Ilham T.MAMEDOV, Dagbeyi M.GULIYEV

REMOVABLE SETS OF SOLUTIONS OF THE SECOND ORDER DEGENERATE ELLIPTIC EQUATIONS OF DIVERGENCE STRUCTURE

Abstract

In the paper necessary and sufficient removability conditions of a compact set are established with respect to the first boundary value problem for the second order degenerate elliptic equations of divergence structure in the space of bounded functions.

Introduction. Let $\mathbb{E}_n$ be n-dimensional Euclidean space of points $x = (x_1, \cdots, x_n), n \geq 3$, D be a bounded domain in $\mathbb{E}_n$ with the boundary ∂D, $0 \in D$. Consider in D the elliptic operator

$$\mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

under the assumption that $\|a_{ij}(x)\|$ is a real symmetrical matrix with measurable in D elements, moreover, for all $\xi \in \mathbb{E}_n$ and almost all $x \in D$ the following condition is fulfilled

$$\gamma \sum_{i=1}^{n} \lambda_i(x) \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \gamma^{-1} \sum_{i=1}^{n} \lambda_i(x) \xi_i^2 . \tag{1}$$

Here $\gamma \in (0,1]$ is a constant, $\lambda_i(x) = (|x|_\alpha)^{\alpha_i}$, $|x|_\alpha = \sum_{i=1}^{n} |x_i|^{\frac{2}{2+\alpha_i}}$,

$$0 \leq \alpha_i < \frac{2}{n-1}; \quad i = 1, ..., n . \tag{2}$$

Denote by $W^1_{2,\alpha}(D)$ Banach space of functions $u(x)$ given on D with finite norm

$$\|u\|_{W^1_{2,\alpha}(D)} = \left(\int_D \left(u^2 + \sum_{i=1}^{n} \lambda_i(x) u_i^2 \right) dx \right)^{\frac{1}{2}} ,$$

where $u_i = \frac{\partial u}{\partial x_i}$; $i = 1, ..., n$. Further let $\overset{\circ}{W}{}^1_{2\alpha}(D)$ be supplement of the set of all functions from $C_0^\infty(D)$ over the norm of space $W^1_{2,\alpha}(D)$. Set of all bounded in D functions will be denoted by $\mathcal{M}(D)$.

The function $u(x) \in W^1_{2,\alpha}(D)$ is called generalized solution of the equation $\mathcal{L}u = f(x)$ on D if for any function $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ the following integral identity holds

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_i \eta_j dx = -\int_D f \eta dx . \tag{3}$$

Here $f(x)$ is given function from $L_2(D)$. Let $E \subset D$ be some compact set. Denote by $\mathcal{A}_E(D)$ the set of all functions $u(x) \in C^\infty(\bar{D})$ such that for each function there exists some neighbourhood of the compact E, in which $u(x) = 0$.

The function $u(x) \in W^1_{2,\alpha}(D\backslash E)$ is called generalized solution of the equation $\mathcal{L}u = f(x)$ in $D\backslash E$ vanishing on ∂D if integral identity (3) is fulfilled for any function $\eta(x) \in \mathcal{A}_E(D)$.

The compact set E is called removable with respect to the first boundary value problem for the operator $\mathcal{L}$ in space $\mathcal{M}(D)$ if any generalized solution of the equation $\mathcal{L}u = 0$ in $D\backslash E$ vanishing on ∂D and belonging to the space $M(D)$, equals to zero identically.

The aim of the present paper is finding necessary and sufficient removability conditions of the compact set E in above mentioned sense.

Note that for the second order uniformly elliptic operators of divergence structure the analogous results have been obtained in the papers [1]-[3]. As to uniformly degenerate elliptic equations of the second order we point at the papers [4]-[5]. We also mention the papers [6]-[9] where questions of removability of compact sets for solutions of second order elliptic equations of nondivergence structure were investigated and also the papers [10]-[11] devoted to removable sets of Neumann problem for divergence elliptic equations.

1. Some known results. We assume that the coefficients of the operator $\mathcal{L}$ are continued onto $\mathbb{E}_n\backslash D$ preserving conditions (1)-(2). For this it is sufficient, for example, to suppose $a_{ij}(x) = \delta_{ij}\lambda_i(x)$ for $x \in \mathbb{E}_n\backslash D;\ i,j = 1,...,n$, where δ_{ij} is Cronecker's symbol. Let further for $x^0 \in \mathbb{E}_n,\ R > 0$ and $k > 0$, $\mathcal{E}_{R;k}(x^0)$ be an ellipsoid $\left\{x : \sum\limits_{i=1}^{n} \frac{(x_i - x_i^0)^2}{R^{\alpha_i}} < (kR)^2\right\}$, $\mathcal{E}$ is an ellipsoid such that $\bar{D} \subset \mathcal{E}$, $\mathcal{B}(\mathcal{E})$ is the set of all functions satisfying on $\overline{\mathcal{E}}$ the uniform Lipschitz condition and vanishing near $\partial\mathcal{E}$. Denote by α vector $(\alpha_1,...,\alpha_n)$, and let $<\alpha>= \alpha_1 + ... + \alpha_n$.

We say that the function $u(x) \in \mathring{W}^1_{2,\alpha}(\mathcal{E})$ is nonnegative on $H \subset \mathcal{E}$ in sense $\mathring{W}^1_{2,\alpha}(\mathcal{E})$ if there exists a sequence of functions $\left\{u_{(m)}(x)\right\};\ m = 1,2,...;$ such that $u_{(m)}(x) \in \mathcal{B}(\mathcal{E})$, $u_{(m)}(x) \geq 0$ for $x \in H$ and $\lim\limits_{m\to\infty} \left\|u_{(m)} - u\right\|_{W^1_{2,\alpha}(\mathcal{E})} = 0$. It is easy to define the inequalities $u(x) \geq const$, $u(x) \geq v(x)\quad u(x) \leq 0$ and also the equality $u(x) = 1$ on the set H in the sense $\mathring{W}^1_{2,\alpha}(\varepsilon)$. In particular, $u(x) = 1$ on H in the sense $\mathring{W}^1_{2,\alpha}(\mathcal{E})$ if $u(x) \geq 1$ and $u(x) \leq 1$ simultaneously on H in the sense $\mathring{W}^1_{2,\alpha}(\mathcal{E})$.

