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## ANISOTROPIC MAXIMAL AND SINGULAR INTEGRAL OPERATORS IN ANISOTROPIC GENERALIZED MORREY SPACES

### Abstract

*In this paper we give the conditions on the pair  $(\varphi_1, \varphi_2)$  which ensures the boundedness of the anisotropic maximal operator and anisotropic singular integral operators from one generalized Morrey space  $\mathcal{M}_{p, \varphi_1}$  to another  $\mathcal{M}_{p, \varphi_2}$ ,  $1 < p < \infty$ , and from the space  $\mathcal{M}_{1, \varphi_1}$  to the weak space  $WM_{1, \varphi_2}$ .*

### 1. Introduction and main results

The theory of boundedness of classical operators of real analysis, such as maximal operator and singular integral operators etc, from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with weighted Lebesgue spaces, general Morrey-type spaces also play an important role.

Let  $P$  be a real  $n \times n$  matrix, all of whose eigenvalues have positive real part. Let  $A_t = t^P$  ( $t > 0$ ), and set  $\gamma = trP$ . Then, there exists a quasi-distance  $\rho$  associated with  $P$  such that

- (a)  $\rho(A_t x) = t\rho(x)$ ,  $t > 0$ , for every  $x \in \mathbb{R}^n$ ;
- (b)  $\rho(0) = 0$ ,  $\rho(x - y) = \rho(y - x) \geq 0$   
and  $\rho(x - y) \leq k(\rho(x - z) + \rho(y - z))$ ;
- (c)  $dx = \rho^{\gamma-1} d\sigma(w) d\rho$ , where  $\rho = \rho(x)$ ,  $w = A_{\rho^{-1}} x$   
and  $d\sigma(w)$  is a  $C^\infty$  measure on the ellipsoid  $S_\rho = \{w : \rho(w) = 1\}$ .

Then,  $\{\mathbb{R}^n, \rho, dx\}$  becomes a space of homogeneous type in the sense of Coifman-Weiss. Moreover, we always assume the following properties on  $\rho$ :

- (d) For every  $x$ ,

$$\begin{aligned} c_1|x|^{\alpha_1} \leq \rho(x) \leq c_2|x|^{\alpha_2} & \text{ if } \rho(x) \geq 1 \\ c_3|x|^{\alpha_3} \leq \rho(x) \leq c_4|x|^{\alpha_4} & \text{ if } \rho(x) \leq 1 \end{aligned}$$

and

$$\rho(\theta x) \leq \rho(x) \text{ for } 0 < \theta < 1.$$

Here  $\alpha_i$  and  $c_i$  ( $i = 1, \dots, 4$ ) are some positive constants. Similar properties hold for  $\rho^*$  which is associated with the matrix  $P^*$ .

There are some important examples for the above spaces:

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1. Let  $(Px, x) \geq (x, x)$  ( $x \in \mathbb{R}^n$ ). In this case,  $\rho(x)$  is defined by the unique solution of  $|A_{t^{-1}}x| = 1$ , and  $k = 1$ . This space is just the one studied by Calderon and Torchinsky in [7].

2. Let  $P$  be a diagonal matrix with positive diagonal entries, and let  $\rho(x)$  be the unique solution of  $|A_{t^{-1}}x| = 1$ .

2<sub>a</sub>) If all diagonal entries are greater than or equal to 1, this space was studied by E.B. Fabes and N.M. Riviere [11]. More precisely they studied the weak  $(1, 1)$  and  $L^p$  estimates of the singular integral operators on this space in 1966.

2<sub>b</sub>) If there are diagonal entries smaller than 1, then  $\rho$  satisfies the above (a) – (d) with  $k \geq 1$ .

Let  $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ . The anisotropic maximal function  $Mf$  is defined by

$$Mf(x) = \sup_{t>0} \frac{1}{|\mathcal{E}_P(x, t)|} \int_{\mathcal{E}_P(x, t)} |f(y)| dy,$$

where  $|\mathcal{E}_P(x, t)|$  is the Lebesgue measure of the ellipsoid  $\mathcal{E}_P(x, t)$  centered at  $x$ .

The boundedness of the maximal operator  $M$  in Morrey spaces  $\mathcal{M}_{p, \lambda}$  was proven in [10] (isotropic case) and in generalized Morrey spaces  $\mathcal{M}_{p, \varphi}$ ,  $p \in (1, \infty)$  with a function  $\varphi(x, r)$  satisfying suitable doubling and integral conditions  $\tilde{\mathcal{Z}}_{p, \gamma}$  (see section 2) in [22]. In more general substations, namely in local and global Morrey type spaces, the boundedness of the maximal operator  $M$  has been investigated in [3]-[6], [14]-[16].

