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ON THE DIRICHLET PROBLEM FOR GILBARG-SERRIN EQUATION

Abstract

In the paper we establish the unique solvability of the first boundary value problem in weight Sobolev space for Gilbarg-Serrin equation whose coefficients don't satisfy the Cordes condition. The counter-examples showing the accuracy of the Cordes condition are given.

1^0. Introduction. Let $\mathbf{E}_n$ be an Euclidean space of points $x=(x_1,...,x_n)$, $n\geq 3$, D be a bounded domain in $\mathbf{E}_n$ with the boundary ∂D. Consider in D the first boundary-value problem

$$\mathcal{M}u=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}=f(x);\ x\in D, \tag{1}$$

$$u\big|_{\partial D}=0 \tag{2}$$

under assumption that $f(x)\in L_2(D)$, $\|a_{ij}(x)\|$ is a real symmetric matrix with measurable in D elements, where for all $x\in D$, $\xi\in\mathbf{E}_n$

$$\alpha|\xi|^2\leq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_i\leq\alpha^{-1}|\xi|^2,$$

where $\alpha\in(0,1]$ is a constant. It's well-known [1-2] that if $\partial D\in C^2$ and

$$\sigma=ess\sup_D\frac{\sum_{i,j=1}^{n}a_{ij}^2(x)}{\left[\sum_{i=1}^{n}a_{ii}(x)\right]^2}<\frac{1}{n-1}, \tag{3}$$

then problem (1)-(2) is uniquely strongly (almost everywhere) solvable in the space $W_2^2(D)$. Condition (3) is called a Cordes condition A counter-example, showing the accuracy of Cordes condition is given in [3]. Let now $0\in D$, and

$$\mathcal{M}=\mathcal{L}=\Delta+\mu\sum_{i,j=1}^{n}\frac{x_ix_j}{|x|^2}\frac{\partial^2}{\partial x_i\partial x_j}$$

be the Gilbarg-Serrin operator [4-5]. Here Δ is a Laplace operator, $\mu>-1$ is a constant. It's easy to see that for $\mu<\frac{n}{n-2}$ condition (3) fulfilled.

According to [6], if $\mu\geq\frac{n}{n-2}$ at $n>4$ and $\mu>\frac{n}{n-2}$ at $n\geq 3$ the unique strong solvability of the Dirichlet problem (1)-(2) holds in weight Sobolev spaces, where a weight function is some negative power of $|x|$ (see also [7]). The aim of the present paper is investigation of the question on unique strong solvability of the Dirichlet problem of Gilbarg-Serrin equation at $\mu=\frac{n}{n-2}$ (i.e. $\sigma=\frac{1}{n-1}$) for $n=3,4$.

We accept now some denotations and definitions. By u_i and u_{ij} we'll denote the derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ respectively, $i, j = 1,...,n$. Let $n = 3$, γ be some number. Denote by $W^1_{2,\gamma}(D)$ and $W^2_{2,\gamma}(D)$ the closures of $A_\gamma(D)$ by norms

$$\|u\|_{W^1_2(D)} = \left(\int_D \left(r^{\gamma-4} u^2 + r^{\gamma-2} \sum_{i=1}^{3} u_i^2 \right) dx \right)^{\frac{1}{2}}$$

and

$$\|u\|_{W^2_{2,\gamma}(D)} = \left(\int_D \left(r^{\gamma-4} u^2 + r^{\gamma-2} \sum_{i=1}^{3} u_i^2 + r^{\gamma} \sum_{i,j=1}^{3} u_{ij}^2 \right) dx \right)^{\frac{1}{2}}$$

respectively. Here $A_\gamma(D)$ is the totality of all functions $u(x)$ from $C^\infty(\overline{D})$, for which the norm $\|u\|_{W^1_{2,\gamma}(D)}$ is finite, and $r = |x|$. Denote by $\overset{\circ}{W}{}^1_{2,\gamma}(D)$ the completion of the set $C_0^\infty(D) \cap A_\gamma(D)$ by the norm of the space $W^1_{2,\gamma}(D)$, and let $\dot{W}^2_{2,\gamma}(D) = W^2_{2,\gamma}(D) \cap \overset{\circ}{W}{}^2_{2,\gamma}(D)$, and $L_{2,\gamma}(D)$ be a Banach space of functions $u(x)$ given on D with the finite norm

$$\|u\|_{L_{2,\gamma}(D)} = \left(\int_D r^\gamma u^2 dx \right)^{\frac{1}{2}}.$$

For $n = 4$ denote by $L_{2,\ln^2 r}(D)$, $W^1_{2,\ln^2 r}(D)$ and $W^2_{2,\ln^2 r}(D)$ the closures of $C^\infty(\overline{D})$ by norms

$$\|u\|_{L_2,\ln^2 r(D)} = \left(\int_D \ln^2 \frac{2d}{r} \cdot u^2 dx \right)^{\frac{1}{2}},$$

$$\|u\|_{W^1_2,\ln^2 r(D)} = \left(\int_D \left(\frac{r^{-4}}{\ln^2 \frac{2d}{r}} u^2 + r^{-2} \sum_{i=1}^{4} u_i^2 \right) dx \right)^{\frac{1}{2}}$$

and

$$\|u\|_{W^2_2,\ln^2 r(D)} = \left(\int_D \left(\frac{r^{-4}}{\ln^2 \frac{2d}{r}} u^2 + r^{-2} \sum_{i=1}^{4} u_i^2 + \ln^2 \frac{2d}{r} \sum_{i,j=1}^{4} u_{ij}^2 \right) dx \right)^{\frac{1}{2}}$$

respectively, and let $\overset{\circ}{W}{}^1_{2,\ln^2 r}(D)$ be a subspace of $W^1_{2,\ln^2 r}(D)$ in which the dense set is $C_0^\infty(D)$, $\dot{W}^2_{2,\ln^2 r}(D) = W^2_{2,\ln^2 r}(D) \cap \overset{\circ}{W}{}^1_{2,\ln^2 r}(D)$. Here $d = \sup_{x \in D} |x|$.

