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# INVESTIGATIONS OF CLASSICAL SOLUTION IM KLEINEN OF INVERSE BOUNDARY-VALUE PROBLEM FOR SEMILINEAR HYPERBOLIC EQUATIONS OF THE SECOND ORDER

**Abstract**

In the paper a theorem on the existence im kleinen of classical solution of one- dimensional boundary- value problem for the semilinear second order differential equations of hyperbolic type was proved by means of Fourier method and contracted mappings principle on a fixed point.

The paper is devoted to investigation of question on the existence of classical solution of one-dimensional inverse boundary- value problem

$$u_{tt}(t,x)-u_{xx}(t,x)=a(t)\,b(t)\,u(t,x)+c(t)\,d(t,x)+$$
$$+f(t,\ x,\ u(t,x),\ u_t(t,x),\ u_x(t,x),\ a(t),\ c(t))\quad (0\le t\le T,\ 0\le x\le 1)\,, \tag{1}$$

$$u(0,x)=\varphi(x)\quad (0\le x\le 1)\,,\quad u_t(0,\ x)=\psi(x)\quad (0\le x\le 1)\,, \tag{2}$$

$$u(t,0)=u(t,1)=0\quad (0\le t\le T)\,, \tag{3}$$

$$u_x(t,0)=g_1(t)\,,\ u_x(1,t)=g_2(t)\ \ (0\le t\le T)\,, \tag{4}$$

where $0<T<+\infty,\ f,\ \varphi,\ \psi,\ g_1,\ g_2$ are given functions and $u(t,x)\,,\ a(t)\,,\ c(t)$ are unknown functions, moreover, as a classical solution of problem (1)- (4) we mean the following.

**Definition.** *We call by classical solution of problem (1)- (4) the triple* $\{u(t,x)\,, a(t)\,,\ c(t)\}$ *of functions* $u(t,x)\,,\ a(t)$ *and* $c(t)$ *having the properties:*

*a)* $u(t,x)\in C^2([0,\ T]\times[0,\ 1])$;

*b)* $a(t)\,,\ c(t)\in C([0,\ T])$;

*c) all the conditions (1)- (4) are satisfied in a general sense [3].*

In order to investigate problem (1)- (4) we will introduce the following spaces

**1.** Denote by $B^{4,3}_{2,T}$ the set of all functions of the form

$$u(t,x)=\sum_{k=1}^{\infty}u_k(t)\sin k\pi x \tag{5}$$

being considered in domain $D_T=\{0\le t\le T,\ 0\le x\le 1\}$, where each of functions $u_k(t)$ is continuously differentiable on $[0,\ T]$ and

$$\|u\|_{B^{4,\ 3}_{2,\ T}}\equiv\left\{\sum_{k=1}^{\infty}\left(k^4\max_{0\le t\le T}|u_k(t)|\right)^2+\sum_{k=1}^{\infty}\left(k^3\max_{0\le t\le T}\left|u_k'(t)\right|\right)^2\right\}^{\frac{1}{2}}<+\infty \tag{6}$$

**2.** Denote by $E_T$ the topological product

$$B^{4,\ 3}_{2,\ T}\times C_{[0,\ T]}\times C_{[0,\ T]}.$$

Denote the norm of element $z=\{u,\ a,\ c\}$ in $E_T$ by

$$\|z\|_{E_T}=\|u\|_{B^{4,\ 3}_{2,\ T}}+\|a\|_{C_{[0,\ T]}}+\|c\|_{C_{[0,\ T]}}\,. \tag{7}$$

It is obvious that spaces $B^{4,\ 3}_{2,\ T}$ and $E_T$ are Banach spaces [1].