**Definition 1.1.** *The function  $k(x; \xi) : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{R}$  is called a variable Calderón-Zygmund type kernel with mixed homogeneity if:*

i) *For every fixed  $x$  the function  $k(x; \cdot)$  is constant kernel satisfying:*

$$i_a) \quad k(x; \cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\});$$

$$i_b) \quad k(x; A_t \xi) = t^{-\gamma} k(x; \xi), \quad t > 0;$$

$$i_c) \quad \int_{S_\rho} k(x; \xi) d\sigma(\xi) = 0 \quad \text{and} \quad \int_{S_\rho} |k(x; \xi)| d\sigma(\xi) < \infty;$$

ii) *For every multiindex  $\beta$ , the inequality  $\sup_{\xi \in \mathbb{S}^{n-1}} |D_\xi^\beta k(x; \xi)| \leq C(\beta)$  is satisfied independently of  $x$ .*

Note that in the isotropic case  $P = I$ , and thus  $\gamma = n$ , Definition 1.1 gives rise to the classical Calderón-Zygmund kernels (see, for example, [8] and [26]). One more example is when  $P = \text{diag}\{\alpha_1, \dots, \alpha_{n-1}, \alpha_n\}$ ,  $\alpha_1 = \dots = \alpha_{n-1} = 1$ ,  $\alpha_n = \bar{\alpha} \geq 1$ . In this case we obtain the parabolic kernels studied by Jones in [17] and discussed in [11].

We consider the following anisotropic singular integral

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x; x - y) f(y) dy \quad (1)$$

with a variable Calderón-Zygmund type kernel  $k(x, \xi)$ ,  $x \in \mathbb{R}^n$ ,  $\xi \in \mathbb{R}^n \setminus \{0\}$ , satisfying a mixed homogeneity condition  $i_b$ ). The boundedness of the operator  $T$  in  $L_p(\mathbb{R}^n)$ ,  $p \in (1, \infty)$  was proven in [1], [11] and in Morrey spaces  $\mathcal{M}_{p, \lambda}$  in [12] (isotropic case). The boundedness of the operator  $T$  in generalized Morrey spaces  $\mathcal{M}_{p, \varphi}$ ,  $p \in (1, \infty)$  with a function  $\varphi(x, r)$  satisfying suitable doubling and integral

conditions  $\tilde{\mathcal{Z}}_{p,\gamma}$  in [25] (isotropic case in [22]). The boundedness of the operator  $T$  from  $\mathcal{M}_{p,\varphi_1}$  to  $\mathcal{M}_{p,\varphi_2}$ ,  $1 < p < \infty$  satisfying integral conditions  $(\varphi_1, \varphi_2) \in \tilde{\mathcal{Z}}_{p,\gamma}$  was proven in [14, 15]. Our goal is to extend results in [14, 15, 16] with a pair  $(\varphi_1, \varphi_2)$  satisfying more large integral conditions  $\mathcal{Z}_{p,\gamma}$ . In [3]-[6], [14] and [15] the boundedness of the singular integral operators in local and global Morrey-type spaces has been investigated. Note that the global Morrey-type space is a more general space than generalized Morrey space.

By  $A \lesssim B$  we mean that  $A \leq CB$  with some positive constant  $C$  independent of appropriate quantities. If  $A \lesssim B$  and  $B \lesssim A$ , we write  $A \approx B$  and say that  $A$  and  $B$  are equivalent.

## 2. Generalized Morrey spaces and Preliminary Results

Morrey spaces  $\mathcal{M}_{p,\lambda}$  were introduced by C. Morrey in 1938 [21] and defined as follows: for  $0 \leq \lambda \leq n$ ,  $1 \leq p \leq \infty$ ,  $f \in \mathcal{M}_{p,\lambda}$  if  $f \in L_p^{\text{loc}}(\mathbb{R}^n)$  and

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty,$$

where  $B(x, r)$  is the open ball centered at  $x$  of radius  $r$ . Note that  $\mathcal{M}_{p,0} = L_p(\mathbb{R}^n)$  and  $\mathcal{M}_{p,n} = L_\infty(\mathbb{R}^n)$ . If  $\lambda < 0$  or  $\lambda > n$ , then  $\mathcal{M}_{p,\lambda} = \Theta$ , where  $\Theta$  is the set of all functions equivalent to 0 on  $\mathbb{R}^n$ .

These spaces appeared to be quite useful in the study of the local behaviour of solutions to partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

We also denote by  $W\mathcal{M}_{p,\lambda}$  the weak Morrey space of all functions  $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$  for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where  $WL_p$  denotes the weak  $L_p$ -space.

If in place of the power function  $r^\lambda$  in the definition of  $\mathcal{M}_{p,\lambda}$  we consider any positive measurable function  $\varphi(x, r)$ , then it becomes generalized Morrey space  $\mathcal{M}_{p,\varphi}$ .

