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ON UNIQUENESS OF STRONG SOLUTION OF DIRICHLET PROBLEM FOR SECOND ORDER QUASILINEAR ELLIPTIC EQUATIONS

Abstract

The first boundary value problem is considered for second order quasilinear elliptic equations of nondivergence form when the leading part of these equations satisfies the Cordes condition. The uniqueness of strong (almost everywhere) solution of the mentioned problem is proved.

Let $\mathbb{E}_n$ be n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$, $n \geq 2$, D be bounded convex domain in $\mathbb{E}_n$ with the boundary $\partial D \in C^2$. Consider the following Dirichlet problem in D

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}\left(x, u\right) u_{ij} = f\left(x\right), \quad x \in D; \tag{1}$$

$$u|_{\partial D} = 0, \tag{2}$$

where $u_i = \dfrac{\partial u}{\partial x_i}$, $u_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$; $i, j = 1, ..., n$; $\|a_{ij}\left(x, z\right)\|$ is a real symmetric matrix whose elements are measurable in D for any fixed $z \in \mathbb{E}_1$, moreover,

$$\mu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\left(x, z\right) \xi_i \xi_j \leq \mu^{-1}|\xi|^2; \; x \in D, \; z \in \mathbb{E}_1, \; \xi \in \mathbb{E}_n; \tag{3}$$

$$\sigma = \sup_{x \in D, \; z \in \mathbb{E}_1} \frac{\sum\limits_{i,j=1}^{n} a_{ij}\left(x, z\right)}{\left[\sum\limits_{i=1}^{n} a_{ii}\left(x, z\right)\right]^2} < \frac{1}{n-1} \; . \tag{4}$$

Here $\mu \in (0, 1]$ is a constant. Condition (4) is called the Cordes condition, and it is understood up to equivalence and nonsingular linear transformation in the following sense: domain D can be covered by a finite number of subdomains $D^1, ..., D^l$ so that in each D^i equation (1) can be replaced by the equivalent equation $\mathcal{L}'u = f'\left(x\right)$, and nonsingular linear transformation of coordinated can be made, at which the coefficients of the image of the operator $\mathcal{L}'$ satisfy condition (4) in the domain D^i; $i = 1, ..., l$.

The aim of the present paper is to prove the uniqueness of strong (almost everywhere) solution of the first boundary value problem (1)-(2) for $f\left(x\right) \in L_2\left(D\right)$ at $n = 2, 3, 4$. In connection with this we refer to the papers [1-2], where the analogous results for the second order linear elliptic equations of nondivergence structure with continuous coefficients were obtained, and we also refer to the papers [3-7], in which some classes of above mentioned equations with discontinuous coefficients were considered. We notice the papers [8-10], where the questions of strong solvability for the second order parabolic equations in nondivergence form were investigated. Existence of strong solution of the first boundary value problem (1)-(2) was established in [11], at that it was done for more general class of equations than (1).

We now agree upon some denotation. For $p \in [1, \infty)$ we denote by $W_p^1(D)$ and $W_p^2(D)$ Banach spaces of functions $u(x)$ given on D with the finite norms

$$\|u\|_{W_p^1(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n |u_i|^p \right) dx \right)^{\frac{1}{p}}$$

and

$$\|u\|_{W_p^2(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p \right) dx \right)^{\frac{1}{p}},$$

respectively. Further, let $\overset{\circ}{W}{}_p^1(D)$ be a completion of $C_0^\infty(D)$ by the norm of space $W_p^1(D)$, $\dot{W}_p^2(D) = W_2^2(D) \cap \overset{\circ}{W}{}_p^1(D)$. Function $u(x) \in \dot{W}_p^2(D)$ is called strong solution of the first boundary-value problem (1)-(2) (for $f(x) \in L_p(D)$) if it satisfies equation (1) almost everywhere in D.

Everywhere below notation $C(...)$ means that the positive constant C depends only on what in parentheses. We now give some known facts which we will need further on.

Theorem 1 ([1]). *Let $1 < p < n$, $1 \leq q \leq \frac{np}{n-p}$. Then for any function $u(x) \in \overset{\circ}{W}{}_p^1(D)$, the following estimate holds*

$$\|u\|_{L_q(D)} \leq C_1(p,q,n) \, \||u_x|\|_{L_p(D)} .$$

If $p > n$, then

$$\|u\|_{L_\infty(D)} \leq C_2(p,n) \, \||u_x|\|_{L_p(D)} .$$

Here $u_x = (u_1, ..., u_n)$.