**3.** Obviously, the first component $u(t,x)$ of each classical solution $\{u(t,\ x),\ a(t),\ c(t)\}$ of the problem (1)- (4) has the form:

$$u(t,x)=\sum_{k=1}^{\infty}u_k(t)\sin k\pi x,\ \ u_k(t)=2\int_0^1 u(t,x)\sin k\pi x. \tag{8}$$

After the formal application of scheme of Fourier method finding of functions $u_k(t)$ $(k=1,\ 2,...)$ is reduced to solving of the following system of nonlinear integro-differential equations

$$u_k(t)=\varphi_k\cos k\pi t+\frac{\psi_k}{k\pi}\sin k\pi t+$$
$$+\frac{2}{k\pi}\int_0^t\int_0^1 F(u(\tau,\ x),\ a(\tau),\ c(\tau))\sin k\pi(t-\tau)\sin k\pi x dx d\tau \tag{9}$$
$$(k=1,2,...,t\in[0,\ T]),$$

where

$$\varphi_k=2\int_0^1\varphi(x)\sin k\pi x dx,\ \ \psi_k=2\int_0^1\psi(x)\sin k\pi x dx. \tag{10}$$

$$F(u(t,\ x),\ a(t),\ c(t))\equiv F(t,\ x,\ u(t,\ x),\ u_t(t,\ x),\ u_x(t,\ x),\ a(t),$$
$$c(t))=a(t)b(x)u(t,\ x)+c(t)d(t,\ x)+f(t,\ x,\ u,\ u_t,\ u_x,\ a,\ c)\,. \tag{11}$$

It is obvious that system (9) can be written in the form of one equation

$$u(t,\ x)=\sum_{k=1}^{\infty}\varphi_k\cos k\pi t\sin k\pi x+\sum_{k=1}^{\infty}\frac{\psi_k}{k\pi}\sin k\pi t\sin k\pi x+$$
$$+\sum_{k=1}^{\infty}\frac{2}{k\pi}\int_0^t\int_0^1 F(u(\tau,\ x),\ a(\tau),\ c(\tau))\sin k\pi(t-\tau)\sin k\pi x dx d\tau\cdot\sin k\pi x\,. \tag{12}$$

Further, in order to obtain the equations also for the rest components $a(t),\ c(t)$ of the classical solution $\{u(t,\ x),\ a(t),\ c(t)\}$ of the problem (1)- (4) substituting expression (12) into condition (4), twice formally differentiating the obtained equalities with respect to $t$, using relations

$$\sum_{k=1}^{\infty}2\pi k\int_0^1 F(u(t,\ x),\ a(t),\ c(t))\sin k\pi x dx=$$
$$=\frac{\partial}{\partial x}\left\{\sum_{k=1}^{\infty}2\int_0^1 F(u(t,\ x),\ a(t),\ c(t))\sin k\pi x dx\right\}\Bigg|_{x=0}= \tag{13}$$
$$=\frac{\partial}{\partial x}\left\{F(u(t,\ x),\ a(t),\ c(t))\right\}\Big|_{x=0},$$

$$\sum_{k=1}^{\infty} 2(-1)^k \pi k \int_0^1 F(u(t,\ x),\ a(t),\ c(t)) \sin k\pi x dx =$$

$$= \frac{\partial}{\partial x} \left\{ \sum_{k=1}^{\infty} 2 \int_0^1 F(u(t,\ x),\ a(t),\ c(t)) \sin k\pi x dx \cdot \sin k\pi x \right\} \Bigg|_{x=1} = \tag{14}$$

$$= \frac{\partial}{\partial x} \left\{ F(u(t,\ x),\ a(t),\ c(t)) \right\} \Big|_{x=1},$$

conditions

$$u(t,\ 0) = u(t,\ 1) = u_{xx}(t,\ 0) = u_{xx}(t,\ 1) = u_t(t,\ 0) = u_t(t,\ 1) = 0$$
$$(0 \le t \le T)$$
$$u_x(t,\ 0) = g_1(t),\ u_x(t,\ 1) = g_2(t),\ u_{tx}(t,\ 0) = g_1'(t),\ u_{tx}(t,\ 1) = g_2'(t) \tag{15}$$
$$(0 \le t \le T)$$