**Definition 2.1.** Let  $\varphi(x, r)$  be a positive measurable function on  $\mathbb{R}^n \times (0, \infty)$  and  $1 \leq p < \infty$ . We denote by  $\mathcal{M}_{p,\varphi}$  the generalized Morrey space, the space of all functions  $f \in L_p^{\text{loc}}(\mathbb{R}^n)$  with finite quasinorm

$$\|f\|_{\mathcal{M}_{p,\varphi}} \equiv \|f\|_{\mathcal{M}_{p,\varphi}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-\frac{1}{p}} \|f\|_{L_p(B(x,t))}.$$

We also denote by  $W\mathcal{M}_{p,\varphi}$  the weak generalized Morrey space of all functions  $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$  for which

$$\|f\|_{W\mathcal{M}_{p,\varphi}} \equiv \|f\|_{W\mathcal{M}_{p,\varphi}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

**Definition 2.2.** Let  $\varphi_1(x, r)$ ,  $\varphi_2(x, r)$  be two positive measurable functions on  $\mathbb{R}^n \times (0, \infty)$ . We say that  $(\varphi_1, \varphi_2)$  belongs to the class  $\mathcal{Z}_{p,m}$ ,  $p \in [0, \infty)$ ,  $m > 0$ , if there is a constant  $C$  such that, for any  $x \in \mathbb{R}^n$  and for any  $t > 0$

$$\left( \int_t^\infty \left( \frac{\text{ess inf}_{r < s < \infty} \varphi_1(x, s)}{r^m} \right)^{\frac{1}{p}} \frac{dr}{r} \right)^p \leq C \frac{\varphi_2(x, t)}{t^m}, \quad \text{if } p \in (0, \infty) \quad (2)$$

and

$$\operatorname{ess\,sup}_{t < r < \infty} \frac{\operatorname{ess\,inf}_{r < s < \infty} \varphi_1(x, s)}{r^m} \leq C \frac{\varphi_2(x, t)}{t^m}, \quad \text{if } p = 0. \quad (3)$$

**Definition 2.3.** Let  $\varphi_1(x, r)$ ,  $\varphi_2(x, r)$  be two positive measurable functions on  $\mathbb{R}^n \times (0, \infty)$ . We say that  $(\varphi_1, \varphi_2)$  belongs to the class  $\tilde{\mathcal{Z}}_{p,m}$ ,  $p \in [0, \infty)$ ,  $m > 0$  if there is a constant  $C$  such that, for any  $x \in \mathbb{R}^n$  and for any  $t > 0$ ,

$$\left( \int_t^\infty \left( \frac{\varphi_1(x, r)}{r^m} \right)^{\frac{1}{p}} \frac{dr}{r} \right)^p \leq C \frac{\varphi_2(x, t)}{t^m}, \quad \text{if } p \in (0, \infty) \quad (4)$$

and

$$\operatorname{ess\,sup}_{t < r < \infty} \frac{\varphi_1(x, r)}{r^m} \leq C \frac{\varphi_2(x, t)}{t^m}, \quad \text{if } p = 0. \quad (5)$$

Note that  $\tilde{\mathcal{Z}}_{p,m} \subset \mathcal{Z}_{p,m}$  for  $p \in [0, \infty)$ ,  $m > 0$ .

The following property for the class  $\mathcal{Z}_{p,m}$ ,  $p \in [0, \infty)$ ,  $m > 0$  is valid.

**Lemma 2.4.** ([16]) For  $m > 0$  the following equality

$$\mathcal{Z}_{0,m} = \bigcup_{0 < p < \infty} \mathcal{Z}_{p,m}$$

is valid.

**Proof.** Assume that  $(\varphi_1, \varphi_2) \in \mathcal{Z}_{p,m}$  for some  $p \in (0, \infty)$ . Then for any  $s \in (t, \infty)$

$$\begin{aligned} \frac{\varphi_2(x, t)}{t^m} &\gtrsim \left( \int_t^\infty \left( \frac{\operatorname{ess\,inf}_{r < \tau < \infty} \varphi_1(x, \tau)}{r^m} \right)^{\frac{1}{p}} \frac{dr}{r} \right)^p \\ &\gtrsim \left( \int_s^\infty \left( \frac{\operatorname{ess\,inf}_{r < \tau < \infty} \varphi_1(x, \tau)}{r^m} \right)^{\frac{1}{p}} \frac{dr}{r} \right)^p \\ &\gtrsim \operatorname{ess\,inf}_{s < \tau < \infty} \varphi_1(x, \tau) \left( \int_s^\infty \frac{dr}{r^{\frac{m}{p}+1}} \right)^p \\ &\approx \frac{\operatorname{ess\,inf}_{s < \tau < \infty} \varphi_1(x, \tau)}{s^m}. \end{aligned}$$

Thus

$$\frac{\varphi_2(x, t)}{t^m} \gtrsim \operatorname{ess\,sup}_{t < s < \infty} \frac{\operatorname{ess\,inf}_{s < \tau < \infty} \varphi_1(x, \tau)}{s^m}.$$

This proves that

$$\bigcup_{0 < p < \infty} \mathcal{Z}_{p,m} \subset \mathcal{Z}_{0,m}.$$

Now assume that  $(\varphi_1, \varphi_2) \in \mathcal{Z}_{0,m}$ . It means that the condition (3) holds. Since

$$\operatorname{ess\,sup}_{t < r < \infty} \frac{\operatorname{ess\,inf}_{r < s < \infty} \varphi_1(x, s)}{r^m} = \lim_{p \rightarrow 0} \left( \int_t^\infty \left( \frac{\operatorname{ess\,inf}_{r < s < \infty} \varphi_1(x, s)}{r^m} \right)^{\frac{1}{p}} \frac{dr}{r} \right)^p,$$

then the remain part of the proof follows from the definition of limit.