Further not specifying this we'll assume that the constant μ of the operator $\mathcal{L}$ is equal to $\frac{n}{n-2}$. The function $u(x) \in \dot{W}^2_{2,\gamma}(D)$ for $n = 3$ or $u(x) \in \dot{W}^2_{2,\ln^2 r}(D)$ for $n = 4$ is

called the strong solution of the first boundary value problem (1)-(2) (at $\mathscr{M}=\mathscr{L}$), if it satisfies equation (1) almost everywhere in D. At that $f(x)\in L_{2,\gamma}(D)$ for $n=3$ and $f(x)\in L_{2,\ln^2 r}(D)$ for $n=4$.

For $x^0\in\mathbf{E}_n$, $R>0$ denote by $B_R(x^0)$ the ball $\{x:|x-x^0|<R\}$, $B_R(0)=B_R$, $|\nabla u|^2=\sum_{i=1}^n u_i^2$, $|\nabla_1 u|^2=\left(\sum_{i=1}^n \frac{x_i}{r}u_i\right)^2$. Everywhere below the notation $C(...)$ means that the positive constant C depends only on contents of brackets.

2^0. Case $n=3$. Let R be such that $\overline{B}_R\subset D$.

Lemma 1. *If $\gamma\in(0,1)$ then for any function $u(x)\in C_0^\infty(B_R)\cap A_\gamma(B_R)$ the following estimation is valid*

$$\int_{B_R} r^{\gamma-2}|\nabla u|^2\,dx\le -C_1(\gamma)\int_{B_R} r^{\gamma-2}u\,\mathscr{L}u\,dx\,. \tag{4}$$

Proof. We have

$$J=-\int_{B_R} r^{\gamma-2}u\mathscr{L}u dx=-\int_{B_R} r^{\gamma-2}u\Delta u dx-3\int_{B_R} r^{\gamma-2}u\sum_{i,j=1}^3\left(\frac{x_ix_j}{r^2}u_j\right)_i dx+$$

$$+6\int_{B_R} r^{\gamma-4}u\sum_{i,j=1}^3 x_iu_i dx=i_1+i_2+i_3\,. \tag{5}$$

Integrating by parts we obtain

$$i_1=\sum_{i=1}^3\int_{B_R}\left(r^{\gamma-2}u\right)_i u_i dx=\int_{B_R} r^{\gamma-2}|\nabla u|^2dx+(\gamma-2)\sum_{i=1}^3\int_{B_R} r^{\gamma-4}x_iuu_i dx=\int_{B_R} r^{\gamma-2}|\nabla u|^2dx-$$

$$-\frac{\gamma-2}{2}\sum_{i=1}^3\int_{B_R}\left(r^{\gamma-4}x_i\right)_i u^2dx=\int_{B_R} r^{\gamma-2}|\nabla u|^2dx-\frac{\gamma-2}{2}(\gamma-1)\int_{B_R} r^{\gamma-4}u^2dx. \tag{6}$$

Analogously we derive

$$i_2=3\sum_{i,j=1}^3\left(r^{\gamma-2}u\right)_i\frac{x_ix_j}{r^2}u_jdx=3\sum_{i,j=1}^3\int_{B_R} r^{\gamma-2}\frac{x_ix_j}{r^2}u_iu_jdx+$$

$$+3(\gamma-2)\sum_{i,j=1}^3\int_{B_R} r^{\gamma-4}u\frac{x_i^2x_j}{r^2}u_jdx=3\int_{B_R} r^{\gamma-2}|\nabla_1 u|^2dx-$$

$$-3\frac{\gamma-2}{2}\sum_{i,j=1}^3\int_{B_R}\left(r^{\gamma-6}x_i^2x_j\right)_j u^2dx=3\int_{B_R} r^{\gamma-2}|\nabla_1 u|^2dx- \tag{7}$$

$$-3\frac{\gamma-2}{2}\sum_{i,j=1}^3\int_{B_R} r^{\gamma-6}u^2\left(x_i^2+2\delta_{ij}x_j^2+(\gamma-6)\frac{x_i^2x_j^2}{r^2}\right)dx=$$

$$=3\int_{B_R} r^{\gamma-2}|\nabla_1 u|^2dx-3\frac{\gamma-2}{2}(\gamma-1)\int_{B_R} r^{\gamma-4}u^2dx,$$

where δ_{ij} is the Cronecker's symbol. Further we obtain

$$i_3=-3\sum_{i=1}^3\int_{B_R}\left(r^{\gamma-4}x_i\right)_i u^2dx=-3(\gamma-1)\int_{B_R} r^{\gamma-4}u^2dx. \tag{8}$$

Taking into account (6)-(8) in (5) we conclude

$$J = \int_{B_R} r^{\gamma-2}\left|\nabla u\right|^2 dx + 3\int_{B_R} r^{\gamma-2}\left|\nabla_1 u\right|^2 dx - (1-\gamma)(1-2\gamma)\int_{B_R} r^{\gamma-4}u^2 dx. \qquad (9)$$

From here we can see that if $\gamma \in \left[\frac{1}{2},1\right)$, then the required estimation (4) is proved. Let now $\gamma \in \left(0,\frac{1}{2}\right)$. For any $\varepsilon > 0$ we have

$$\varepsilon \int_{B_R} r^{\gamma-4}u^2 dx + \frac{1}{\varepsilon}\int_{B_R} r^{\gamma-2}\left|\nabla_1 u\right|^2 dx \geq 2\sum_{i=1}^{3}\int_{B_R} r^{\gamma-4}x_i u u_i dx =$$

$$= -\sum_{i=1}^{3}\int_{B_R}\left(r^{\gamma-4}x_i\right)_i u^2 dx = (1-\gamma)\int_{B_R} r^{\gamma-4}u^2 dx.$$