relations

$$\frac{\partial}{\partial x} \left\{ F(u(t,\ x),\ a(t),\ c(t)) \right\} \Big|_{x=0} = a(t) b(0) g_1(t) + c(t) d_x(t,\ 0) +$$
$$+ \frac{\partial}{\partial x} \left\{ f(u(t,\ x),\ a(t),\ c(t)) \right\} \Big|_{x=0},$$
$$\frac{\partial}{\partial x} \left\{ F(u(t,\ x),\ a(t),\ c(t)) \right\} \Big|_{x=0} = a(t) b(1) g_2(t) + c(t) d_x(t,\ 1) + \tag{16}$$
$$+ \frac{\partial}{\partial x} \left\{ f(u(t,\ x),\ a(t),\ c(t)) \right\} \Big|_{x=1},$$

assuming that

$$\forall t \in [0,\ T] \qquad \Delta(t) \equiv \begin{vmatrix} b(0) g_1(t) & d_x(t,\ 0) \\ b(1) g_2(t) & d_x(t,\ 1) \end{vmatrix} \neq 0 \tag{17}$$

and solving the system of two obtained equations with respect to $a(t)$ and $c(t)$ we will obtain

$$a(t)=(\Delta(t))^{-1}d_x(1,\ t)\left\{g_1''(t)+\sum_{k=1}^{\infty}(k\pi)^3\varphi_k\cos k\pi t+\right.$$
$$+\sum_{k=1}^{\infty}(k\pi)^2\psi_k\sin k\pi t-\left.\frac{\partial f}{\partial x}\right|_{x=0}+2\sum_{k=1}^{\infty}(k\pi)^2\times$$
$$\left.\times\int_0^t\int_0^1 F(u(\tau,x),a(\tau),\ c(\tau))\sin k\pi(t-\tau)\sin k\pi x dxd\tau\right\}-$$
$$-(\Delta(t))^{-1}d_x(0,\ t)\left\{g_2''(t)+\sum_{k=1}^{\infty}(-1)^k(k\pi)^3\varphi_k\cos k\pi t+\right. \tag{18}$$
$$+\sum_{k=1}^{\infty}(-1)^k(k\pi)^2\psi\sin k\pi t-\left.\frac{\partial f}{\partial x}\right|_{x=1}+2\sum_{k=1}^{\infty}(-1)^k(k\pi)^2\times$$
$$\left.\times\int_0^t\int_0^1 F(u(\tau,\ x),\ a(\tau),\ c(\tau))\sin k\pi(t-\tau)\sin k\pi x dxd\tau\right\},$$

$$c(t)=(\Delta(t))^{-1}b(0)g_1(t)\left\{g_2''(t)+\sum_{k=1}^{\infty}(-1)^k(k\pi)^3\varphi_k\cos k\pi t+\right.$$
$$+\sum_{k=1}^{\infty}(-1)^k(k\pi)^2\psi_k\sin k\pi t-\left.\frac{\partial f}{\partial x}\right|_{x=1}+2\sum_{k=1}^{\infty}(-1)^k(k\pi)^2\times$$
$$\left.\times\int_0^t\int_0^1 F(u(\tau,\ x),\ a(\tau),\ c(\tau))\sin k\pi(t-\tau)\sin k\pi x dxd\tau\right\}-$$
$$-(\Delta(t))^{-1}b(1)g_2(t)\left\{g_1''(t)+\sum_{k=1}^{\infty}(k\pi)^3\varphi_k\cos k\pi t+\right. \tag{19}$$
$$+\sum_{k=1}^{\infty}(k\pi)^2\psi\sin k\pi t-\left.\frac{\partial f}{\partial x}\right|_{x=0}+2\sum_{k=1}^{\infty}(k\pi)^2\times$$
$$\left.\times\int_0^t\int_0^1 F(u(\tau,\ x),\ a(\tau),\ c(\tau))\sin k\pi(t-\tau)\sin k\pi x dxd\tau\right\},$$

Thus, we reduced the solving of the problem (1)- (4) to the solving of system (12), (18), (19) with respect to unknown functions $u(t,\ x),\ a(t),\ c(t)$ [2].