In [25] Softova proved the following statement, containing in the isotropic case Nakai's result in [22].

**Theorem 2.5.** *Let  $1 \leq p < \infty$ . Moreover, let  $\varphi(t), t > 0$  be a positive measurable function satisfying the following conditions: there exists  $c > 0$  such that*

$$0 < r \leq t \leq 2r \Rightarrow c^{-1}\varphi(r) \leq \varphi(t) \leq c\varphi(r) \tag{6}$$

and  $(\varphi, \varphi) \in \widetilde{\mathcal{Z}}_{1,\gamma}$ .

Then for  $1 < p < \infty$  the operators  $M$  and  $T$  are bounded from  $\mathcal{M}_{p,\varphi}$  to  $\mathcal{M}_{p,\varphi}$  and for  $p = 1$  the operators  $M$  and  $T$  are bounded from  $\mathcal{M}_{1,\varphi}$  to  $W\mathcal{M}_{1,\varphi}$ .

The following statement, containing Softova results in [25] was proved by Guliyev in [14] for singular integrals defined on homogeneous Folland-Stein groups [13] (see also [15]).

**Theorem 2.6.** *Let  $1 \leq p < \infty$  and  $(\varphi_1, \varphi_2) \in \widetilde{\mathcal{Z}}_{p,\gamma}(\mathbb{R}^n)$ . Then for  $1 < p < \infty$  the operators  $M$  and  $T$  are bounded from  $\mathcal{M}_{p,\varphi_1}$  to  $\mathcal{M}_{p,\varphi_2}$  and for  $p = 1$  the operators  $M$  and  $T$  are bounded from  $\mathcal{M}_{1,\varphi_1}$  to  $W\mathcal{M}_{1,\varphi_2}$ .*

Sufficient conditions on  $\varphi$  for the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces  $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$  have been obtained in [3]-[6], [14]-[16], [18]-[20], [22]-[25].

Let  $\mathfrak{M}(0, \infty)$  be the set of all Lebesgue-measurable functions on  $(0, \infty)$  and  $\mathfrak{M}^+(0, \infty)$  its subset consisting of all nonnegative functions on  $(0, \infty)$ . We denote by  $\mathfrak{M}^+(0, \infty; \uparrow)$  the cone of all functions in  $\mathfrak{M}^+(0, \infty)$  which are non-decreasing on  $(0, \infty)$  and

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0, \infty; \uparrow) : \lim_{t \rightarrow 0^+} \varphi(t) = 0 \right\}.$$

Let  $u$  be a continuous and non-negative function on  $(0, \infty)$ . We define the supremal operator  $\overline{S}_u$  on  $g \in \mathfrak{M}(0, \infty)$  by

$$(\overline{S}_u g)(t) := \|u g\|_{L_\infty(t, \infty)}, \quad t \in (0, \infty).$$

The following theorem was proved in [6].

**Theorem 2.7.** *Let  $v_1, v_2$  be non-negative measurable functions satisfying  $0 < \|v_1\|_{L_\infty(t, \infty)} < \infty$  for any  $t > 0$  and  $u$  be a continuous non-negative function on  $(0, \infty)$*

*Then the operator  $\overline{S}_u$  is bounded from  $L_{\infty, v_1}(0, \infty)$  to  $L_{\infty, v_2}(0, \infty)$  on the cone  $\mathbb{A}$  if and only if*

$$\left\| v_2 \overline{S}_u \left( \|v_1\|_{L_\infty(\cdot, \infty)}^{-1} \right) \right\|_{L_\infty(0, \infty)} < \infty. \tag{7}$$

We are going to use the following statement on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t} \int_0^t g(r) dr, \quad 0 < t < \infty.$$

**Theorem 2.8.** ([9]) *The inequality*

$$\operatorname{ess\,sup}_{t>0} w(t)Hg(t) \leq c \operatorname{ess\,sup}_{t>0} v(t)g(t) \quad (8)$$

holds for all non-negative and non-increasing  $g$  on  $(0, \infty)$  if and only if

$$A := \sup_{t>0} \frac{w(t)}{t} \int_0^t \frac{ds}{\operatorname{ess\,sup}_{0<y<s} v(y)} < \infty, \quad (9)$$

and  $c \approx A$ .

### 3. The anisotropic maximal operator in generalized Morrey spaces

We need the following two lemmas.