Let $\omega(x)$ be a measurable in D function, finite and positive for almost all $x \in D$. Denote by $L_{p,\omega}(D)$ a Banach space of functions $u(x)$, given on D, with finite norm

$$\|u\|_{L_{p,\omega}(D)} = \left(\int\limits_D (\omega(x))^{\frac{p}{2}} |u|^p dx\right)^{\frac{1}{p}}; \quad 1 < p < \infty .$$

Finally, let $W^1_{p,\alpha}(D)$ be Banach space of functions $u(x)$, given on D, with finite norm

$$\|u\|_{W^1_{p,\alpha}(D)} = \left(\int\limits_D \left(|u|^p + \sum_{i=1}^{n} (\lambda_i(x))^{\frac{p}{2}} |u_i|^p\right) dx\right)^{\frac{1}{p}}; \quad 1 < p < \infty .$$

By analogy to $\mathring{W}^1_{2,\alpha}(\mathcal{E})$ the subspace $\mathring{W}^1_{p,\alpha}(D)$ for $p \in (1,\infty)$ is introduced. Space, conjugate to $\mathring{W}^1_{p,\alpha}(D)$, will be denoted by $W^{1,*}_{p,\alpha}(D)$. Everywhere below notation $C(...)$ means that the positive constant C depends only on what in the parentheses.

Let $h(x) \in W^1_{2,\alpha}(D)$, $f^0(x) \in L_2(D)$, $f^i(x) \in L_{2,\lambda_i^{-1}}(D)$; $i = 1, ..., n$ be given functions. Consider the first boundary value problem

$$\mathcal{L}u = f^0(x) + \sum_{i=1}^{n} \frac{\partial f^i}{\partial x_i}, \quad x \in D, \tag{4}$$

$$(u(x) - h(x)) \in \overset{\circ}{W}{}^1_{2,\alpha}(D). \tag{5}$$

Function $u(x) \in W^1_{2,\alpha}(D)$ is called a generalized solution of problem (4)-(5) if for any function $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ the integral identity

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_i \eta_j dx = \int_D \left(-f^0 \eta + \sum_{i=1}^{n} f^i \eta_i \right) dx$$

holds.

Theorem 1. ([12], [13]). *If the coefficients of the operator $\mathcal{L}$ satisfy conditions (1)-(2), then first boundary value problem (4)-(5) has a unique generalized solution $u(x)$ for any $h(x) \in W^1_{2,\alpha}(D)$, $f^0(x) \in L_2(D)$, $f^i(x) \in L_{2,\lambda_i^{-1}}(D)$; $i = 1, ..., n$. At that there exists $p_0(\alpha, n)$ such that if $p > p_0$, $h(x) \in W^1_{p,\alpha}(D)$, $f^0(x) \in L_p(D)$, $f^i(x) \in L_{p,\lambda_i^{-1}}(D)$; $i = 1, ..., n$, $\partial D \in C^1$, then solution $u(x)$ is Hölder continuous on D.*

Theorem 2. ([13]). *Let the coefficients of the operator $\mathcal{L}$ satisfy conditions (1)-(2). Then any generalized solution of the equation $\mathcal{L}u = 0$ on D is Hölder continuous in each strictly interior subdomain of domain D.*

Theorem 3 ([14]). *Let the coefficients of the operator $\mathcal{L}$ satisfy conditions (1)-(2) and $\overline{\mathcal{E}}_{R;1}(0) \subset D$. Then for any positive generalized solution $u(x)$ of equation $\mathcal{L}u = 0$ on D, Harnack type inequality*

$$\sup_{\mathcal{E}_{R;1}(0)} \leq C_1(\gamma, \alpha, n) \inf_{\mathcal{E}_{R;1}(0)} u \tag{6}$$

holds. Moreover if $y \in \partial\mathcal{E}_{R;2}(0)$ and $\overline{\mathcal{E}}_{R;1}(y) D$, then inequality of the form (6) is valid in the ellipsoid $\mathcal{E}_{R;1}(y)$.

Theorem 4. ([15]). *Let the coefficients of the operator $\mathcal{L}$ satisfy conditions (1)-(2), and $u(x)$ be a generalized solution of the first boundary value problem (4)-(5) at $f^i(x) \equiv 0$; $i = 0, ..., n$. If $h(x)$ is bounded on ∂D in the sense $W^1_{2,\alpha}(D)$, then for the solution $u(x)$ the following maximum principle is valid*

$$\inf_{\partial D} h \leq \inf_D u \leq \sup_D u \leq \sup_{\partial D} h,$$

where $\inf_{\partial D} h \left(\sup_{\partial D} h \right)$ is the exact upper (lower) bound of those numbers a, for which $h(x) \geq a$ $(h(x) \leq a)$ on ∂D in the sense $W^1_{2,d}(D)$.

2. Capacity, Green's function and fundamental solution. The statement in the present point follows the scheme of the paper [2].

Let $H \subset \mathcal{E}$ be some compact set, V_H be the set of all functions $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ such that $\varphi(x) \geq 1$ on H in the sense $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$. Consider the functional

$$J_{\mathcal{L}}(\varphi) = \int_{\mathcal{E}} \sum_{i,j=1}^{n} a_{ij}(x)\,\varphi_i \varphi_j dx \ ; \ \varphi(x) \in V_H .$$

Quantity $\inf\limits_{\varphi \in V_H} J_{\mathcal{L}}(\varphi)$ is called $\mathcal{L}$-capacity of the compact set H with respect to the ellipsoid $\mathcal{E}$ and is denoted by $cap_{\mathcal{L}}^{(\mathcal{E})}(H)$. In the case $\mathcal{E} = \mathbb{E}_n$ the corresponding quantity is called $\mathcal{L}$-capacity of the compact set H and is denoted by $cap_{\mathcal{L}}(H)$.