Based on the definition of the classical solution of the problem (1)- (4) the following lemma is easily proved.

**Lemma.** *If* $\{u(t,\ x),\ a(t),\ c(t)\}$ *is any classical solution of the problem (1)-(4), then functions* $u_k=2\int_0^1 u(t,\ x)\sin k\pi x dx\ \ (k=1,2,...)$ *satisfy the system (9) on* $[0,\ T]$.

The main result of the given paper is the following theorem on the existence im kleinen of the classical solution of the problem (1)- (4).

**Theorem.** *Let*

*1.* $\varphi(x)\in C^{(3)}_{([0,1])},\ \varphi^{IV}(x)\in L_2(0,1)$ *and* $\varphi(0)=\varphi(1)=\varphi''(0)=\varphi''(1)=0.$

*2.* $\psi(x) \in C^{(2)}_{([0,1])}$, $\psi^{III}(x) \in L_2(0,1)$ *and* $\psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0$.

*3.* $g_1(t)$, $g_2(t) \in C^{(2)}_{([0,T])}$.

*4. Fitting conditions for initial and boundary functions* $\varphi(x), \psi(x)$ *and* $g_1(t)$, $g_2(t)$*:*

$\varphi'(0) = g_1(0), \varphi'(1) = g_2(0)$, $\psi'(0) = g_1'(0), \psi'(1) = g_2'(0)$.

*5. a)* $b(x) \in C^{(3)}_{([0,1])}$ *and* $b'(0) = b'(1) = 0$*;*

*b)* $d(t,x)$, $d_x(t,x)$, $d_{xx}(t,x)$, $d_{xxx}(t,x) \in C_{([0,T]\times[0,1])}$ *and* $d(t,0) = d(t,1) = d_{xx}(t,0) = d_{xx}(t,1) = 0$*;*

*c)* $\forall t \in [0,\ T]$ $\quad \Delta(t) \equiv b(0)\, g_1(t)\, d_x(0,t) - b(1)\, g_2(t)\, d_x(1,t) \neq 0$.

*6.* $\frac{\partial^s f(t,\xi_0,\xi_1,\ldots,\xi_5)}{\partial\xi_0^{\alpha_0}\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}\partial\xi_3^{\alpha_3}}\ (s = \overline{0,3}) \in C_{\left([0,T]\times[0,1]\times(-\infty;\infty)^5\right)}$ *and functions* $f$, $f_{\xi_3}$, $f_{\xi_0\xi_0}$, $f_{\xi_0\xi_1}$, $f_{\xi_0\xi_2}$, $f_{\xi_1\xi_2}$ *are equal to zero at points* $(t,0,0,0,g_1(t),\xi_4,\xi_5)$ *and* $(t,1,0,0,g_2(t),\xi_4,\xi_5)$ *for any* $t \in [0,T]$, $\xi_4,\xi_5 \in (-\infty;\infty)$, *where* $\xi_0,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5$ *are denotations of those arguments of the function* $f$ *instead of which in the right-hand side of equation (1)*$x, u, u_t, u_x, a, c$ *are placed respectively.*

*7. For any fixed* $r$ $(0 < r < \infty)$ *for any* $t \in [0,\ T]$ *and* $\xi_3, a_1, a_2, c_1, c_2 \in [-r;r]$*:*

$$\left|f_{\xi_i}(t,0,0,0,g_1(t),a_1,c_1) - f_{\xi_i}(t,0,0,0,g_2(t),a_2,c_2)\right|,$$

$$\left|f_{\xi_i}(t,1,0,0,g_2(t),a_1,c_1) - f_{\xi_i}(t,1,0,0,g_2(t),a_2,c_2)\right| \leq$$

