Vugar E. ISMAILOV

ON BEHAVIOUR OF THE BEST APPROXIMATION AS A FUNCTION OF AN APPROXIMATION SET

Abstract

In this paper the best approximation $E(f,Q)$ of a continuous function $f(x,y)$ by sums $\varphi(x)+\psi(y)$ of continuous $\varphi(x)$ and $\psi(y)$ is considered as a function depending on an approximation set Q. The order relations between $E(f,Q)$ and $E(f,Q_1)+E(f,Q_2)$ are established for some sets $Q\subset R^2,\ Q_1\subset Q\ ,\ Q_2\subset Q\ ,\ Q_1\cup\cup Q_2=Q$ and some classes of functions from $C(Q)$.

Let Q be a compact set on R^2. Q_x and Q_y be projections of Q onto coordinate axes x and y accordingly. Consider an approximation of a continuous function $f(x,y)$ on Q by the manifold $\{\varphi(x)+\psi(y)\}$, where $\varphi(x)\in C(Q_x)$ and $\psi(y)\in C(Q_y)$. It is evident that the best approximation

$$E(f,Q)=\inf_{\varphi+\psi}\|f(x,y)-\varphi(x)-\psi(y)\|_{C(Q)}$$

depends not only on f, but also on Q. Sufficiently great number of interesting properties of $E(f,Q)$ considered as a functional of an approximated function f have been investigated for the last 50 years. One of these properties is the well-known property of semiadditivity:

$$E(f_1+f_2,Q)\le E(f_1,Q)+E(f_2,Q) \tag{1}$$

for any functions $f_1,\ f_2$ from $C(Q)$ (see [1], p.17).

In this work we begin to study the behaviour of $E(f,Q)$ as a function depending on an approximation set Q. For the sake of the clarity and simplicity of presentation let us denote the best approximation $E(f,Q)$ with fixed function f by $E_f(Q)$. First of all it is obvious that the function $E_f(Q)$ is defined on compact subsets of Q and not decreasing, i.e. if $Q_1\subset Q_2$ then $E_f(Q_1)\le E_f(Q_2)$. Further, inequality (1) raises the following fair question: Is any property of the kind (1) true for the $E_f(Q)$? In other words if there exist any order relation between $E_f(Q)$ and $E_f(Q_1)+E_f(Q_2)$, where $Q_1,\ Q_2$ are compact subsets of Q and $Q_2=\overline{Q\backslash Q_1}$ ($\bar{A}$ denotes a closue of A)? As we show none of inequalities

$$E_f(Q)\le E_f(Q_1)+E_f(Q_2)\,,$$

$$E_f(Q)\ge E_f(Q_1)+E_f(Q_2)\,,$$

is valid in general case for an arbitrary continuous function $f(x,y)$ defined on some compact set $Q\subset R^2$ and compact subsets $Q_1\subset Q\ ,\ \ Q_2=\overline{Q\backslash Q_1}$. But each of the relations

$$E_f(Q)<E_f(Q_1)+E_f(Q_2)\,,$$

$$E_f(Q)=E_f(Q_1)+E_f(Q_2)\,,$$

$$E_f(Q)>E_f(Q_1)+E_f(Q_2)$$

is valid for certain classes of continuous functions and sets Q , Q_1 , Q_2, which follows from the theorem given below. To formulate the main result we need the following definitions.

Let $\Pi = [a_1, b_1; a_2 b_2]$ be a closed rectangle.

Definition 1[2]**.** *We say that a continuous function $f(x, y)$ on Π belongs to the class $M(\Pi)$ if the inequality*

$$f(x', y') + f(x'', y'') - f(x', y'') - f(x'', y') \geq 0$$

is valid for any rectangle $\Pi'[x', x''; y', y''] \subset \Pi$.