Lemma 1. *There exists a unique function $u(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ such that $u(x) \geq 1$ on H in the sense $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ and $cap_{\mathcal{L}}^{(\mathcal{E})}(H) = J_{\mathcal{L}}(u)$. Moreover, $u(x) = 1$ on H in the sense $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$.*

Proof. It is clear that V_H is a convex closed set in $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$. There exists a unique function $u(x) \in V_H$ on which exact lower bound of functional $J_{\mathcal{L}}(\varphi)$ is achieved because $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ is a Hilbert space. Further let $\{u(x)\}^1 = \left\{ \begin{array}{l} u(x)\,,\ if\ u(x) \leq 1 \\ 1,\ if\ u(x) > 1 \end{array} \right\}$. It is clear that $\{u(x)\}^1 \in \overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$. Moreover, $\{u(x)\}^1 \in V_H$. Denote by $A^+ = \{x : x \in \mathcal{E},\ \ u(x) > 1\}$. We have

$$J_L\left(\{u\}^1\right) = \left(\int_{A^+} + \int_{\mathcal{E}\backslash A^+}\right) \sum_{i,j=1}^{n} a_{ij}(x)\,\{u\}^1_i\,\{u\}^1_j\,dx = \int_{\mathcal{E}\backslash A^+} \sum_{i,j=1}^{n} a_{ij}(x)\,u_i u_j dx \ . \quad (7)$$

On the other hand, according to (1)

$$\int_{A^+} \sum_{i,j=1}^{n} a_{ij}(x)\,u_i u_j dx \geq 0 \ . \quad (8)$$

From (7) and (8) we conclude

$$J_{\mathcal{L}}\left(\{u\}^1\right) \leq J_{\mathcal{L}}(u) = \inf_{\varphi \in V_H} J_{\mathcal{L}}(\varphi)\,,$$

i.e. $J_{\mathcal{L}}\left(\{u\}^1\right) = J_{\mathcal{L}}(u)$. From the uniqueness of extreme function it follows that $\{u(x)\}^1 = u(x)$, and the lemma is proved. The function $u(x)$, on which exact lower bound of functional $J_{\mathcal{L}}(\varphi)$ on the set V_H is achieved, is called $\mathcal{L}$-capacitary potential of the compact set H with respect to the ellipsoid $\mathcal{E}$.

Lemma 2. *$\mathcal{L}$-capacitary potential $u(x)$ of compact set H with respect to $\mathcal{E}$ is generalized solution of the equation $\mathcal{L}u = 0$ on $\mathcal{E}\backslash H$. This solution equals to 0 on $\partial\mathcal{E}$ and to 1 on ∂H in the sense $W^1_{2,\alpha}(\mathcal{E})$.*

Proof. It is sufficient to show the validity of the first part of lemma's statement. Let $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ and $\eta(x) \geq 0$ on H in the sense $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$. Then for any $\varepsilon > 0$ $(u(x) + \varepsilon\eta(x)) \in V_H$. Therefore

$$J_{\mathcal{L}}(u + \varepsilon\eta) \geq J_{\mathcal{L}}(u) \ .$$

Thus,

$$J_{\mathcal{L}}(u)+\varepsilon^2 J_{\mathcal{L}}(\eta)+2\varepsilon\int\limits_{\mathcal{E}}\sum_{i,,j=1}^{n}a_{ij}(x)\,u_i\eta_j dx\geq J_{\mathcal{L}}(u)$$

i.e.

$$\varepsilon J_{\mathcal{L}}(\eta)+2\int\limits_{\mathcal{E}}\sum_{i,j=1}^{n}a_{ij}(x)\,u_i\eta_j dx\geq 0\ .$$

Tending ε to zero we conclude

$$\int\limits_{\mathcal{E}}\sum_{i,j=1}^{n}a_{ij}(x)\,u_i\eta_j dx\geq 0. \tag{9}$$

It is clear that as $\eta(x)$ in (9) we can take any function from $C^1\left(\overline{\mathcal{E}}\right)$ with compact support in $\mathcal{E}\backslash H$. Then

$$\int\limits_{\mathcal{E}\backslash H}\sum_{i,j=1}^{n}a_{ij}(x)\,u_i\eta_j dx\geq 0\ .$$

Replacing $\eta(x)$ by $-\eta(x)$, we come to equality

$$\int\limits_{\mathcal{E}\backslash H}\sum_{i,j=1}^{n}a_{ij}(x)\,u_i\eta_j dx=0\ .$$

The lemma is proved.

Let μ be charge of bounded variation given on $\mathcal{E}$. We will say that the function $u(x)\in L_1(\mathcal{E})$ is a weak solution of the equation $\mathcal{L}u=-\mu$, vanishing on $\partial\mathcal{E}$ if for any function $\varphi(x)\in \overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})\cap C\left(\overline{\mathcal{E}}\right)$, $\mathcal{L}\varphi(x)\in C\left(\overline{\mathcal{E}}\right)$ the following integral identity holds

$$\int\limits_{\mathcal{E}}u\mathcal{L}\varphi dx=\int\limits_{\mathcal{E}}\varphi d\mu\ .$$

According to theorem 1 (for $h=0$) there exists a continuous linear operator χ from $W^{1,*}_{2,\alpha}(\mathcal{E})$ to $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ such that for any functional $T\in W^{1,*}_{2,\alpha}(\mathcal{E})$, the function $u=\chi(T)$ is a unique in $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ generalized solution of the equation $\mathcal{L}u=T$.

The operator χ is called Green's operator.

By theorem 1 this operator at $p>p_0$ transforms $W^{1,*}_{2,\alpha}(\mathcal{E})$ to $C\left(\overline{\mathcal{E}}\right)$. It is clear that the function $u(x)$ is the weak solution of the equation $\mathcal{L}u=-\mu$, vanishing on $\partial\mathcal{E}$, iff for any function $\psi(x)\in C\left(\overline{\mathcal{E}}\right)$ the following integral identity holds

$$\int\limits_{\mathcal{E}}u\psi dx=\int\limits_{\mathcal{E}}\chi(\psi)\,d\mu\ . \tag{10}$$

By analogy to [2] one can show that for every measure μ on $\mathcal{E}$, there exists a unique weak solution of the equation $\mathcal{L}u=-\mu$, vanishing on $\partial\mathcal{E}$.

We say that charge $\mu\in W^{1,*}_{2,\alpha}(\mathcal{E})$ if there exists the vector

$$\bar{f}(x)=\left(f^0(x),f^1(x),...,f^n(x)\right)\ \left(f^0(x)\in L_2(\mathcal{E}),\ f^i(x)\in L_{2,\lambda_i^{-1}}(\mathcal{E});\ i=1,..,n\right)$$

such that for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E}) \cap C(\overline{\mathcal{E}})$ the integral identity

$$\mu(\varphi) = \int_{\mathcal{E}} \varphi d\mu = \int_{\mathcal{E}} \left(f^0 \varphi - \sum_{i=1}^{n} f^i \varphi_i \right) dx$$

holds.