Choosing $\varepsilon = \frac{1-\gamma}{2}$ we obtain

$$\int_{B_R} r^{\gamma-4}u^2 dx \leq \frac{4}{(1-\gamma)^2}\int_{B_R} r^{\gamma-2}\left|\nabla_1 u\right|^2 dx. \qquad (10)$$

From (9), (10) and the inequality $\left|\nabla_1 u\right|^2 \leq \left|\nabla u\right|^2$ it follows that

$$J \geq \int_{B_R} r^{\gamma-2}\left|\nabla u\right|^2 dx + \left(\frac{4\gamma}{1-\gamma}-1\right)\int_{B_R} r^{\gamma-2}\left|\nabla_1 u\right|^2 dx \geq \frac{4\gamma}{1-\gamma}\int_{B_R} r^{\gamma-2}\left|\nabla u\right|^2 dx \text{ , if } \frac{4\gamma}{1-\gamma} < 1$$

$$\text{and } J \geq \int_{B_R} r^{\gamma-2}\left|\nabla u\right|^2 dx, \text{ if } \frac{4\gamma}{1-\gamma} \geq 1,$$

and the lemma is proved.

Corollary. *If $\gamma \in (0,1)$, then for any function $u(x) \in C_0^\infty(B_R) \cap A_\gamma(B_R)$ the following estimation is valid*

$$\int_{B_R}\left(r^{\gamma-4}u^2 + r^{\gamma-2}\left|\nabla u\right|^2\right)dx \leq C_2(\gamma)\int_{B_R} r^\gamma (\mathcal{L}u)^2 dx.$$

In fact, from (4) and (10) for any $\varepsilon_1 > 0$ we conclude

$$\int_{B_R} r^{\gamma-2}\left|\nabla u\right|^2 dx \leq \frac{C_1\varepsilon_1}{2}\int_{B_R} r^{\gamma-4}u^2 dx + \frac{C_1}{2\varepsilon_1}\int_{B_R} r^\gamma (\mathcal{L}u)^2 dx \leq$$

$$\leq \frac{2C_1\varepsilon_1}{(1-\gamma)^2}\int_{B_R} r^{\gamma-2}\left|\nabla u\right|^2 dx + \frac{C_1}{2\varepsilon_1}\int_{B_R} r^\gamma (\mathcal{L}u)^2 dx.$$

Choosing now $\varepsilon_1 = \frac{(1-\gamma)^2}{4C_1}$ we obtain

$$\int_{B_R} r^{\gamma-2}\left|\nabla u\right|^2 dx \leq \frac{4C_1^2}{(1-\gamma)^2}\int_{B_R} r^\gamma (\mathcal{L}u)^2 dx.$$

Again taking into account estimation (10) we arrive at the statement of the corollary.

Lemma 2. *If $\gamma \in (0,1)$, then for any function $u(x) \in C_0^\infty(B_R) \cap A_\gamma(B_R)$ the following estimation is valid*

$$\int\limits_{B_R} r^{\gamma} \sum_{i,j=1}^{3} u_{ij}^2 dx \le C_3(\gamma) \int\limits_{B_R} r^{\gamma} (\mathcal{L}u)^2 dx. \tag{11}$$

Proof. We fix arbitrary natural k, $1 \le k \le n$ and consider the function ru_k. It's clear that this function has a compact support in B_R, and for it the norm $\|ru_k\|_{W^1_{2,\gamma}(B_R)}$ is finite. Therefore, lemma 1 is applicable to it, according to which

$$\int\limits_{B_R} r^{\gamma-2} \sum_{i=1}^{3} (ru_k)_i^2 dx \le -C_1 \int\limits_{B_R} r^{\gamma-1} u_k \mathcal{L}(ru_k) dx. \tag{12}$$

On the other hand,

$$\int\limits_{B_R} r^{\gamma-2} \sum_{i=1}^{3} (ru_k)_i^2 dx = -\gamma \int\limits_{B_R} r^{\gamma-2} u_k^2 dx + \int\limits_{B_R} r^{\gamma} \sum_{i=1}^{3} u_{ki}^2 dx. \tag{13}$$

Taking into account that

$$\mathcal{L}(ru_k) = r\mathcal{L}u_k + u_k \mathcal{L}r + 2 \sum_{i,j=1}^{3} \left(\delta_{ij} + 3\frac{x_i x_j}{r^2} \right) r_i u_{kj}, \quad \mathcal{L}r = \frac{2}{r},$$

we obtain

$$-C_1 \int\limits_{B_R} r^{\gamma-1} u_k \mathcal{L}(ru_k) dx = -C_1 \int\limits_{B_R} r^{\gamma} u_k \mathcal{L}u_k dx - 2C_1 \int\limits_{B_R} r^{\gamma-2} u_k^2 dx -$$

$$-2C_1 \sum_{i=1}^{3} \int\limits_{B_R} r^{\gamma-1} x_i u_k u_{ki} dx - 6C_1 \sum_{i,j=1}^{3} \int\limits_{B_R} r^{\gamma-4} x_i^2 x_j u_k u_{kj} dx = j_1 + j_2 + j_3 + j_4. \tag{14}$$

Further we have

$$j_1 = -C_1 \int\limits_{B_R} r^{\gamma} u_k (\mathcal{L}u)_k dx - C_1 \int\limits_{B_R} r^{\gamma} u_k \left[\mathcal{L}u_k - (\mathcal{L}u)_k \right] dx = j_{11} + j_{12}.$$

But for any $\varepsilon_2 > 0$

$$j_{11} = C_1 \int\limits_{B_R} \left(r^{\gamma} u_k \right)_k \mathcal{L}u dx = C_1 \int\limits_{B_R} r^{\gamma} u_{kk} \mathcal{L}u dx + C_1 \gamma \int\limits_{B_R} r^{\gamma-2} x_k u_k \mathcal{L}u dx \le$$

$$\le \frac{C_1 \varepsilon_2}{2} \int\limits_{B_R} r^{\gamma} u_{kk}^2 dx + \left(\frac{C_1}{2\varepsilon_2} + \frac{C_1 \gamma}{2} \right) \int\limits_{B_R} r^{\gamma} (\mathcal{L}u)^2 dx + \frac{C_1 \gamma}{2} \int\limits_{B_R} r^{\gamma-2} u_k^2 dx. \tag{15}$$