Consider the following equation in D

$$\mathcal{M}u = \sum_{i,j=1}^n a_{ij}(x, u, u_x) \, u_{ij} = f(x), \quad x \in D \tag{1'}$$

and suppose that the elements of the real symmetric matrix $\|a_{ij}(x,z,\upsilon)\|$ are measurable in D for any fixed $z \in \mathbb{E}_1$, $\upsilon \in \mathbb{E}_n$,

$$\mu|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x,z,\upsilon) \, \xi_i \xi_j \leq \mu^{-1} |\xi|^2; \ x \in D, \ z \in \mathbb{E}_1, \ \upsilon \in \mathbb{E}_n, \ \xi \in \mathbb{E}_n; \tag{3'}$$

$$\sigma = \sup_{x \in D, \ z \in \mathbb{E}_1, \upsilon \in \mathbb{E}_n} \frac{\sum\limits_{i,j=1}^n a_{ij}^2(x,z,\upsilon)}{\left[\sum\limits_{i=1}^n a_{ii}(x,z,\upsilon) \right]^2} < \frac{1}{n-1} ; \tag{4'}$$

and

$$\left| a_{ij}(x, z^1, \upsilon^1) - a_{ij}(x, z^2, \upsilon^2) \right| \leq H \left(\left| z^1 - z^2 \right|^\alpha + \left| \upsilon^1 - \upsilon^2 \right|^\alpha \right);$$
$$x \in D; \ z^1, \ z^2 \in \mathbb{E}_1; \ \upsilon^1, \upsilon^2 \in \mathbb{E}_n; \ i,j = 1, ..., n \tag{5}$$

with constants $H \geq 0$ and $\alpha \in (0, 1]$.

Theorem 2 ([11]). *Let the coefficients of the operator $\mathcal{M}$ satisfy conditions (3'), (4'), (5'). Then there exists $p_1(\mu,\sigma,n) \in \left(\frac{3}{2},2\right)$ such that at any $p \in [p_1,2]$ for any function $u(x) \in \dot{W}_p^2(D)$ the following estimate holds*

$$\|u\|_{W_p^2(D)} \le C_3(\mu,\sigma,n,\partial D)\,\|\mathcal{M}u\|_{L_p(D)} \ . \tag{6}$$

At that for any $A>0$ there exists $d_A = d_A(\mu,\sigma,n,\partial D,H,\alpha,A)$ such that if $mesD \le d_A$, then the first boundary value problem (1')-(2') has a strong solution from space $\dot{W}_2^2(D)$ for any function $f(x) \in L_2(D)$, whenever $\|f\|_{L_2(D)} \le A$.

Remark. Actually, in [11] theorem 2 was proved under the condition that in (5) $\alpha \in (0,\alpha^0]$, where $\alpha^0 = \frac{n(2-p_1)}{2(n-p_1)}$. But one can see from the proof that by virtue of boundedness of the coefficients $a_{ij}(x,z,\upsilon)\,;\ i,j=1,...,n$ the statement of the theorem is valid for $\alpha \in (0,1]$.

Theorem 3. *Let $3 \le n \le 4$, and let the coefficients of the operator $\mathcal{L}$ satisfy conditions (3)-(4) and*

$$|a_{ij}(x,z^1) - a_{ij}(x,z^2)| \le H_1|z^1 - z^2|;\ x\in D;\ z^1,z^2 \in \mathbb{E}_1;\ i,j=1,...,n \tag{7}$$

with some nonnegative constant H_1. Then for any $s\in(2,\infty)$ for $n=4$, $s\in[2,\infty)$, for $n=3$ and $A>0$ there exists $\rho_A = \rho_A(\mu,\sigma,n,\partial D,H_1,s,A)$ such that if $mesD \le \rho_A$, $f(x)\in L_s(D)$ and $\|f\|_{L_S(D)} \le A$ then the first boundary value problem (1)-(2) has a unique strong solution $u(x) \in \dot{W}_2^2(D)$.

Proof. The constant ρ_A will be chosen such that $\rho_A \le d_A$. Therefore, according to theorem 2 we must prove only the the uniqueness of the solution. Let $u^1(x)$ and $u^2(x)$ be two strong solutions of the first boundary problem (1)-(2) from space $\dot{W}_2^2(D)$,

$$\mathcal{L}_{(1)} = \sum_{i.j=1}^{n} a_{ij}(x)\,u^{(1)}(x)\,\frac{\partial^2}{\partial x_i \partial x_j} \ .$$

We have

$$\begin{gathered}\mathcal{L}_{(1)}\left(u^1-u^2\right) = \sum_{i,j=1}^{n} a_{ij}\left(x,u^1\right)u_{ij}^1 - \sum_{i,j=1}^{n}\left[a_{ij}\left(x,u^1\right) - a_{ij}\left(x,u^2\right)\right]u_{ij}^2 - \\ -f(x) = -\sum_{i,j=1}^{n}\left[a_{ij}\left(x,u^1\right) - a_{ij}\left(x,u^2\right)\right]u_{ij}^2 = F(x) \ .\end{gathered} \tag{8}$$