At that it is obvious that

$$\left| \int_{\mathcal{E}} \varphi d\mu \right| \leq C_2 \left(\bar{f}\right) \|\varphi\|_{W^1_{2,\alpha}(\mathcal{E})} \ .$$

Lemma 3. *The weak solution $u(x)$ of the equation $\mathcal{L}u = -\mu$, vanishing on $\partial\mathcal{E}$, belongs to $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ iff $\mu \in W^{1,*}_{2,\alpha}(\mathcal{E})$.*

Proof. At first we will show that if the function $u(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$ satisfies integral identity

$$\int_{\mathcal{E}} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx = -\int_{\mathcal{E}} \varphi d\mu \tag{11}$$

for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E}) \cap C(\overline{\mathcal{E}})$, then it is a weak solution of the equation $\mathcal{L}u = -\mu$, vanishing on $\partial\mathcal{E}$. In fact, supposing $\varphi = \chi(\psi)$, $\psi(x) \in C(\overline{\mathcal{E}})$ we obtain subject to (11)

$$\int_{\mathcal{E}} \chi(\psi) d\mu = \int_{\mathcal{E}} \varphi d\mu = -\int_{\mathcal{E}} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx =$$

$$= \int_{\mathcal{E}} u \sum_{i,j=1}^{n} \left(a_{ij}(x) \varphi_j\right) = \int_{\mathcal{E}} u \mathcal{L} \varphi dx = {}_i dx \int_{\mathcal{E}} u \psi dx \ ,$$

and now it is sufficient to use identity (10). Let us show now that $\mu \in W^{1,*}_{2,\alpha}(\mathcal{E})$. For this it is sufficient to prove that if $f^j(x) = \sum\limits_{i=1}^{n} a_{ij}(x) u_i(x)$, then $f^j(x) \in L_{2,\lambda_j^{-1}}(\mathcal{E})$; $j = 1, ..., n$. In condition (1) we suppose $\xi_1 = ... = \xi_{i-1} = \xi_{i+1} = ... = \xi_n = 0$, $\xi_i = \frac{1}{\sqrt{\lambda_i(x)}}$. We obtain

$$\gamma \leq \frac{a_{ii}(x)}{\lambda_i(x)} \leq \gamma^{-1}; \quad i = 1, ..., n \ . \tag{12}$$

Let $i \neq j$. Assuming in (1) $\xi_k = 0$ for $k \neq i$ and $k \neq j$, $\xi_i = \frac{1}{\sqrt{\lambda_i(x)}}$, $\xi_j = \frac{1}{\sqrt{\lambda_j(x)}}$ we get

$$2\gamma \leq \frac{a_{ii}(x)}{\lambda_i(x)} + \frac{a_{jj}(x)}{\lambda_j(x)} + \frac{2a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}} \leq 2\gamma^{-1} \ .$$

Using (12) we conclude

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \leq \gamma^{-1} - \gamma; \quad i, j = 1, ..., n; \ i \neq j \ . \tag{13}$$

From (12) and (13) it follows that

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \le \gamma^{-1}; \quad i,j=1,...,n\ . \tag{14}$$

Thus, from (14) we derive for $j=1,...,n$.

$$\int_{\mathcal{E}} \frac{1}{\lambda_j(x)} \left(f^j\right)^2 dx = \int_{\mathcal{E}} \frac{1}{\lambda_j(x)} \left(\sum_{i=1}^n a_{ij}(x) u_i\right)^2 dx \le \gamma^{-2} n \sum_{i=1}^n \int_{\mathcal{E}} \lambda_i(x) u_i^2 dx < \infty\ .$$

Thus, $\mu \in W_{2,\alpha}^{1,*}(\mathcal{E})$. And v.v. if $u(x)$ is a weak solution of the equation $\mathcal{L}u = -\mu$ vanishing on $\partial\mathcal{E}$ and $\mu \in W_{2,\alpha}^{1,*}(\mathcal{E})$, then there exist functions $f^0(x) \in L_2(\mathcal{E})$, $f^i(x) \in L_{2,\lambda_i^{-1}}(\mathcal{E})$; $i=1,...,n$ such that

$$\int_{\mathcal{E}} \left(f^0\varphi - \sum_{i=1}^n f^i(\varphi_i)\right) dx = \int_{\mathcal{E}} \varphi d\mu = \int_{\mathcal{E}} u\mathcal{L}\varphi dx$$

$$= \int_{\mathcal{E}} u \sum_{i,j=1}^n \left(a_{ij}(x)\varphi_j\right)_i dx = -\int_{\mathcal{E}} \sum_{i,j=1}^n a_{ij}(x) u_i \varphi_j dx$$

for any function $\varphi(x) \in \overset{\circ}{W}{}_{2,\alpha}^1(\mathcal{E}) \cap C\left(\overline{\mathcal{E}}\right), \mathcal{L}\varphi(x) \in C\left(\overline{\mathcal{E}}\right)$.

Then from theorem 1 we obtain that $u(x) \in \overset{\circ}{W}{}_{2,\alpha}^1(\mathcal{E})$. The lemma is proved.

Let now $\delta(x)$ be Dirac measure concentrated at the point 0, y be arbitrary fixed point $\mathcal{E}$.

The weak solution $g(x,y)$ of the equation $\mathcal{L}g = -\delta(x-y)$, vanishing on $\partial\mathcal{E}$, is called Green's function of the operator $\mathcal{L}$ on $\mathcal{E}$.

In the case $\mathcal{E} = \mathbb{E}_n$ the corresponding function is called fundamental solution of the operator $\mathcal{L}$ and is denoted by $G(x,y)$.

According to proved above, if $\psi(x)$ is arbitrary function from $C\left(\overline{\mathcal{E}}\right)$, then generalized solution $\varphi(x) \in \overset{\circ}{W}{}_{2,\alpha}^1(\mathcal{E})$ of equation $\mathcal{L}\varphi = -\psi$ can be represented in the form

$$\varphi(y) = \int_{\mathcal{E}} g(x,y)\psi(\xi)dx\ .$$

One can show that $g(x,y)$ is nonnegative in $\mathcal{E}\times\mathcal{E}$, moreover, $g(x,y) = g(y,x)$.

Lemma 4. *For any charge of bounded variation on $\mathcal{E}$ integral*

$$u(x) = \int_{\mathcal{E}} g(x,y)d\mu(y)$$

exists, it is finite a.e. on $\mathcal{E}$. Moreover $u(x)$ is weak solution of the equation $\mathcal{L}u = -\mu$, vanishing on $\partial\mathcal{E}$.