Definition 2[3]**.** *We say that a function $f(x, y)$ from $M(\Pi)$ belongs to the class $M_c(\Pi)$, where $c \in (a_1, b_1]$, if the inequality*

$$f(c, y'') - f(c, y') \geq \frac{1}{2}\left[f(a_1, y'') - f(a_1, y') + f(b_1, y'') - f(b_1, y')\right]$$

is valid for any $y', y'' : a_2 \leq y' \leq y'' \leq b_2$.

Definition 3[3]**.** *We say that a continuous function $f(x, y)$ on Π belongs to the class $\Phi_c(\Pi)$, where $c \in (a_1, b_1]$, if the function*

$$\tilde{f}(x, y) = \begin{cases} f(x, y) - f(c, y) \ , & (x, y) \in \Pi_1[a_1, c; a_2 b_2] , \\ -f(x, y) + f(c, y) \ , & (x, y) \in \Pi_2[c, b_1; a_2 b_2] \end{cases}$$

is from $M_c(\Pi)$.

Some properties of classes $M(\Pi)$, $M_c(\Pi)$ and $\Phi_c(\Pi)$ and results related to these classes have been set out in [3].

Theorem. *Let Π , Π_1 , Π_2 be closed rectangles with sides parallel to coordinate axes, $\Pi_1 \subset \Pi$, $\Pi_2 \subset \Pi$. Then the following propositions are true:*

1) *if* $\Pi = \Pi_1 \cup \Pi_2$, $\Pi_2 = \overline{\Pi \backslash \Pi_1}$ *and* $f(x, y) \in M(\Pi)$ *then*

$$E_f(\Pi) = E_f(\Pi_1) + E_f(\Pi_2);$$

2) *if* $\Pi = \Pi_1 \cup \Pi_2$, $\underset{x}{pr}(\Pi_1 \cap \Pi_2) = c$ *and* $f(x, y) \in \Phi_c(\Pi)$ *then*

$$E_f(\Pi) < E_f(\Pi_1) + E_f(\Pi_2);$$

3) *if* $\Pi \neq \Pi_1 \cup \Pi_2$, $\Pi_1 \cup \Pi_2$ *contains all vertices of* Π *and* $f(x, y) \in M(\Pi)$ *then*

$$E_f(\Pi_1 \cup \Pi_2) > E_f(\Pi_1) + E_f(\Pi_2).$$

Proof. 1) Let $\Pi = [a_1, b_1; a_2, b_2]$ and Π_1, Π_2 be such rectangles that $\Pi_1 \cup \Pi_2 = \Pi$, $\Pi_2 = \overline{\Pi \backslash \Pi_1}$. It is clear that Π_1 and Π_2 are of the forms

$$\Pi_1 = [a_1, c_1; a_2, b_2] \ , \ \Pi_2 = [c_1, b_1; a_2, b_2] \ , \quad \text{where} \quad a_1 < c_1 < b_1 \tag{2}$$

or

$$\Pi_1 = [a_1, b_1; a_2, c_2] \ , \ \Pi_2 = [a_1, b_1; c_2, b_2] \ , \quad \text{where} \quad a_2 < c_2 < b_2. \tag{3}$$

For definiteness, let Π_1 and Π_2 be of the forms (2). By theorem 1 from [2]

$$E_f(\Pi_1) = \frac{1}{4}\left[f(a_1, a_2) + f(c_1, b_2) - f(a_1, b_2) - f(c_1, a_2)\right], \tag{4}$$

$$E_f(\Pi_2) = \frac{1}{4}\left[f(c_1,a_2) + f(b_1,b_2) - f(c_1,b_2) - f(b_1,a_2)\right], \tag{5}$$

Adding equalities (4) and (5) we obtain

$$E_f(\Pi_1) + E_f(\Pi_2) = \frac{1}{4}\left[f(a_1,a_2) + f(b_1,b_2) - f(a_1,b_2) - f(b_1,a_2)\right].$$