**Lemma 3.1.** *Let  $1 < p < \infty$ . Then for any ellipsoid  $\mathcal{E}_P = \mathcal{E}_P(x, r)$  in  $\mathbb{R}^n$  the inequality*

$$\|Mf\|_{L_p(\mathcal{E}_P(x, r))} \lesssim \|f\|_{L_p(\mathcal{E}_P(x, 2r))} + r^{\frac{\gamma}{p}} \sup_{t>2r} t^{-\gamma} \|f\|_{L_1(\mathcal{E}_P(x, t))} \quad (10)$$

holds for all  $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ .

Moreover, the inequality

$$\|Mf\|_{WL_1(\mathcal{E}_P(x, r))} \lesssim \|f\|_{L_1(\mathcal{E}_P(x, 2r))} + r^\gamma \sup_{t>2r} t^{-\gamma} \|f\|_{L_1(\mathcal{E}_P(x, t))} \quad (11)$$

holds for all  $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ .

**Proof.** Let  $1 < p < \infty$ . It is obvious that for any ellipsoid  $\mathcal{E}_P = \mathcal{E}_P(x, r)$  the following inequality holds

$$\|Mf\|_{L_p(\mathcal{E}_P)} \leq \|M(f\chi_{(2\mathcal{E}_P)})\|_{L_p(\mathcal{E}_P)} + \|M(f\chi_{\mathbb{R}^n \setminus (2\mathcal{E}_P)})\|_{L_p(\mathcal{E}_P)}.$$

By continuity of the operator  $M : L_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)$ ,  $1 < p < \infty$  we have

$$\|M(f\chi_{(2\mathcal{E}_P)})\|_{L_p(\mathcal{E}_P)} \lesssim \|f\|_{L_p(2\mathcal{E}_P)}.$$

Let  $y$  be an arbitrary point from  $\mathcal{E}_P$ . If  $\mathcal{E}_P(y, t) \cap \{\mathbb{R}^n \setminus (2\mathcal{E}_P)\} \neq \emptyset$ , then  $t > r$ . Indeed, if  $z \in \mathcal{E}_P(y, t) \cap \{\mathbb{R}^n \setminus (2\mathcal{E}_P)\}$ , then  $t > \rho(z, y) \geq \rho(z, x) - \rho(x, y) > 2r - r = r$ .

On the other hand  $\mathcal{E}_P(y, t) \cap \{\mathbb{R}^n \setminus (2\mathcal{E}_P)\} \subset \mathcal{E}_P(x, 2t)$ . Indeed,  $z \in \mathcal{E}_P(y, t) \cap \{\mathbb{R}^n \setminus (2\mathcal{E}_P)\}$ , then we get  $\rho(z, x) \leq \rho(z, y) + \rho(y, x) < t + r < 2t$ . Hence

$$\begin{aligned} M(f\chi_{\mathbb{R}^n \setminus (2\mathcal{E}_P)})(y) &= \sup_{t>0} \frac{1}{|\mathcal{E}_P(y, t)|} \int_{\mathcal{E}_P(y, t) \cap \{\mathbb{R}^n \setminus (2\mathcal{E}_P)\}} |f(y)| dy \\ &\leq 2^\gamma \sup_{t>r} \frac{1}{|\mathcal{E}_P(x, 2t)|} \int_{\mathcal{E}_P(x, 2t)} |f(y)| dy \\ &= 2^\gamma \sup_{t>2r} \frac{1}{|\mathcal{E}_P(x, t)|} \int_{\mathcal{E}_P(x, t)} |f(y)| dy. \end{aligned}$$

Therefore, for all  $y \in \mathcal{E}_P$  we have

$$M(f\chi_{\mathbb{R}^n \setminus (2\mathcal{E}_P)})(y) \leq 2^\gamma \sup_{t>2r} \frac{1}{|\mathcal{E}_P(x, t)|} \int_{\mathcal{E}_P(x, t)} |f(y)| dy. \quad (12)$$

Thus

$$\|Mf\|_{L_p(\mathcal{E}_P)} \lesssim \|f\|_{L_p(2\mathcal{E}_P)} + |\mathcal{E}_P|^{\frac{1}{p}} \left( \sup_{t>2r} \frac{1}{|\mathcal{E}_P(x,t)|} \int_{\mathcal{E}_P(x,t)} f(y) dy \right)$$

Let  $p = 1$ . It is obvious that for any ellipsoid  $\mathcal{E}_P = \mathcal{E}_P(x, r)$  the following inequality holds

$$\|Mf\|_{WL_1(\mathcal{E}_P)} \leq \|M(f\chi_{(2\mathcal{E}_P)})\|_{WL_1(\mathcal{E}_P)} + \|M(f\chi_{\mathbb{R}^n \setminus (2\mathcal{E}_P)})\|_{WL_1(\mathcal{E}_P)}.$$

By continuity of the operator  $M : L_1(\mathbb{R}^n) \rightarrow WL_1(\mathbb{R}^n)$  we have

$$\|M(f\chi_{(2\mathcal{E}_P)})\|_{WL_1(\mathcal{E}_P)} \lesssim \|f\|_{L_1(2\mathcal{E}_P)}.$$

Then by (12), we get the inequality (11).