Proof. Without loss of generality, we assume that charge μ is a measure in $\mathcal{E}$. Let $\psi(x) \in C\left(\overline{\mathcal{E}}\right)$, $\psi(x) \ge 0$ on $\mathcal{E}$. Denote by $\varphi(x) \in \overset{\circ}{W}{}_{2,\alpha}^1(\mathcal{E})$ generalized solution of equation $\mathcal{L}\varphi = -\psi(x)$. Then by theorem 1 $\varphi(x) \in C\left(\overline{\mathcal{E}}\right)$, and $\varphi(x) \ge 0$ according to theorem 4. At that

$$\varphi(y) = \int_{\mathcal{E}} g(x,y)\psi(x)dx\ .$$

Then by Fubini's theorem we conclude that the integral $\int\limits_{\mathcal{E}} g(x,y)\,d\mu(y)$ exists for almost all $x\in\mathcal{E}$, moreover,

$$\int\limits_{\mathcal{E}}\chi(\psi)\,d\mu(y)=\int\limits_{\mathcal{E}}\varphi(y)\,d\mu(y)=\iint\limits_{\mathcal{E}\times\mathcal{E}} g(x,y)\,\psi(x)\,dx d\mu(y)=\int\limits_{\mathcal{E}}\psi(x)\,u(x)\,dx\ . \quad (15)$$

Note that equality (15) is fulfilled for any nonnegative and continuous in $\bar{\varepsilon}$ function $\psi(x)$. Now it suffices to recall identity (10), and the lemma is proved.

Consider now $\mathcal{L}$-capacitary potential $u(x)$ of compact set H with respect to ellipsoid $\mathcal{E}$. It was earlier shown that $u(x)$ satisfies inequality (9) for any nonnegative on H function $\eta(x)\in C_0^{\infty}(\mathcal{E})$. By the Schwartz theorem [16] there exists measure μ on H such that

$$\int\limits_{\mathcal{E}}\sum_{i,j=1}^{n}a_{ij}(x)\,u_i\eta_j dx=\int\limits_{\mathcal{E}}\eta d\mu. \quad (16)$$

Further since $u=1$ on H in the sense $\overset{\circ}{W}{}^1_{2,\alpha}(\mathcal{E})$, then support of measure μ lies on ∂H. Measure μ is called $\mathcal{L}$-capacitary distribution of the compact set H. According to lemma 4, $\mathcal{L}$-capacitary potential $u(x)$ is a weak solution of the equation $\mathcal{L}u=-\mu$ vanishing on $\partial\mathcal{E}$, and it can be represented in the form

$$u(x)=\int\limits_{\mathcal{E}} g(x,z)\,d\mu(z)\,. \quad (17)$$

On the other hand, there exists sequence of functions $\left\{\eta^{(m)}(x)\right\};\ m=1,2,...$, such that $\eta^{(m)}(x)\in\mathcal{B}(\mathcal{E})\,,\ \eta^{(m)}(x)=1$ for $x\in H$ and $\lim\limits_{m\to\infty}\left\|\eta^{(m)}-u\right\|_{W^1_{2,\alpha}(\mathcal{E})}=0$. Putting $\eta^{(m)}(x)$ instead of $\eta(x)$ in equality (16), we conclude that its right hand side equals to $\mu(H)$ for any natural m, whereas the left hand side tends to $cap_{\mathcal{L}}^{(\mathcal{E})}(H)$ as $m\to\infty$. Thus,

$$cap_{\mathcal{L}}^{(\mathcal{E})}(H)=\mu(H)\,. \quad (18)$$

Lemma 5. *Let the coefficients of the operator $\mathcal{L}$ satisfy conditions (1)-(2), $y\in\partial\mathcal{E}_{R;2}(0)\,,\ \overline{\mathcal{E}}_{R;1}(y)\subset D,\ x\in\partial\mathcal{E}_{R;1}(y)$. Then for Greens function $g(x,y)$ the following estimates holds.*

$$C_3(\gamma,\alpha,n)\left[cap_{\mathcal{L}}^{(\mathcal{E})}\left(\overline{\mathcal{E}}_{R;1}(y)\right)\right]^{-1}\le g(x,y)\le C_4(\gamma,\alpha)\left[cap_{\mathcal{L}}^{(\mathcal{E})}\left(\overline{\mathcal{E}}_{R;1}(y)\right)\right]^{-1}. \quad (19)$$

If $\overline{\mathcal{E}}_{R;1}(0)\subset D,\ x\in\partial\mathcal{E}_{R;1}(0)$, then

$$C_3\left[cap_{\mathcal{L}}^{(\mathcal{E})}\left(\overline{\mathcal{E}}_{R;1}(0)\right)\right]^{-1}\le g(x,0)\le C_4\left[cap_{\mathcal{L}}^{(\mathcal{E})}\left(\overline{\mathcal{E}}_{R;1}(0)\right)\right]^{-1}. \quad (20)$$

Proof. Without loss of generality we assume that the coefficients of the operator $\mathcal{L}$ are infinitely differentiable on $\overline{\mathcal{E}}$. The general case is obtained by means of passage to the limit. Then for $x\neq y$ function $g(x,y)$ is continuous with respect to x and y, moreover,

$$\lim_{x\to y} g(x,y)=\infty. \quad (21)$$

Let a be a positive number, which will be chosen later, $K_a=\{x: g(x,y)\ge a\}$, where y is arbitrary fixed point on $\partial\mathcal{E}_{R;2}(0)$. From (21) it follows that y is an interior point of the compact set K_a. Then $\mathcal{L}$-capacitary potential of K_a, represented in the form (17), is continuous at the point y, that is it equals to 1 on it. Thus,

$$1 = \int\limits_{\mathcal{E}} g\left(y,z\right) d\mu_a\left(z\right) \ ,$$

where μ_a is L$\mathcal{L}$-capacitary distribution of the compact K_a. Taking into account that support of measure μ_a lies on ∂K_a, where $g\left(y,z\right) = a$, and using (18), we obtain

$$\mu_a\left(K_a\right) = cap_{\mathcal{L}}^{(\mathcal{E})}\left(K_a\right) = \frac{1}{a} \ . \tag{22}$$

Suppose now $a = \inf\limits_{x\in\partial\mathcal{E}_{R;1}(y)} g\left(x,y\right)$. By the maximum principle $\overline{\mathcal{E}}_{R;1}\left(y\right) \subset K_a$. Therefore, from (22) we conclude

$$cap_{\mathcal{L}}^{(\mathcal{E})}\left(\overline{\mathcal{E}}_{R;1}\left(y\right)\right) \le cap_{\mathcal{L}}^{(\mathcal{E})}\left(K_a\right) = \frac{1}{\inf\limits_{x\in\partial\mathcal{E}_{R;1}(y)} g\left(x,y\right)}. \tag{23}$$

If we suppose $b = \sup\limits_{x\in\partial\mathcal{E}_{R;1}(y)} g\left(x,y\right)$, then $\overline{\mathcal{E}}_{R;1}\left(y\right) \supset K_b$, i.e.