On the other hand, according to (7)

$$|F(x)| \le H_1\left|u^1-u^2\right|\sum_{i,j=1}^{n}|u_{ij}^2| \ . \tag{9}$$

At first consider the case $n=4$. Let $q_1 = \frac{4p_1}{4-p_1}$, $q_2 = \frac{2p_1}{2-p_1}$. From theorem 1 we obtain

$$\left\|u^1-u^2\right\|_{L_{q_2}(D)} \le C_1(p_1)\left\|u^1-u^2\right\|_{W_{q_1}^1(D)} \le C_1^2\left\|u^1-u^2\right\|_{W_{p_1}^2(D)} \cdot \tag{10}$$

Now applying theorem 2, from (8)-(10) we conclude

$$\left\|u^1-u^2\right\|_{L_{q_2}(D)}\leq C_1^2C_3\left\|F\right\|_{L_{p_1}(D)}\leq$$
$$\leq C_1^2C_3H_1\left[\int_D\left|u^1-u^2\right|^{p_1}\left(\sum_{i,j=1}^{4}|u_{ij}^2|\right)^{p_1}dx\right]^{\frac{1}{p_1}}\leq$$
$$\leq C_1^2C_3H_1\left\|u^1-u^2\right\|_{L_{q_2}(D)}\left[\int_D\left(\sum_{i,j=1}^{n}|u_{ij}^2|\right)^{2}dx\right]^{\frac{1}{2}}\leq \tag{11}$$
$$\leq 4C_1^2C_3H_1\left\|u^1-u^2\right\|_{L_{q_2}(D)}\left\|u^2\right\|_{W_2^2(D)}\leq 4C_1^2C_3^2H_1\left\|u^1-u^2\right\|_{L_{q_2}(D)}\times$$
$$\times\left\|f\right\|_{L_2(D)}\leq 4C_1^2C_3^2H_1\left\|u^1-u^2\right\|_{L_{q_2}(D)}\leq 4C_1^2C_3^2H_1\left\|u^1-u^2\right\|_{L_{q_2}(D)}\times$$
$$\times\left(mesD\right)^{\frac{S-2}{29}}\left\|f\right\|_{L_S(D)}\leq 4C_1^2C_3^2H_1A\rho^{\frac{S-2}{2S}}\left\|u^1-u^2\right\|_{L_{q_2}(D)}\ .$$

Let ρ' be such that

$$4C_1^2C_3^2H_1A\left(\rho'\right)^{\frac{S-2}{2S}}=\frac{1}{2}\ .$$

We choose $\rho_A=\min\{d_A,\rho'\}$. Then from (11) it follows that

$$\left\|u^1-u^2\right\|_{L_{q_2}(D)}\leq\frac{1}{2}\left\|u^1-u^2\right\|_{L_{q_2}(D)},$$

i.e. $u^1(x)=u^2(x)$ a.e. in D.

Let $n=3,\ q_3=\frac{3p_1}{3-p_1}$. It is clear that $q_3>3$. Therefore, according to theorem 1,

$$\left\|u^1-u^2\right\|_{L_\infty(D)}\leq C_2\left(p_1\right)\left\|u^1-u^2\right\|_{W_{q_3}^1(D)}\leq C_2^2\left\|u^1-u^2\right\|_{W_{p_1}^2(D)}\ .$$

Further, applying theorem 2, we obtain

$$\left\|u^1-u^2\right\|_{L_\infty(D)}\leq C_2^2C_3\left\|F\right\|_{Lp_1(D)}\leq C_2^2C_3H_1\left\|u^1-u^2\right\|_{L_\infty(D)}\times$$
$$\times\left[\int_D\left(\sum_{i,j=1}^{3}\left|u_{ij}^2\right|\right)^{p_1}dx\right]^{\frac{1}{p_1}}\leq 9^{\frac{p_1-1}{p_1}}C_2^2C_3H_1\left\|u^1-u^2\right\|_{L_\infty(D)}\left\|u^2\right\|_{W_{p_1}^2(D)}\leq$$
$$\leq 9^{\frac{p_1-1}{p_1}}C_2^2C_3^2H_1\left\|u^1-u^2\right\|_{L_\infty(D)}\left\|f\right\|_{Lp_1(D)}\leq 9^{\frac{p_1-1}{p_1}}C_2^2C_3^2H_1\left\|u^1-u^2\right\|_{L_\infty(D)}\times$$
$$\times\left(mesD\right)^{\frac{2-p_1}{2p_1}}\left\|f\right\|_{L_2(D)}\leq 9^{\frac{p_1-1}{p_1}}C_2^2C_3^2H_1\rho_A^{\frac{2-p_1}{2p_1}}A\left\|u^1-u^2\right\|_{L_\infty(D)}. \tag{12}$$