Besides, for any $\varepsilon_3 > 0$

$$j_{12} = 3C_1 \sum_{i,j=1}^{3} \int\limits_{B_R} r^{\gamma} u_k \left(\frac{x_i x_j}{r^2} \right)_k u_{ij} dx = 3C_1 \sum_{i,j=1}^{3} \int\limits_{B_R} r^{\gamma} u_k \left[\frac{\delta_{ik} x_j}{r^2} + \frac{\delta_{jk} x_i}{r^2} - 2\frac{x_i x_j x_k}{r^4} \right] u_{ij} dx =$$

$$= -3C_1(\gamma+1) \int\limits_{B_R} r^{\gamma-2} u_k^2 dx - 6C_1 \sum_{i,j=1}^{3} \int\limits_{B_R} r^{\gamma-4} u_k x_i x_j x_k u_{ij} dx \le 3C_1 \varepsilon_3 \sum_{i,j=1}^{3} \int\limits_{B_R} r^{\gamma} u_{ij}^2 dx +$$

$$+ \left(\frac{3C_1}{\varepsilon_2} - 3C_1(\gamma+1) \right) \int\limits_{B_R} r^{\gamma-2} u_k^2 dx. \tag{16}$$

Analogously we obtain

$$j_3 = C_1(\gamma+1) \int\limits_{B_R} r^{\gamma-2} u_k^2 dx; \quad j_4 = 3C_1(\gamma+1) \int\limits_{B_R} r^{\gamma-2} u_k^2 dx. \tag{17}$$

Taking into account (15)-(17) we conclude

$$-C_1 \int\limits_{B_R} r^{\gamma-1} u_k \mathcal{L}(ru_k) dx \le \frac{C_1 \varepsilon_2}{2} \int\limits_{B_R} r^{\gamma} u_{kk}^2 dx + 3C_1 \varepsilon_3 \sum_{i,j=1}^{3} \int\limits_{B_R} r^{\gamma} u_{ij}^2 dx +$$

$$+ C_4(\gamma,\varepsilon_2)\int_{B_R} r^\gamma (\mathcal{L}u)^2\,dx + C_5(\gamma,\varepsilon_3)\int_{B_R} r^{\gamma-2}u_k^2\,dx\,. \tag{18}$$

Using now (18) and (13) in (12) and summing over k from 1 to 3 we obtain the estimation

$$\int_{B_R} r^\gamma \sum_{i,j=1}^{3} u_{ij}^2\,dx \le \frac{C_1\varepsilon_2}{2}\int_{B_R} r^\gamma \sum_{i,j=1}^{3} u_{ij}^2\,dx + 9C_1\varepsilon_3\int_{B_R} r^\gamma u_{ij}^2\,dx +$$

$$+ 3C_4\int_{B_R} r^\gamma (\mathcal{L}u)^2\,dx + C_5\int_{B_R} r^{\gamma-2}|\nabla u|^2\,dx\,.$$

Fixing $\varepsilon_2 = \dfrac{1}{2C_1}$ and $\varepsilon_3 = \dfrac{1}{36C_1}$ we obtain

$$\int_{B_R} r^\gamma \sum_{i,j=1}^{3} u_{ij}^2\,dx \le 6C_4\int_{B_R} r^\gamma (\mathcal{L}u)^2\,dx + 2C_5\int_{B_R} r^{\gamma-2}|\nabla u|^2\,dx\,.$$

Hence, subject to the corollary of lemma 1 the required inequality (11) follows.

The lemma is proved.

Corollary. *If* $\gamma\in(0,1)$, *then for any function* $u(x)\in C_0^\infty(B_R)\cap A_\gamma(B_R)$ *the following estimation is valid*

$$\|u\|_{W^2_{2,\gamma}(B_R)} \le C_6(\gamma)\|\mathcal{L}u\|_{L_{2,\gamma}(B_R)}.$$

Lemma 3. *Let* $\gamma\in(0,1)$. *Then for any function* $u(x)\in A_\gamma(B_R)$ *for any* $\delta>0$ *the following estimation holds*

$$\|u\|_{W^2_{2,\gamma}\left(B_{\frac{R}{2}}\right)} \le C_6\|\mathcal{L}u\|_{L_{2,\gamma}(B_R)} + \delta\|u\|_{W^2_{2,\gamma}(B_R)} + C_7(\gamma,\delta,R)\|u\|_{L_{2,\gamma}(B_R)}. \tag{19}$$

Proof. Consider the function $\eta(x)\in C_0^\infty(B_R)$ such that $\eta(x)=1$ at $x\in B_{\frac{R}{2}}$, $0\le\eta(x)\le 1$ and

$$|\eta_i| \le \frac{C_8}{R},\ |\eta_{ij}| \le \frac{C_8}{R^2};\ i,j=1,...,n, \tag{20}$$

where C_8 is an absolute constant. We'll apply the corollary of lemma 2 to the function $u(x)\eta(x)$.