$$cap_{\mathcal{L}}^{(\mathcal{E})}\left(\overline{\mathcal{E}}_{R;1}\left(y\right)\right) \ge cap_{\mathcal{L}}^{(\mathcal{E})}\left(K_b\right) = \frac{1}{\sup\limits_{x\in\partial\mathcal{E}_{R;1}(y)} g\left(x,y\right)}. \tag{24}$$

From (23) and (24) it follows that

$$\inf_{x\in\partial\mathcal{E}_{R;1}(y)} g\left(x,y\right) \le \left[cap_{\mathcal{L}}^{(\mathcal{E})}\left(\overline{\mathcal{E}}_{R;1}\left(y\right)\right)\right]^{-1} \le \sup_{x\in\partial\mathcal{E}_{R;1}(y)} g\left(x,y\right). \tag{25}$$

On the other hand, according to theorem 3,

$$\sup_{x\in\partial\mathcal{E}_{R;1}(y)} g\left(x,y\right) \le C_5\left(\gamma,\alpha,n\right) \inf_{x\in\partial\mathcal{E}_{R;1}(y)} g\left(x,y\right). \tag{26}$$

Now required estimates (19) follow from (25) and (26). Analogously one can show that inequalities (20) are valid. The lemma is proved.

Corollary. *Let conditions of lemma be fulfilled, and* $y \in \partial\mathcal{E}_{R;2}\left(0\right),\ \overline{\mathcal{E}}_{R;1}\left(y\right) \subset D,\ x \in \partial\mathcal{E}_{R;1}\left(y\right)$ or $y = 0,\ \overline{\mathcal{E}}_{R;1}\left(0\right) \subset D,\ x \in \partial\mathcal{E}_{R;1}\left(0\right)$. *Then for fundamental solution* $G\left(x,y\right)$ *the following estimates hold*

$$C_3\left[cap_{\mathcal{L}}\left(\overline{\mathcal{E}}_{R;1}\left(y\right)\right)\right]^{-1} \le G\left(x,y\right) \le C_4\left[cap_{\mathcal{L}}\left(\overline{\mathcal{E}}_{R;1}\left(y\right)\right)\right]^{-1}. \tag{27}$$

3. Removability criteria of a compact set in space $\mathcal{M}\left(D\right)$

Theorem 5. *Let the coefficients of operator* $\mathcal{L}$ *satisfy conditions (1)-(2). Then for removability of a compact set* $E \subset D$ *with respect to the first boundary value problem for the operator* $\mathcal{L}$ *in the space* $\mathcal{M}\left(D\right)$ *it is necessary and sufficient that*

$$cap_{\mathcal{L}}\left(E\right) = 0. \tag{28}$$

Proof. Let the ellipsoid $\mathcal{E}$ have the same meaning as above. It is clear that if condition (28) is satisfied, then

$$cap_{\mathcal{L}}^{(\mathcal{E})}(E) = 0\ . \qquad (29) \tag{29}$$

Without loss of generality we can restrict ourselves to the case when the coefficients of the operator $\mathcal{L}$ are infinitely differentiable on $\overline{\mathcal{E}}$. We fix arbitrary $\varepsilon > 0$ and $x^0 \in D\backslash E$. By (29) there exists a neighbourhood H of the compact set E such that

$$cap_{\mathcal{L}}^{(\mathcal{E})}\left(\bar{H}\right) < \varepsilon. \tag{30}$$

At that we can assume ε so small that

$$dist\left(x^0, \bar{H}\right) \geq \frac{1}{2} dist\left(x^0, E\right). \tag{31}$$

Denote by $\upsilon_H(x)$ and μ_H $\mathcal{L}$-capacitary potential of the compact set $\bar{H}$ with respect to ellipsoid $\mathcal{E}$ and $\mathcal{L}$-capacitary distribution of $\bar{H}$, respectively. According to the proved above

$$\upsilon_H(x) = \int\limits_{\mathcal{E}} g(x,y)\, d\mu_H(y)\,,$$

at that function $\upsilon_H(x)$ is generalized solution of the equation $\mathcal{L}\upsilon_H = 0$ in $\mathcal{E}\backslash\bar{H}$, vanishing on $\partial\mathcal{E}$ and turning to 1 on ∂H in the sense $W_{2,\alpha}^1(\mathcal{E})$. Let now $u(x) \in \mathcal{M}(D)$ be arbitrary solution of the equation $\mathcal{L}u = 0$ on $D\backslash E$ vanishing on ∂D, $M = \sup\limits_D |u|$. It is clear, that the function $\upsilon_H(x)$ is nonnegative on ∂D in the sense $W_{2,\alpha}^1(D)$. Hence, the function $u(x) - M\upsilon_H(x)$ which is a generalized solution of the equation $\mathcal{L}u = 0$ on $D\backslash\bar{H}$, is nonpositive on $\partial\left(D\backslash\bar{H}\right)$. By the theorem 4 $u(x) - M\upsilon_H(x) \leq 0$ on $D\backslash H$ and, particularly,

$$u\left(x^0\right) \leq M\upsilon_H\left(x^0\right) \leq M \sup_{y\in\partial H} g\left(x^0,y\right)\mu_H\left(\bar{H}\right) = M \sup_{y\in\partial H} g\left(x^0,y\right) cap_{\mathcal{L}}^{(\mathcal{E})}\left(\bar{H}\right). \tag{32}$$

By continuity of the function $g(x,y)$ for $x \neq y$ and inequality (31) we obtain

$$\sup_{y\in\partial H} g\left(x^0,y\right) \leq C_6\left(\gamma,\alpha,n,x^0,E\right)\ .$$

Thus, from (30) and (32) we conclude

$$u\left(x^0\right) \leq MC_6\varepsilon\ .$$

Using the arbitrariness of ε, we come to the inequality

$$u\left(x^0\right) \leq 0\ . \tag{33}$$

Analogously reasoning for the function $u(x) + M\upsilon_H(x)$, we obtain

$$u\left(x^0\right) \geq 0. \tag{34}$$

From (33)-(34) and the arbitrariness of the point x^0 it follows that $u(x) \equiv 0$ in $D\backslash E$. By that the sufficiency of condition (28) is proved. Let us prove its necessity. Suppose $cap_{\mathcal{L}}(E) > 0$. Denote by $\mathcal{E}'$ ellipsoid such that $\overline{\mathcal{E}}' \subset \mathcal{E},\ E \subset \mathcal{E}'$. Suppose