**Lemma 3.2.** *Let  $1 < p < \infty$ . Then for any ellipsoid  $\mathcal{E}_P = \mathcal{E}_P(x, r)$  in  $\mathbb{R}^n$ , the inequality*

$$\|Mf\|_{L_p(\mathcal{E}_P(x,r))} \lesssim r^{\frac{\gamma}{p}} \sup_{t>2r} t^{-\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}_P(x,t))} \quad (13)$$

holds for all  $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ .

Moreover, the inequality

$$\|Mf\|_{WL_1(\mathcal{E}_P(x,r))} \lesssim r^{\gamma} \sup_{t>2r} t^{-\gamma} \|f\|_{L_1(\mathcal{E}_P(x,t))} \quad (14)$$

holds for all  $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ .

**Proof.** Let  $1 < p < \infty$ . Denote by

$$\begin{aligned} \mathcal{M}_1 &:= |\mathcal{E}_P|^{\frac{1}{p}} \left( \sup_{t>2r} \frac{1}{|\mathcal{E}_P(x,t)|} \int_{\mathcal{E}_P(x,t)} |f(y)| dy \right), \\ \mathcal{M}_2 &:= \|f\|_{L_p(2\mathcal{E}_P)}. \end{aligned}$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1 \lesssim |\mathcal{E}_P|^{\frac{1}{p}} \left( \sup_{t>2r} \frac{1}{|\mathcal{E}_P(x,t)|^{\frac{1}{p}}} \left( \int_{\mathcal{E}_P(x,t)} |f(y)|^p dy \right)^{\frac{1}{p}} \right).$$

On the other hand,

$$\begin{aligned} &|\mathcal{E}_P|^{\frac{1}{p}} \left( \sup_{t>2r} \frac{1}{|\mathcal{E}_P(x,t)|^{\frac{1}{p}}} \left( \int_{\mathcal{E}_P(x,t)} |f(y)|^p dy \right)^{\frac{1}{p}} \right) \\ &\gtrsim |\mathcal{E}_P|^{\frac{1}{p}} \left( \sup_{t>2r} \frac{1}{|\mathcal{E}_P(x,t)|^{\frac{1}{p}}} \right) \|f\|_{L_p(2\mathcal{E}_P)} \approx \mathcal{M}_2. \end{aligned}$$

Since by Lemma 3.1

$$\|Mf\|_{L_p(\mathcal{E}_P)} \leq \mathcal{M}_1 + \mathcal{M}_2,$$

we arrive at (13).

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Let  $p = 1$ . The inequality (14) directly follows from (11).

**Theorem 3.3.** *Let  $p \in [1, \infty)$  and  $(\varphi_1, \varphi_2) \in \mathcal{Z}_{0,\gamma}(\mathbb{R}^n)$ . Then for  $p > 1$  the operator  $M$  is bounded from  $\mathcal{M}_{p,\varphi_1}$  to  $\mathcal{M}_{p,\varphi_2}$ , and for  $p = 1$  the operator  $M$  is bounded from  $\mathcal{M}_{1,\varphi_1}$  to  $W\mathcal{M}_{1,\varphi_2}$ .*

**Proof.** By Lemma 3.2 and Theorem 2.7 we get

$$\begin{aligned} \|Mf\|_{\mathcal{M}_{p,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-\frac{1}{p}} r^{\frac{\gamma}{p}} \left( \sup_{t > r} t^{-\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}_P(x,t))} \right) \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x, r)^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}_P(x,t))} = \|f\|_{\mathcal{M}_{p,\varphi_1}(\mathbb{R}^n)}, \end{aligned}$$

if  $p \in (1, \infty)$  and

$$\begin{aligned} \|Mf\|_{W\mathcal{M}_{1,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} r^\gamma \left( \sup_{t > r} t^{-\gamma} \|f\|_{L_1(\mathcal{E}_P(x,t))} \right) \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x, r)^{-1} \|f\|_{L_1(\mathcal{E}_P(x,t))} = \|f\|_{\mathcal{M}_{1,\varphi_1}(\mathbb{R}^n)}, \end{aligned}$$

if  $p = 1$ .

#### 4. The anisotropic singular operator in generalized Morrey spaces

The following Lemma has been proved in [14] (see also [5, 15]). For the sake of completeness we give the proof.