$$\leq l_{i,\ r}\left[|a_1 - a_2| + |c_1 - c_2|\right] \quad (i = 0,1,2),$$

*at that*

$$N_r \equiv \left\|\Delta^{-1}(t)\right\|_{C_{[0,T]}} \left\{ l_{1,r}\left(\|d_x(1,t)\|_{C_{[0,T]}} + |b(1)|\,\|g_2(t)\|_{C_{[0,T]}}\right) \times\right.$$

$$\times\left(1 + \|g_1(t)\|_{C_{[0,T]}} + \|g_1'(t)\|_{C_{[0,T]}}\right) + l_{2,r}\left(\|d_x(0,t)\|_{C_{[0,T]}} + |b(0)|\,\|g_1(t)\|_{C_{[0,T]}}\right) \times$$

$$\left.\times\left(1 + \|g_2(t)\|_{C_{[0,T]}} + \|g_2'(t)\|_{C_{[0,T]}}\right)\right\} < \frac{1}{2}.$$

*8. For every* $R$ $(0 < R < \infty)$ *in domain* $D_T \times [-R,R]^5$:

$$\left|\frac{\partial^s f(t,x,u,v,w,a_1,c_1)}{\partial x^{\alpha_0}\partial u^{\alpha_1}\partial v^{\alpha_2}\partial w^{\alpha_3}} - \frac{\partial^s f(t,x,u,v,w,a_2,c_2)}{\partial x^{\alpha_0}\partial u^{\alpha_1}\partial v^{\alpha_2}\partial w^{\alpha_3}}\right| \leq$$

$$\leq C_R(t,x)\left(|a_1 - a_2| + |c_1 - c_2|\right) \quad (s = 1,2),$$

$$\left|\frac{\partial^3 f(t,x,u,v,w,a_1,c_1)}{\partial x^{\alpha_0}\partial u^{\alpha_1}\partial v^{\alpha_2}\partial w^{\alpha_3}} - \frac{\partial^3 f(t,x,u_2,v_2,w_2,a_2,c_2)}{\partial x^{\alpha_0}\partial u^{\alpha_1}\partial v^{\alpha_2}\partial w^{\alpha_3}}\right| \leq$$

$$\leq C_R(t,x)\left(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2| + |a_1 - a_2| + |c_1 - c_2|\right),$$

*where* $C_R(t,x) \in L_2(D_T)$.

*9. For some* $R_0$ $(0 < R_0 < \infty)$*:*

$$C_0 + D_{R_0}\max_{0\leq t\leq T}\left|\Delta^{-1}(t)\right| < R_0,$$

*where*

$$C_0 \equiv \frac{\sqrt{6(1+\pi^2)}}{\pi^4} + \frac{1}{\sqrt{3}} \left\| \Delta^{-1}(t) \right\|_{C_{[0,T]}} \Big( \| d_x(0,t) \|_{C_{[0,T]}} + \| d_x(1,t) \|_{C_{[0,T]}} +$$

$$+ |b(0)| \, \| g_1(t) \|_{C_{[0,T]}} + |b(1)| \, \| g_2(t) \|_{C_{[0,T]}} \Big) \cdot \left( \left\| \varphi^{IV}(x) \right\|_{L_2(0,1)} + \left\| \psi^{III}(x) \right\|_{L_2(0,1)} \right),$$

$$D_{R_0} \equiv \| d_x(1,t) \|_{C_{[0,T]}} \max_{\substack{0 \le t \le T \\ -R_0 \le a, \ c \le R_0}} |f_x(t,0,0,0,g_1(t),a,c) + f_u(\ldots) \, g_1(t) +$$

$$+ f_v(...) \, g_1'(t)| + |b(0)| \, \| g_1(t) \|_{C_{[0,T]}} \max_{\substack{0 \le t \le T \\ -R_0 \le a, \ c \le R_0}} |f_x(t,1,0,0,g_2(t),a,c) +$$

$$+ f_u(...) \, g_2(t) + f_v(...) \, g_2'(t)| \, .$$

*Then for sufficiently small values of $T$ problem (1)- (4) has a classical solution.*
*The theorem is proved by means of principle of contraction mappings.*

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