$D = \mathcal{E}'$. It is clear that $cap_{\mathcal{L}}^{(\mathcal{E}')}(E) > 0$. Let further $u_E(x)$ and v_E be $\mathcal{L}$-capacitary potential of compact set E with respect to ellipsoid $\mathcal{E}'$ and $\mathcal{L}$-capacitary distribution of E, respectively. Following to [17] we can give the equivalent definition of Vallee-Poussin type of $\mathcal{L}$-capacity of the compact set E with respect to ellipsoid $\mathcal{E}'$. Let $g(x,y)$ be Green's function of operator $\mathcal{L}$ on $\mathcal{E}'$. We call measure μ on E $\mathcal{L}$-admissible if supp $\mu \subset E$, and

$$U_\mu^E(x) = \int_{\mathcal{E}'} g(x,y)\, d\mu(y) \le 1 \qquad \text{for} \qquad x \in \operatorname{supp} \mu. \tag{35}$$

Quantity $\sup \mu(E) = cap_{\mathcal{L}}^{(\mathcal{E}')}(E)$, where exact upper bound is taken over all $\mathcal{L}$-admissible measures is called $\mathcal{L}$-capacity of the compact set E with respect to the ellipsoid $\mathcal{E}'$.

$\mathcal{L}$-capacity $cap_{\mathcal{L}}(E)$ is defined analogously.

At that it is shown by standard method that there exists a unique measure, on which exact upper bound of functional $\mu(E)$ over the set of all $\mathcal{L}$-admissible measures μ is achieved. This measure is $\mathcal{L}$-capacitary distribution of the compact set E.

According to above mentioned function $u_E(x)$ is a generalized solution of the equation $\mathcal{L}u_E = 0$ on $\mathcal{E}' \backslash E$ vanishing on $\partial\mathcal{E}'$. Besides from (35) and maximum principle it follows that $u_E(x) \in \mathcal{M}(\mathcal{E}')$. On the other hand $u_E(x) \not\equiv 0$, because $v_H(E) > 0$. The theorem is proved.

Lemma 6. *Let the coefficients of the operator $\mathcal{L}$ satisfy condition (1). Then if $y \in \partial\mathcal{E}_{R;2}(0)$, then*

$$C_7(\gamma,\alpha,n)\, R^{n+\frac{<\alpha>}{2}-2} \le cap_{\mathcal{L}}\left(\overline{\mathcal{E}}_{R;1}(y)\right) \le C_8(\gamma,\alpha,n)\, R^{n+\frac{<\alpha>}{2}-2}. \tag{36}$$

Proof. Let $\mathcal{L}_0 = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\lambda_i(x)\frac{\partial}{\partial x_i}\right)$. Then according to (1)

$$\gamma\; cap_{\mathcal{L}_0}\left(\overline{\mathcal{E}}_{R;1}(y)\right) \le cap_{\mathcal{L}}\left(\overline{\mathcal{E}}_{R;1}(y)\right) \le \gamma^{-1} cap_{\mathcal{L}_0}\left(\overline{\mathcal{E}}_{R;1}(y)\right). \tag{37}$$

Let $u(x) \in C_0^\infty\left(\mathcal{E}_{R;\frac{3}{2}}(y)\right)$, $u(x) = 1$ for $x \in \mathcal{E}_{R;1}(y)$, moreover

$$|u_i(x)| \le \frac{C_9(\alpha,n)}{R^{1+\frac{\alpha_i}{2}}}; \quad i = 1,...,n. \tag{38}$$

Then

$$cap_{\mathcal{L}_0}\left(\overline{\mathcal{E}}_{R;1}(y)\right) \le \int_{\mathcal{E}_{R;\frac{3}{2}}(y)} \sum_{i=1}^{n} \lambda_i(x)\, u_i^2 dx. \tag{39}$$

On the other hand, since $y \in \partial\mathcal{E}_{R;2}(0)$, then $\sum_{i=1}^{n} \frac{y_i^2}{R^{\alpha_i}} = 4R^2$ and by that

$$|y_i| \le 2R^{1+\frac{\alpha_i}{2}}; \quad i = 1,...,n$$

Besides, since $x \in \mathcal{E}_{R;\frac{3}{2}}(y)$, then

$$|x_i - y_i| \le \frac{3}{2}R^{1+\frac{\alpha_i}{2}}; \quad i = 1,...,n.$$

Thus,

$$|x_i| \le |y_i| + |x_i - y_i| \le \frac{7}{2} R^{1+\frac{\alpha_i}{2}}; \quad i = 1, ..., n.$$

Hence,

$$|x|_\alpha \le R \sum_{i=1}^{n} \left(\frac{7}{2}\right)^{\frac{2}{2+\alpha_i}} = C_{10}(\alpha, n) R.$$

Therefore,

$$\lambda_i(x) \le C_{10}^{\alpha_i} R^{\alpha_i} \le C_{10}^{\alpha+} R^{\alpha_i}; \quad i = 1, ..., n, \tag{40}$$

where $\alpha^+ = \max\{\alpha_1, ..., \alpha_n\}$.

Taking into account (38) and (40) in (39) we obtain

$$cap_{\mathcal{L}_0}\left(\overline{\mathcal{E}}_{R;1}(y)\right) \le C_{10}(\alpha, n) R^{-2} mes\left(\mathcal{E}_{R;\frac{3}{2}}(y)\right) = C_{11}(\alpha, n) R^{n+\frac{<\alpha>}{2}-2},$$

and by virtue of (37) the upper estimate in (36) is proved.

In order to show the validity of the lower estimate in (36) we note that

$$cap_{\mathcal{L}_0}\left(\overline{\mathcal{E}}_{R;1}(y)\right) \ge cap_{\mathcal{L}_0}\left(\overline{\mathcal{E}}_{R;\frac{1}{2\sqrt{n}}}(y)\right). \tag{41}$$

Besides, reasoning by the same way as in [2], we conclude

$$cap_{\mathcal{L}_0}\left(\overline{\mathcal{E}}_{R;\frac{1}{2\sqrt{n}}}(y)\right) \ge C_{12}(\alpha, n)\, cap_{\mathcal{L}_0}^{(\mathcal{E}_0)}\left(\overline{\mathcal{E}}_{R;\frac{1}{2\sqrt{n}}}(y)\right), \tag{42}$$

where $\varepsilon_0 = \varepsilon_{R;\frac{1}{\sqrt{n}}}(y)$. Let

$$W = \left\{u(x) : u(x) \in C_0^\infty(\varepsilon_0),\ u(x) = 1 \text{ for } x \in \mathcal{E}_{R;\frac{1}{2\sqrt{n}}}(y)\right\}.$$

