A_p}^{\frac{1}{p}} w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

Thus we complete the proof of Lemma 4.1.

Theorem 4.2. *Suppose that Ω be satisfies the conditions* (1), (2) *and* $\Omega \in L_\infty(S^{n-1})$. *Let also,* $1 < p < \infty$, $w \in A_p(\mathbb{R}^n)$ *and the pair* (φ_1, φ_2) *satisfy the condition* (9). *Then the operator* μ_Ω *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$ *for* $p > 1$. *Moreover*

$$\|\mu_\Omega(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. By Lemma 4.1 and Theorem 2.3 with $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,t))^{-\frac{1}{p}}$, $\nu_2(r) = \varphi_2(x,r)^{-1}$ and $w(r) = w(B(x,t))^{-\frac{1}{p}}$ we have for $p > 1$

$$\begin{aligned} \|\mu_\Omega(f)\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_\Omega(f)\|_{L_{p,w}(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_{r}^{\infty} \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}B(x,r)} \\ &= \|f\|_{M_{p,\varphi_1}(w)}. \end{aligned}$$

Remark 4.3. Note that, in the case $w = 1$ the boundedness of the parametric Marcinkiewicz operator on generalized Morrey spaces were study in [20].

5. Marcinkiewicz operator $\mu^L_{j,\Omega}$ in the spaces $M_{p,\varphi}(w)$

In this section, we prove the boundedness of the Marcinkiewicz operator μ^L_j on $M_{p,\varphi}(w)$ spaces. For $x \in \mathbb{R}^n$, the function $m_V(x)$ is defined by

$$\rho(x) = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V\ (y)dy\ \leq 1 \right\}.$$

Lemma 5.1. [29] *Let $V \in B_q$ with $q \geq n/2$. Then there exists $l_0 > 0$ such that*

$$\frac{l}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \approx \rho(y)$, if $|x-y| < C\rho(x)$.

Lemma 5.2. [29] *Let $V \in B_q$ with $q \geq n/2$. For any $l > 0$, there exists $C_l > 0$ such that*

$$\left|K^L_j(x,y)\right| \leq \frac{C_l}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^l} \frac{1}{|x-y|^{n-1}},$$

and

$$\left|K^L_j(x,y) - K_j(x-y)\right| \leq\ C\frac{\rho(x)^{-1}}{|x-y|^{n-2}}.$$

The following theorem in the case $w = 1$ was proved in [2].

Theorem 5.3. *Suppose that $\Omega \in L_\infty(S^{n-1})$ satisfies the conditions* (1), (2) *and $V \in B_n$. Then for every $w \in A_p(\mathbb{R}^n)$ there is a constant C independent of f such that*

$$\|\mu^L_{j,\Omega}(f)\|_{L_{p,w}} \leq C\|f\|_{L_{p,w}}.$$

In the proof of the Theorem 5.3 the validity of the following inequality is proved

$$\mu^L_{j,\Omega}f(x) \leq \mu_{j,\Omega}f(x) + CM_\Omega f(x), a.e.\ \ x \in \mathbb{R}^n.$$

Note that the operators M_Ω and $\mu_{j,\Omega}$ which are sublinear operators satisfies the condition (8) and bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$. Statements of the Lemma 4.1 for the operators M_Ω and $\mu_{j,\Omega}$ is provided. Then we get that the statements of the Lemma 4.1 also true for the operators $\mu^L_{j,\Omega}$, $j = 1, \ldots, n$. From Lemma 4.1 and Theorem 5.3 the following corollary is obtained.

Corollary 5.4.*Suppose that $\Omega \in L_\infty(S^{n-1})$ satisfies the conditions* (1), (2) *and $V \in B_n$. Then for every $w \in A_p(\mathbb{R}^n)$ the inequality*

$$\|\mu^L_{j,\Omega}(f)\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, and for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Corollary 5.5.*Suppose that $\Omega \in L_\infty(S^{n-1})$ satisfies the conditions* (1), (2) *and $V \in B_n$. Let also, for $w \in A_p(\mathbb{R}^n)$ the pair (φ_1, φ_2) satisfies the condition* (9)*. Then the operator $\mu^L_{j,\Omega}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$. Moreover*

$$\|\mu^L_{j,\Omega}(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

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