**Lemma 4.1.** *Let  $p \in [1, \infty)$ ,  $f \in L_p^{\text{loc}}(\mathbb{R}^n)$  and for any  $x_0 \in \mathbb{R}^n$*

$$\int_1^\infty t^{-\frac{\gamma}{p}+1} \|f\|_{L_p(\mathcal{E}_P(x_0,t))} dt < \infty.$$

*Then  $Tf$  exists for a.e.  $x \in \mathbb{R}^n$  and for any  $x_0 \in \mathbb{R}^n$ ,  $r > 0$  and  $p \in (1, \infty)$*

$$\|Tf\|_{L_p(\mathcal{E}_P(x_0,r))} \leq C r^{\frac{\gamma}{p}} \int_{2r}^\infty t^{-\frac{\gamma}{p}+1} \|f\|_{L_p(\mathcal{E}_P(x_0,t))} dt, \quad (15)$$

*where constant  $C > 0$  does not depend on  $x_0$ ,  $r$  and  $f$ .*

*Moreover, for any  $x_0 \in \mathbb{R}^n$  and  $r > 0$*

$$\|Tf\|_{WL_1(\mathcal{E}_P(x_0,r))} \leq C r^\gamma \int_{2r}^\infty t^{-\gamma+1} \|f\|_{L_1(\mathcal{E}_P(x_0,t))} dt, \quad (16)$$

*where constant  $C > 0$  does not depend on  $x_0$ ,  $r$  and  $f$ .*

**Proof.** Let  $p \in (1, \infty)$ . For arbitrary  $x_0 \in \mathbb{R}^n$ , set  $\mathcal{E}_P = \mathcal{E}_P(x_0, r)$  for the ellipsoid centered at  $x_0$  and of radius  $r$ . Write  $f = f_1 + f_2$  with  $f_1 = f\chi_{2\mathcal{E}_P}$  and  $f_2 = f\chi_{\mathbb{R}^n \setminus (2\mathcal{E}_P)}$ . Since  $f_1 \in L_p(\mathbb{R}^n)$ ,  $Tf_1(x)$  exists for a.e.  $x \in \mathbb{R}^n$  and from the boundedness of  $T$  in  $L_p(\mathbb{R}^n)$  (see [12]) it follows that

$$\|Tf_1\|_{L_p(\mathcal{E}_P)} \leq \|Tf_1\|_{L_p(\mathbb{R}^n)} \leq C \|f_1\|_{L_p(\mathbb{R}^n)} = C \|f\|_{L_p(2\mathcal{E}_P)},$$

where the constant  $C > 0$  is independent of  $f$ .

Now we prove that the non-singular integral  $Tf_2(x)$  exists for all  $x \in \mathcal{E}_P$ .



It is clear that  $x \in \mathcal{E}_P$ ,  $y \in \mathbb{R}^n \setminus (2\mathcal{E}_P)$  implies  $\rho(x - y) \sim \rho(x_0 - y)$  and we get

$$|Tf_2(x)| \lesssim \int_{\mathbb{R}^n \setminus (2\mathcal{E}_P)} \frac{|f(y)|}{\rho(x_0 - y)^\gamma} dy.$$

By Fubini's theorem we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus (2\mathcal{E}_P)} \frac{|f(y)|}{\rho(x_0 - y)^\gamma} dy &\approx \int_{\mathbb{R}^n \setminus (2\mathcal{E}_P)} |f(y)| \int_{\rho(x_0 - y)}^\infty \frac{dt}{t^{\gamma+1}} dy \\ &\approx \int_{2r}^\infty \int_{2r \leq \rho(x_0 - y) < t} |f(y)| dy \frac{dt}{t^{\gamma+1}} \\ &\lesssim \int_{2r}^\infty \int_{\mathcal{E}_P(x_0, t)} |f(y)| dy \frac{dt}{t^{\gamma+1}}. \end{aligned}$$

Applying Hölder's inequality, we get

$$\int_{\mathbb{R}^n \setminus (2\mathcal{E}_P)} \frac{|f(y)|}{\rho(x_0 - y)^\gamma} dy \lesssim \int_{2r}^\infty \|f\|_{L_p(\mathcal{E}_P(x_0, t))} \frac{dt}{t^{\frac{\gamma}{p}+1}}.$$

Therefore  $Tf_2(x)$  exists for all  $x \in \mathcal{E}_P$ . Since  $\mathbb{R}^n = \bigcup_{r>0} \mathcal{E}_P(x_0, r)$ , we get existence of  $Tf(x)$  for a.e.  $x_0 \in \mathbb{R}^n$ .

Moreover, for all  $p \in [1, \infty)$  the inequality

$$\|Tf_2\|_{L_p(\mathcal{E}_P)} \lesssim r^{\frac{\gamma}{p}} \int_{2r}^\infty \|f\|_{L_p(\mathcal{E}_P(x_0, t))} \frac{dt}{t^{\frac{\gamma}{p}+1}}. \quad (17)$$

is valid. Thus

$$\|Tf\|_{L_p(\mathcal{E}_P)} \lesssim \|f\|_{L_p(2\mathcal{E}_P)} + r^{\frac{\gamma}{p}} \int_{2r}^\infty \|f\|_{L_p(\mathcal{E}_P(x_0, t))} \frac{dt}{t^{\frac{\gamma}{p}+1}}.$$