We obtain

$$\|u\|_{W^2_{2,\gamma}\left(B_{\frac{R}{2}}\right)} \le \|u\eta\|_{W^2_{2,\gamma}(B_R)} \le C_6\|\mathcal{L}(u\eta)\|_{L_{2,\gamma}(B_R)}\,. \tag{21}$$

But on the other hand subject to (20)

$$|\mathcal{L}(u\eta)| \le \eta|\mathcal{L}u| + |u||\mathcal{L}\eta| + 2\left|\sum_{i,j=1}^{3}\left(\delta_{ij} + 3\frac{x_i x_j}{r^2}\right)u_i\eta_j\right| \le$$

$$\le |\mathcal{L}u| + C_9(R)\left(|u| + \sum_{i=1}^{n}|u_i|\right).$$

Therefore, from (21) we conclude

$$\|u\|_{W^2_{2,\gamma}\left(B_{\frac{R}{2}}\right)} \le C_6 \|\mathcal{L}(u)\|_{L_{2,\gamma}(B_R)} + C_{10}(\gamma,R)\left(\|u\|_{L_{2,\gamma}(B_R)} + \left(\int\limits_{B_R} r^\gamma \sum_{i=1}^{3} u_i^2 dx \right)^{\frac{1}{2}} \right). \qquad (22)$$

According to the interpolation inequality [8] for any $\delta_1 > 0$

$$\left(\int\limits_{B_R} r^\gamma \sum_{i=1}^{3} u_i^2 dx \right)^{\frac{1}{2}} \le \delta_1 \left(\int\limits_{B_R} r^\gamma \sum_{i,j=1}^{3} u_{ij}^2 dx \right)^{\frac{1}{2}} + C_{11}(\gamma,\delta_1)\|u\|_{L_{2,\gamma}(B_R)}. \qquad (23)$$

We'll fix arbitrary $\delta > 0$ and choose $\delta_1 = \dfrac{\delta}{C_{10}}$ then the required estimation (19) follows from (22)-(23).

The lemma is proved.

Corollary. *Estimation of the form (19) is valid also for balls* $B_{\frac{R}{2}}(x^0)$ *and* $B_R(x^0)$ *respectively, if* $\overline{B}_R(x^0) \subset D$, $0 \notin \overline{B}_R(x^0)$.

Denote for $\rho > 0$ the set $\{x : x \in D, dist(x,\partial D) > \rho\}$ by D_ρ.

$$\|u\|_{W^2_{2,\gamma}(D_\rho)} \le C_{12}(\gamma,\rho,D)\|\mathcal{L}u\|_{L_{2,\gamma}(D)} + \delta\|u\|_{W^2_{2,\gamma}(D)} + C_{13}(\gamma,\rho,D,\delta)\|u\|_{L_{2,\delta}(D)}$$

Lemma 4. *Let* $\gamma \in (0,1)$. *Then for any function* $u(x) \in A_\gamma(D)$ *at any* $\rho > 0$ *and* $\delta > 0$ *the following estimation is valid*

$$\|u\|_{W^2_{2,\gamma}(D_\rho)} \le C_{12}(\gamma,\rho,D)\|\mathcal{L}u\|_{L_{2,\gamma}(D)} + \delta\|u\|_{W^2_{2,\gamma}(D)} + C_{13}(\gamma,\rho,D,\delta)\|u\|_{L_{2,\gamma}(D)}.$$

The statement of the lemma immediately follows from lemma 3 and its corollary.

Lemma 5. *Let* $\gamma \in (0,1)$, $\partial D \in C^2$. *Then for any function* $u(x) \in A_\gamma(D)$, $u|_{\partial D} = 0$ *at any* $\rho > 0$ *and* $\delta > 0$ *the following estimation is valid*

$$\|u\|_{W^2_{2,\gamma}(D\setminus\overline{D}_\rho)} \le C_{14}(\gamma,\rho,D)\|Lu\|_{L_{2,\gamma}(D)} + \delta\|u\|_{W^2_{2,\gamma}(D)} + C_{15}(\gamma,\rho,D,\delta)\|u\|_{L_{2,\gamma}(D)}.$$

In order to prove this it is sufficient to apply a standard technique of local rectification of the boundary ∂D and odd extension of the function $u(x)$, and then use the corollary of lemma 3 (see [8]) .

Theorem 1. *Let* $\gamma \in (0,1)$, $\partial D \in C^2$. *Then for any function* $u(x) \in \overset{\circ}{W}{}^2_{2,\gamma}(D)$ *the following estimation is valid*

$$\|u\|_{W^2_{2,\gamma}(D)} \le C_{16}(\gamma,D)\|\mathcal{L}u\|_{L_{2,\gamma}(D)}. \qquad (24)$$

Proof. It follows from lemmas 4 and 5 that for any function $u(x) \in \dot{W}{}^2_{2,\gamma}(D)$ at any $\delta > 0$ the inequality

$$\|u\|_{W^2_{2,\gamma}(D)} \le C_{17}(\gamma,D)\|\mathcal{L}u\|_{L_{2,\gamma}(D)} + \delta\|u\|_{W^\gamma_{2,\gamma}(D)} + C_{18}(\gamma,D,\delta)\|u\|_{L_{2,\gamma}(D)}$$

holds.

Choosing $\delta = \dfrac{1}{2}$ we obtain

$$\|u\|_{W^2_{2,\gamma}(D)} \le C_{19}(\gamma,D)\left(\|\mathcal{L}u\|_{L_{2,\gamma}(D)} + \|u\|_{L_{2,\gamma}(D)} \right). \qquad (25)$$

But on the other hand, according to the corollary of lemma 1

$$\|u\|_{L_{2,\gamma}(D)} \le C_{20}(D)\|u\|_{L_{2,\gamma-4}(D)} \le C_{21}(\gamma, D)\|\mathcal{L}u\|_{L_{2,\gamma}(D)},$$

and the required estimation (24) follows from (25). The theorem is proved.

Theorem 2. *Let* $\gamma \in (0,1)$, $\partial D \in C^2$. *Then the first boundary value problem (1)-(2) is uniquely strongly solvable in* $\dot{W}^2_{2,\gamma}(D)$ *for any* $f(x) \in L_{2,\gamma}(D)$. *At that for solution* $u(x)$ *the estimation*

$$\|u\|_{W^2_{2,\gamma}(D)} \le C_{16}\|f\|_{L_{2,\gamma}(D)} \tag{26}$$

is valid.