Let ρ'' be such that

$$9^{\frac{p_1-1}{p_1}}C_2^2C_3^2H_1A\left(\rho''\right)^{\frac{2-\rho_1}{2p_1}}=\frac{1}{2}\ .$$

Then choosing $\rho_A=\min\{d_A,\rho''\}$, from (12) we conclude

$$\left\|u^1-u^2\right\|_{L_\infty(D)}\leq\frac{1}{2}\left\|u^1-u^2\right\|_{L_\infty(D)},$$

i.e. $u^1(x) = u^2(x)$ a.e. in D. The theorem is proved.

Theorem 4. *Let $n = 2$, and let the coefficients of the operator $\mathcal{M}$ satisfy conditions (3') and (5) (for $\alpha = 1$). Then for any $s \in (2, \infty)$ and $A > 0$ there exists $\rho_A = \rho_A(\mu, \partial D, H, s, A)$ such that if $mesD \le \rho_A$, $f(x) \in L_s(D)$* and $\|f\|_{L_S(D)} \le A$ *then the first boundary value problem (1')-(2) has a unique strong solution $u(x) \in \dot{W}_2^2(D)$.*

Proof. It suffices to show the uniqueness of the solution. Note that for $n = 2$, condition (4') follows from (3'). Let $u^1(x)$ and $u^2(x)$ be two strong solutions of the first boundary value problem (1')-(2) from the space $\dot{W}_2^2(D)$,

$$\mathcal{M}_{(1)} = \sum_{i,j=1}^{2} a_{ij}\left(x, u^1(x), u_x^1(x)\right) \frac{\partial^2}{\partial x_i \partial x_j} .$$

We have

$$\mathcal{M}_{(1)}\left(u^1 - u^2\right) = \sum_{i,j=1}^{2} a_{ij}\left(x, u^1, u_x^1\right) u_{ij}^1 -$$
$$- \sum_{i,j=1}^{2} \left[a_{ij}\left(x, u^1, u_x^1\right) - a_{ij}\left(x, u^2, u_x^2\right)\right] u_{ij}^2 - f(x) = \tag{13}$$
$$= - \sum_{i,j=1}^{2} \left[a_{ij}\left(x, u^1, u_x^1\right) - a_{ij}\left(x, u^2, u_x^2\right)\right] u_{ij}^2 = F_1(x) .$$

On the other hand, according to (5)

$$|F_1(x)| \le H\left(|u^1 - u^2| + |u_x^1 - u_x^2|\right) \sum_{i,j=1}^{2} |u_{ij}^2| . \tag{14}$$

Let $q_4 = \frac{2p_1}{2-p_1}$. From theorem 1 we obtain

$$\left\|u^1 - u^2\right\|_{W_{q_4(D)}^1} \le C_1(p_1) \left\|u^1 - u^2\right\|_{W_{p_1}^2(D)} . \tag{15}$$

Applying theorem 2, from (13)-(15) we conclude

$$\left\|u^1 - u^2\right\|_{W_{q_4}^1(D)} \le C_1 C_3 \|F_1\|_{L_{p_1}(D)} \le$$
$$\le C_1 C_3 H \left[\int_D \left(|u^1 - u^2| + |u_x^1 - u_x^2|\right)^{p_1} \left(\sum_{i,j=1}^{2} |u_{ij}^2|\right)^{p_1} dx\right]^{\frac{1}{p_1}} \le$$
$$\le C_4(\mu, \partial D, H) \left\|u^1 - u^2\right\|_{W_{q_4}^1(D)} \|f\|_{L_2(D)} \le \tag{16}$$

$$\le C_4 \left\|u^1 - u^2\right\|_{W_{q_4}^1(D)} (mesD)^{\frac{S-2}{2S}} \|f\|_{L_S(D)} \le C_4 \rho_A^{\frac{S-2}{2S}} A \left\|u^1 - u^2\right\|_{W_{q_4}^1(D)} .$$

Let $\bar{\rho}$ be so that

$$C_4 A \, \bar{\rho}^{\frac{S-2}{2S}} = \frac{1}{2} .$$

We choose $\rho_A = \min\{d_A, \bar{\rho}\}$. Then from (16) it follows that

$$\left\|u^1 - u^2\right\|_{W_{q_4}^1(D)} \le \frac{1}{2} \left\|u^1 - u^2\right\|_{W_{q_4}^1(D)} ,$$

i.e. $u^1(x) = u^2(x)$ a.e. in D. The theorem is proved.

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