From the last equality on strength of the same theorem from [2] we get

$$E_f(\Pi_1) + E_f(\Pi_2) = E_f(\Pi).$$

2) Let $\Pi = [a_1,b_1;a_2,b_2]$ and $\Pi_1,\ \Pi_2$ be such rectangles that $\Pi = \Pi_1 \cup \Pi_2$, $\underset{x}{pr}(\Pi_1 \cap \Pi_2) = c$. Then it is clear that Π_1 and Π_2 are of the forms

$$\Pi_1 = [a_1,c;a_2,b_2]\ ,\quad \Pi_2 = [c,b_1;a_2,b_2] \quad \text{where} \quad a_1 < c < b_1.$$

By theorem 1 from [3]

$$E_f(\Pi) = \frac{1}{4}\left[f(a_1,a_2) + f(c,b_2) - f(a_1,b_2) - f(c,a_2)\right]. \tag{6}$$

On the strength of fact 2 from [3] the function $f(x,y)$ is from $M(\Pi_1)$. Hence by theorem 1 from [2]:

$$E_f(\Pi_1) = \frac{1}{4}\left[f(a_1,a_2) + f(c,b_2) - f(a_1,b_2) - f(c,a_2)\right]. \tag{7}$$

Taking into account (6), (7) and the fact that $E_f(\Pi_2) > 0$ (see [4]) we conclude

$$E_f(\Pi) < E_f(\Pi_1) + E_f(\Pi_2).$$

3) Let $\Pi = [a_1,b_1;a_2,b_2]$ and $\Pi_1,\ \Pi_2$ be such rectangles that $\Pi_1 \cup \Pi_2, \neq \Pi$, but the set $\Pi_1 \cup \Pi_2$ contains all verticies of Π. Then it is clear that Π_1 and Π_2 are of the forms

$$\Pi_1 = [a_1,c_1;a_2,b_2]\ ,\quad \Pi_2 = [d_1,b_1;a_2,b_2]\ ,\quad \text{where} \quad a_1 < c_1 < d_1 < b_1 \tag{8}$$

or

$$\Pi_1 = [a_1,b_1;a_2,c_2]\ ,\quad \Pi_2 = [a_1,b_1;d_2,b_2]\ ,\quad \text{where} \quad a_2 < c_2 < d_2 < b_2. \tag{9}$$

For definiteness let Π_1 and Π_2 be of forms (8). By the same way as in proof of proposition 1) one can show that

$$E_f(\Pi) = E_f(\Pi_1) + E_f(\Pi_2) + E_f(\Pi_3) \tag{10}$$

where $\Pi_3 = \overline{\Pi \backslash \Pi_1 \backslash \Pi_2}$.

Now show that

$$E_f(\Pi) = E_f(\Pi_1 \cup \Pi_2).$$

Consider the functional

$$L(f) = \frac{1}{4}\left[f(a_1,a_2) + f(b_1,b_2) - f(a_1,b_2) - f(b_1,a_2)\right]$$

It is not difficult to verify that the functional $L(f)$ annihilates functions of the form $\varphi(x)+\psi(y)$, i.e. $L(\varphi+\psi)=0$. So, $L(f)=L(f-\varphi-\psi)$. Then

$$L(f) \le \|f-\varphi-\psi\|_{C(\Pi_1 \cup \Pi_2)}, \text{ for any sum } \varphi+\psi.$$

As $\varphi+\psi$ is arbitrary, we deduce from here that

$$L(f) \le E_f(\Pi_1 \cup \Pi_2).$$

By theorem 1 from [2] $L(f)=E_f(\Pi)$. Thus

$$E_f(\Pi) \le E_f(\Pi_1 \cup \Pi_2). \tag{11}$$

on the other hand since $\Pi_1 \cup \Pi_2 \subset \Pi$,

$$E_f(\Pi_1 \cup \Pi_2) \le E_f(\Pi). \tag{12}$$

Considering (11) and (12) we obtain

$$E_f(\Pi) = E_f(\Pi_1 \cup \Pi_2).$$

Now from (10) we finally conclude that

$$E_f(\Pi_1 \cup \Pi_2) > E_f(\Pi_1) + E_f(\Pi_2).$$

The theorem has been proved.

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Vugar E. Ismailov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: 39-47-20 (off.).

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