Proof. At first we suppose that $f(x) \in C^\infty(\overline{D})$. Consider for natural m the sequence of Dirichlet problems

$$\mathcal{L}u_m = f(x),\ x \in D_m\ ;\ u_m|_{\partial D_m} = 0, \tag{27}$$

where $D_m = D \setminus \overline{B}_{m^{-1}}$ (without losing generality, we assume that $\overline{B}_1 \subset D$). Since for each m coefficients of the operator $\mathcal{L}$ are infinitely differentiable in $\overline{D}_m$, then classical solution $u_m(x)$ of problems (27) exists. It can be also shown [9] that for $x \in D \setminus \{0\}$ the limit $u(x) = \lim\limits_{x\to\infty} u_m(x)$ exists, at that the function $u(x)$ is a classical solution of the Dirichlet problem

$$\mathcal{L}u = f(x),\ x \in D \setminus \{0\};\ u|_{\partial D} = 0,\ u(0) = 0.$$

We'll prove now that for $|x| < 1$

$$|u(x)| \le C_{22}(f)|x|^{\frac{1}{2}}. \tag{28}$$

To this end at first by direct checking we make sure that the function $\vartheta(x) = |x|^{\frac{1}{2}}$ is a solution of equation $\mathcal{L}u = 0$ in $D \setminus \{0\}$. Consider in $D \setminus \{0\}$ the auxiliary function

$$w(x) = u(x) + C_{23}|x|^2,$$

where the constant $C_{23} > 0$ will be chosen later. Since $f(x) \in C^\infty(\overline{D})$ that there exists such M, that $|f(x)| \le M$ for $x \in D$. Therefore, for $x \in B_1 \setminus \{0\}$

$$\mathcal{L}w = f + 6C_{23} + 6C_{23}\sum_{i=1}^{3}\frac{x_i^2}{r^2} \ge -M + 12C_{23},$$

and if we'll choose $C_{23} = \dfrac{M}{12}$, then $\mathcal{L}w \ge 0$.

Let now $P(x) = w(x) - C_{24}|x|^{\frac{1}{2}}$ for $x \in B_1 \setminus \{0\}$, where the constant $C_{24} > 0$ will be chosen later. According to above mentioned $\mathcal{L}P \ge 0$. Besides this, $P|_{\partial B_1} \le M_1 + C_{23} - C_{24}$, where $M_1 = \sup\limits_D |u|$. Assume $C_{24} = M_1 + C_{23}$. Then $P|_{\partial B_1} \le 0$. By the maximum principle $P(x) \le 0$ for $x \in B_1 \setminus \{0\}$ and, in particular, $u(x) \le C_{24}|x|^{\frac{1}{2}}$. The estimation $u(x) \ge -C_{24}|x|^{\frac{1}{2}}$ for $x \in B_1 \setminus \{0\}$ is obtained quite similarly. Hence it follows that inequality (28) is valid, and

$$\int_D r^{\gamma-4}u^2\,dx < \infty,$$

if $\gamma \in (0,1)$. Analogously it can be shown that $u(x) \in \dot{W}^2_{2,\gamma}(D)$. So, the function $u(x)$ is a strong solution of the Dirichlet problem (1)-(2). The uniqueness of this solution and estimation (26) follow from theorem 1. Let now $f(x) \in L_{2,\gamma}(D)$.

Consider the sequence of functions $f_k(x)$, $k = 1,2,...$ such that $f_k(x) \in C^\infty(\overline{D})$ and $\lim\limits_{k\to\infty} \| f_k - f \|_{L_{2,\gamma}(D)} = 0$.

Let further $\{u^k(x)\}$ be a sequence of strong solutions of Dirichlet problems

$$\mathcal{L}u^k = f^k(x),\ x \in D;\ u^k\big|_{\partial D} = 0. \tag{29}$$

According to the fact proved earlier strong solutions of problems (29) exist. Then by theorem 1

$$\| u^k \|_{W^2_{2,\gamma}(D)} \le C_{16} \| f^k \|_{L_{2,\gamma}(D)} \le C_{25}(\gamma, D, f).$$

So, the sequence $\{u^k(x)\}$ is bounded by the norm of $\overset{\circ}{W}{}^2_{2,\gamma}(D)$. Hence, in particular, it follows that there exist sequence $k_l \to \infty$ for $l \to \infty$ and the function $u(x) \in \dot{W}^2_{2,\gamma}(D)$ such that

$$< \mathcal{L}u^{k_l}, \varphi > \to < \mathcal{L}u, \varphi >,\ l \to \infty$$

for any $\varphi(x) \in C^\infty(\overline{D})$. Here $< g, h > = \int\limits_D r^\gamma gh\, dx$.

But

$$< \mathcal{L}u^{k_l}, \varphi > = < f^{k_l}, \varphi > \to < f, \varphi > \text{ as } l \to \infty.$$

So, $< \mathcal{L}u, \varphi > = < f, \varphi >$, i.e. $\mathcal{L}u = f$ almost everywhere in D. Thus, $u(x)$ is a strong solution of the first boundary value problem (1)-(2), and the theorem is proved.

3⁰. Case $n = 4$**.** Without losing generality we'll assume that $d = \frac{1}{2}$.