On the other hand,

$$\begin{aligned} \|f\|_{L_p(2\mathcal{E}_P)} &\approx r^{\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}_P)} \int_{2r}^\infty \frac{dt}{t^{\frac{\gamma}{p}+1}} \\ &\lesssim r^{\frac{\gamma}{p}} \int_{2r}^\infty \|f\|_{L_p(\mathcal{E}_P(x_0, t))} \frac{dt}{t^{\frac{\gamma}{p}+1}}. \end{aligned}$$

Thus

$$\|Tf\|_{L_p(\mathcal{E}_P)} \lesssim r^{\frac{\gamma}{p}} \int_{2r}^\infty \|f\|_{L_p(\mathcal{E}_P(x_0, t))} \frac{dt}{t^{\frac{\gamma}{p}+1}}.$$

Let  $p = 1$ . From the weak (1,1) boundedness of  $T$  (see [2]) it follows that

$$\|Tf_1\|_{WL_1(\mathcal{E}_P)} \leq \|Tf_1\|_{WL_1(\mathbb{R}^n)} \leq C \|f_1\|_{L_1(\mathbb{R}^n)} = C \|f\|_{L_1(2\mathcal{E}_P)},$$

where the constant  $C > 0$  is independent of  $f$ .

Then by (17) we get the inequality (16).

**Theorem 4.2.** *Let  $p \in [1, \infty)$  and  $(\varphi_1, \varphi_2) \in \mathcal{Z}_{p, \gamma}$ . Then the anisotropic singular integral  $Tf$  exists for a.e.  $x \in \mathbb{R}^n$ ; and for  $p > 1$  the operator  $T$  is bounded from*

$\mathcal{M}_{p,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}_{p,\varphi_2}(\mathbb{R}^n)$ , and for  $p = 1$  the operator  $T$  is bounded from  $\mathcal{M}_{1,\varphi_1}(\mathbb{R}^n)$  to  $W\mathcal{M}_{1,\varphi_2}(\mathbb{R}^n)$ . Moreover, for  $p > 1$

$$\|Tf\|_{\mathcal{M}_{p,\varphi_2}} \lesssim \|f\|_{\mathcal{M}_{p,\varphi_1}},$$

and for  $p = 1$

$$\|Tf\|_{W\mathcal{M}_{1,\varphi_2}} \lesssim \|f\|_{\mathcal{M}_{1,\varphi_1}}.$$

**Proof.** By Lemma 4.1 and Theorem 2.8 we have for  $p > 1$

$$\begin{aligned} \|Tf\|_{\mathcal{M}_{p,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-\frac{1}{p}} r^{\frac{\gamma}{p}} \int_r^\infty \|f\|_{L_p(\mathcal{E}_P(x,t))} \frac{dt}{t^{\frac{\gamma}{p}+1}} \\ &\approx \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-\frac{1}{p}} r^{\frac{\gamma}{p}} \int_0^{r^{-\frac{\gamma}{p}}} \|f\|_{L_p(\mathcal{E}_P(x,t^{-\frac{p}{\gamma}}))} dt \\ &= \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r^{-\frac{p}{\gamma}})^{-\frac{1}{p}} \frac{1}{r} \int_0^r \|f\|_{L_p(\mathcal{E}_P(x,t^{-\frac{p}{\gamma}}))} dt \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x, r^{-\frac{p}{\gamma}})^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}_P(x,r^{-\frac{p}{\gamma}}))} \\ &= \|f\|_{\mathcal{M}_{p,\varphi_1}(\mathbb{R}^n)} \end{aligned}$$

and for  $p = 1$

$$\begin{aligned} \|Tf\|_{W\mathcal{M}_{1,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} r^\gamma \int_r^\infty \|f\|_{L_1(\mathcal{E}_P(x,t))} \frac{dt}{t^{\gamma+1}} \\ &\approx \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} r^\gamma \int_0^{r^{-\gamma}} \|f\|_{L_1(\mathcal{E}_P(x,r^{-\frac{1}{\gamma}}))} dt \\ &= \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r^{-\frac{1}{\gamma}})^{-1} \frac{1}{r} \int_0^r \|f\|_{L_1(\mathcal{E}_P(x,t^{-\frac{1}{\gamma}}))} dt \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x, r^{-\frac{1}{\gamma}})^{-1} \|f\|_{L_1(\mathcal{E}_P(x,r^{-\frac{1}{\gamma}}))} \\ &= \|f\|_{\mathcal{M}_{1,\varphi_1}(\mathbb{R}^n)}. \end{aligned}$$

**Corollary 4.3.** [14] Let  $p \in [1, \infty)$  and  $(\varphi_1, \varphi_2) \in \tilde{\mathcal{Z}}_{p,\gamma}(\mathbb{R}^n)$ . Then for  $p > 1$  the operator  $T$  is bounded from  $\mathcal{M}_{p,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}_{p,\varphi_2}(\mathbb{R}^n)$  and for  $p = 1$  the operator  $T$  is bounded from  $\mathcal{M}_{1,\varphi_1}$  to  $W\mathcal{M}_{1,\varphi_2}$ .

Note that Theorem 2.6 and Corollary 4.3 coincide.

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