Lemma 6. *For any function* $u(x) \in C_0^\infty(B)$ *the following estimation is valid*

$$\int\limits_B r^{-2} |\nabla u|^2 dx \le 4 \int\limits_B \ln^2 r (\mathcal{L}u)^2 dx. \tag{30}$$

Proof. It's easy to see that

$$-\int\limits_B r^{-2} u \mathcal{L}u dx = \int\limits_B r^{-2} |\nabla u|^2 dx + 2\int\limits_B r^{-2} |\nabla_1 u|^2 dx.$$

So, for any $\varepsilon_4 > 0$ and $\varepsilon_5 > 0$

$$\int\limits_B r^{-2} |\nabla u|^2 dx \le \frac{\varepsilon_4}{2} \int\limits_B \frac{r^{-4}}{\ln^2 r} u^2 dx + \frac{1}{2\varepsilon_4} \int\limits_B \ln^2 r (\mathcal{L}u)^2 dx, \tag{31}$$

$$\varepsilon_5 \int\limits_B \frac{r^{-4}}{\ln^2 r} u^2 dx + \frac{1}{\varepsilon_5} \int\limits_B r^{-2} |\nabla_1 u|^2 dx \ge 2 \sum_{i=1}^4 \int\limits_B \frac{r^{-4}}{\ln r} x_i u u_i dx =$$

$$= -\sum_{i=1}^4 \int\limits_B \left(\frac{r^{-4}}{\ln r} x_i \right)_i u^2 dx = \int\limits_B \frac{r^{-4}}{\ln^2 r} u^2 dx. \tag{32}$$

Assuming $\varepsilon_5 = \frac{1}{2}$ we obtain from (32)

$$\int_B \frac{r^{-4}}{\ln^2 r} u^2 dx \le 4\int_B r^{-2}\left|\nabla_1 u\right|^2 dx \le 4\int_B r^{-2}\left|\nabla u\right|^2 dx . \qquad (33)$$

Taking into account (33) in (31) at $\varepsilon_4 = \frac{1}{4}$ we conclude

$$\int_B r^{-2}\left|\nabla u\right|^2 dx \le \frac{1}{2}\int_B r^{-2}\left|\nabla u\right|^2 dx + 2\int_B \ln^2 r\left(\mathcal{L}u\right)^2 dx ,$$

whence, the required estimation (30) follows

Corollary. *For any function* $u(x)\in C_0^\infty(B)$ *the inequality*

$$\int_B \left(\frac{r^{-4}}{\ln^2 r} u^2 + r^{-2}\left|\nabla u\right|^2 \right) dx \le 20\int_B \ln^2 r\left(\mathcal{L}u\right)^2 dx$$

holds.

Lemma 6. *For any function* $u(x)\in C_0^\infty(B)$ *the estimation*

$$\int_B \ln^2 r \sum_{i,j=1}^{4} u_{ij}^2 dx \le C_{26}\int_B \ln^2 r\left(\mathcal{L}u\right)^2 dx$$

holds, where C_{26} *is an absolute constant.*

The proof is analogous to the one of lemma 2.

Corollary. *Under conditions of the lemma*

$$\|u\|_{W^2_{2,\ln^2 r}(B)} \le C_{27}\|\mathcal{L}u\|_{L_{2,\ln^2 r}(B)},$$

where C_{27} *is an absolute constant .*

The following two theorems are proved similarly as the corresponding statements in the case $n=3$.

Theorem 3. *Let* $\partial D\in C^2$. *Then for any function* $u(x)\in \dot{W}^{\,2}_{2,\ln^2 r}(D)$ *the following inequality is valid*

$$\|u\|_{W^2_{2,\ln^2 r}(D)} \le C_{28}(D)\|\mathcal{L}u\|_{L_{2,\ln^2 r}(D)}.$$

Theorem 4. *Let* $\partial D\in C^2$. *Then the first boundary value problem (1)-(2) is uniquely strongly solvable in* $\dot{W}^{\,2}_{2,\ln^2 r}(D)$ *for any* $f(x)\in L_{2,\ln^2 r}(D)$. *At that for solution* $u(x)$ *the following estimation is valid*

$$\|u\|_{W^2_{2,\ln^2 r}(D)} \le C_{28}\|f\|_{L_{2,\ln^2 r}(D)}.$$

4⁰. Counter-examples. Let $n=3$ the function $u_1(x)=|x|^{\frac{1}{2}}-1$ satisfies the equation B_1 almost everywhere in $\mathcal{L}u_1=0$ and vanishes in ∂B_1. The same is valid for the function $u_2(x)\equiv 0$. But the strong solution of the boundary value problem

$$\mathcal{L}u=0,\ x\in B_1;\ u|_{\partial B_1}=0$$

is only the function $u_2(x)$, since at $\gamma\in(0,1)$, $u_1(x)\notin \dot{W}^2_{2,\gamma}(B_1)$. Let $n=4$, $f(x)=|x|^{-2}\ln^{-\frac{3}{2}}\frac{1}{|x|}$, $x\in B_{\frac{1}{2}}$. It's clear that $f(x)\in L_2\left(B_{\frac{1}{2}}\right)$, but $f(x)\notin L_{2,\ln^2 r}\left(B_{\frac{1}{2}}\right)$. Consider for $t\in\left[0,\frac{1}{2}\right]$ and $\varepsilon\in\left(0,\frac{1}{2}\right)$ functions

$$g_\varepsilon(t)=\begin{cases}2, \ if \ \ t>\varepsilon,\\ 2\dfrac{t}{\varepsilon}, \ if \ \ t\le\varepsilon\end{cases}$$

and

$$f_\varepsilon(t)=\begin{cases}t^{-2}\ln^{-\frac{3}{2}}\dfrac{1}{t}, \ if \ \ t>\varepsilon,\\ \varepsilon^{-2}\ln^{-\frac{3}{2}}\dfrac{1}{\varepsilon}, \ if \ \ t\le\varepsilon.\end{cases}$$

Let further $\mathcal{L}_\varepsilon=\Delta+g_\varepsilon(|x|)\sum\limits_{i,j=1}^{4}\dfrac{x_ix_j}{|x|^2}\dfrac{\partial^2}{\partial x_i\partial x_j}$. We'll look for the classical solution $u^\varepsilon(x)$ of the problem

$$\mathcal{L}u^\varepsilon=f_\varepsilon(|x|),\ x\in B_{\frac{1}{2}};\ u^\varepsilon\big|_{\partial B_{\frac{1}{2}}}=0; \tag{34}$$

in the form $u^\varepsilon(x)=u^\varepsilon(r)$. Denoting for simplicity u^ε by u from (34) we obtain

$$(1+g_\varepsilon(r))u'''+\frac{3}{r}u'=f_\varepsilon(r),\ u\left(\frac{1}{2}\right)=0. \tag{35}$$

Assume $u'(r)=-\vartheta(r)$, i.e. $u(r)=\int\limits_r^{\frac{1}{2}}\vartheta(\tau)d\tau$. Then from (35) it follows that

$$(1+g_\varepsilon(r))\vartheta'+\frac{3}{r}\vartheta=-f_\varepsilon(r). \tag{36}$$

It's easy to see that a general solution of the homogeneous equation

$$(1+g_\varepsilon(r))\vartheta'+\frac{3}{r}\vartheta=0$$

is represented in the form

$$\vartheta(r)=H\exp\left[3\int\limits_r^{\frac{1}{2}}\frac{d\tau}{\tau(1+g_\varepsilon(\tau))}\right],$$

where H is an arbitrary constant. We'll seek solution of equation (36) in the form

$$\vartheta(r)=H(r)\exp\left[3\int\limits_r^{\frac{1}{2}}\frac{d\tau}{\tau(1+g_\varepsilon(\tau))}\right]$$

with the unknown function $H(r)$. From (36) we conclude

$$H'(r)=-f_\varepsilon(r)\exp\left[-3\int\limits_r^{\frac{1}{2}}\frac{d\tau}{\tau(1+g_\varepsilon(\tau))}\right].$$

So,

$$H(r)=-\int_0^r f_\varepsilon(\tau)\exp\left[-3\int_\tau^{\frac{1}{2}}\frac{dz}{z(1+g_\varepsilon(z))}\right]d\tau,$$

and by the same taken

$$\vartheta(r)=-\exp\left[3\int_r^{\frac{1}{2}}\frac{d\tau}{\tau(1+g_\varepsilon(\tau))}\right]\int_0^r f_\varepsilon(\tau)\exp\left[-3\int_\tau^{\frac{1}{2}}\frac{dz}{z(1+g_\varepsilon(z))}\right]d\tau.$$

Finally, we find the solution $u(r)$ of problem (35)

$$u(r)=-\int_r^{\frac{1}{2}}\exp\left[3\int_\tau^{\frac{1}{2}}\frac{dz}{z(1+g_\varepsilon(z))}\right]\int_0^\tau f_\varepsilon(y)\exp\left[-3\int_\tau^{\frac{1}{2}}\frac{dz}{z(1+g_\varepsilon(z))}\right]dy\,d\tau. \tag{37}$$

But on the other hand,

$$\sum_{i,j=1}^4 u_{ij}^2=(u'')^2+\frac{3}{r^2}(u')^2\geq\frac{(u')^2}{r^2}. \tag{38}$$

Reverting again to the denotation $u^\varepsilon(r)$ from (37) and (38) we obtain

$$\left\|u^\varepsilon\right\|_{W_2^2\left(B_{\frac{1}{2}}\right)}\geq\int_{B_{\frac{1}{2}}}\sum_{i,j=1}^4\left(u^\varepsilon\right)_{ij}^2dx\geq C_{29}\int_0^{\frac{1}{2}}r\left[\left(u^\varepsilon\right)'\right]^2dr=$$

$$=C_{29}\int_0^{\frac{1}{2}}r\exp\left[6\int_r^{\frac{1}{2}}\frac{d\tau}{\tau(1+g_\varepsilon(\tau))}\right]\left(\int_0^r f_\varepsilon(z)\exp\left[-3\int_z^{\frac{1}{2}}\frac{d\tau}{\tau(1+g_\varepsilon(\tau))}\right]dz\right)^2dr\geq$$

$$\geq C_{29}\int_\varepsilon^{\frac{1}{2}}r\exp\left[2\int_r^{\frac{1}{2}}\frac{d\tau}{\tau}\right]\left(\int_\varepsilon^r z^{-2}\ln^{-\frac{3}{2}}\frac{1}{z}\exp\left[-\int_z^{\frac{1}{2}}\frac{d\tau}{\tau}\right]dz\right)^2dr, \tag{39}$$

where C_{29} is the surface area of the unit sphere in $\mathbf{E}_4$.

From (39) it follows that

$$\left\|u^\varepsilon\right\|_{W_2^2\left(B_{\frac{1}{2}}\right)}\geq C_{30}\int_\varepsilon^{\frac{1}{2}}\frac{dr}{r}\left(\int_\varepsilon^r\frac{dz}{z\ln^{\frac{3}{2}}\frac{1}{z}}\right)^2=$$

$$=4C_{30}\left(\int_\varepsilon^{\frac{1}{2}}\frac{dr}{r\ln\frac{1}{r}}+\frac{1}{\ln\frac{1}{\varepsilon}}\int_\varepsilon^{\frac{1}{2}}\frac{dr}{r}-\frac{2}{\ln^{\frac{1}{2}}\frac{1}{\varepsilon}}\int_\varepsilon^{\frac{1}{2}}\frac{dr}{r\ln^{\frac{1}{2}}\frac{1}{r}}\right)=4C_{30}\left(p_1(\varepsilon)+p_2(\varepsilon)+p_\varepsilon(\varepsilon)\right), \tag{40}$$

where C_{30} is an absolute constant. It's easy to see that

$$\lim_{\varepsilon\to0}p_1(\varepsilon)=\infty,\ \lim_{\varepsilon\to0}p_2(\varepsilon)=1,\ \lim_{\varepsilon\to0}p_\varepsilon(\varepsilon)=-4. \tag{41}$$

Taking into account (41) in (40) we conclude

$$\lim_{\varepsilon\to 0}\left\|u^{\varepsilon}\right\|_{W_2^2\left(B_{\frac{1}{2}}\right)}=\infty$$

and, consequently,

$$\lim_{\varepsilon\to 0}\left\|u^{\varepsilon}\right\|_{W^2_{2,\ln^2 r}\left(B_{\frac{1}{2}}\right)}=\infty.$$

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