Then

$$cap_{\mathcal{L}_0}^{(\mathcal{E}_0)}\left(\mathcal{E}_{R;\frac{1}{2\sqrt{n}}}(y)\right) = \inf_{u \in W} \int_{\mathcal{E}_0} \sum_{i=1}^{n} \lambda_i(x) u_i^2 dx. \tag{43}$$

On the other hand, if $y \in \partial\mathcal{E}_{R;2}(0)$, then there exists $i_0,\ 1 \le i_0 \le n$ such that $y_{i_0}^2 \ge \frac{4R^{2+\alpha_{i_0}}}{n}$, i.e.

$$|y_{i_0}| \ge \frac{2R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}.$$

Besides, if $x \in \mathcal{E}_0$, then

$$|x_{i_0} - y_{i_0}| \le \frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}.$$

Therefore

$$|x_{i_0}| \ge |y_{i_0}| - |x_{i_0} - y_{i_0}| \ge \frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}.$$

By that

$$\lambda_i(x) \ge n^{-\frac{1}{2+\alpha_{i_0}}} R \ge n^{-\frac{1}{2+\alpha^-}} R; \quad i = 1, ..., n; \tag{44}$$

where $\alpha^- = \min\{\alpha_1, ..., \alpha_n\}$.

Taking into account (44) in (43) we get

$$cap_{\mathcal{L}_0}^{(\mathcal{E}_0)}\left(\overline{\mathcal{E}}_{R;\frac{1}{2\sqrt{n}}}(y)\right) \geq C_{13}(\alpha, n) \inf_{u\in W} \int\limits_{\mathcal{E}_0} \sum_{i=1}^{n} R^{\alpha_i} u_i^2 dx. \tag{45}$$

Denote by $B_R(z)$ the ball $\{x : |x-z| < R\}$. Let us make transformation of variables in (45) $\vartheta_i = \frac{x_i}{R^{1+\frac{\alpha_i}{2}}}$; $i = 1, ..., n$, and let $\tilde{y}$ be the image of the point y, $\tilde{W} = \left\{\tilde{u}(\vartheta) : \tilde{u}(\vartheta) \in C_0^\infty(B_0),\ \tilde{u}(\vartheta) = 1 \text{ for } \vartheta \in B_{\frac{1}{2\sqrt{n}}}(\tilde{y})\right\}$, where $B_0 = B_{\frac{1}{\sqrt{n}}}(\tilde{y})$. Then from (45) we derive

$$cap_{\mathcal{L}_0}^{(\mathcal{E}_0)}\left(\mathcal{E}_{R;\frac{1}{2\sqrt{n}}}(y)\right) \geq C_{13} R^{n+\frac{<\alpha>}{2}-2} \inf_{\tilde{u}\in\tilde{W}} \int\limits_{B_0} \sum_{i=1}^{n} \left(\frac{\partial\tilde{u}}{\partial\vartheta_i}\right)^2 d\vartheta =$$

$$= C_{13} R^{n+\frac{<\alpha>}{2}-2} cap^{(B_0)}\left(\bar{B}_{\frac{1}{2\sqrt{n}}}(\tilde{y})\right), \tag{46}$$

where we denote by $cap^{(B_0)}\left(\bar{B}_{\frac{1}{2\sqrt{n}}}(\tilde{y})\right)$ the Wiener capacity of the compact set $\bar{B}_{\frac{1}{2\sqrt{n}}}(\tilde{y})$ with respect to the ball B_0. Now it suffices to note that $cap^{(B_0)}\left(\bar{B}_{\frac{1}{2\sqrt{n}}}(\tilde{y})\right) = C_{14}(n)$, and the required estimate follows from (41), (42) and (46). The lemma is proved.

Lemma 7. *Let the coefficients of the operator $\mathcal{L}$ satisfy condition (1). Then*

$$C_{15}(\gamma, \alpha, n) R^{n+\frac{<\alpha>}{2}-2} \leq cap_{\mathcal{L}}\left(\overline{\mathcal{E}}_{R;1}(0)\right) \leq C_{16}(\gamma, \alpha, n) R^{n+\frac{<\alpha>}{2}-2}. \tag{47}$$

The upper estimate in (47) is proved similarly to the corresponding estimate in (36). To prove the lower estimate it suffices to note that $\mathcal{E}_{R;\frac{1}{4}}(\bar{y}) \subset \mathcal{E}_{R;1}(0)$, i.e.

$$cap_{\mathcal{L}}\left(\overline{\mathcal{E}}_{R;\frac{1}{4}}(\bar{y})\right) \leq cap_{\mathcal{L}}\left(\overline{\mathcal{E}}_{R;1}(0)\right),$$

where $\bar{y} = \left(\frac{1}{2}R^{1+\frac{\alpha_1}{2}}, 0, ..., 0\right)$ and we must repeat reasonings of the proof of the previous lemma

Corollary 1. *If conditions (1)-(2) are fulfilled, $y \in \partial\mathcal{E}_{R;2}(0)$, then for any $\rho \in (0, R]$ the following estimate holds*

$$cap_{\mathcal{L}}\left(\mathcal{E}_{\rho;1}(y)\right) \leq C_{17}(\gamma, \alpha, n) \rho^{n+\frac{<\alpha>}{2}-2} \left(1 + \sum_{i=1}^{n} \left(\frac{R}{\rho}\right)^{\alpha_i}\right). \tag{48}$$

In fact, let $\upsilon(x) \in C_0^\infty\left(\mathcal{E}_{\rho;\frac{3}{2}}(y)\right)$, $\upsilon(x) = 1$ for $x \in \mathcal{E}_{\rho;1}(y)$, moreover,

$$|\upsilon_i(x)| \leq \frac{C_{18}(\alpha, n)}{\rho^{1+\frac{\alpha_i}{2}}}; \quad i = 1, ..., n.$$

Then

$$cap_{\mathcal{L}}\left(\overline{\mathcal{E}}_{\rho;1}(y)\right) \leq \gamma^{-1} C_{18}^2 \rho^{-2} \int\limits_{\mathcal{E}_{\rho;\frac{3}{2}(y)}} \sum_{i=1}^{n} \lambda_i(x) \rho^{-\alpha_i} dx\ . \tag{49}$$

On the other hand, reasoning by the same way as in the proof of lemma 6, we come to the inequality

$$\lambda_i(x) \le C_{19}(\alpha, n)(R+\rho)^{\alpha_i}, \; x \in \mathcal{E}_{\rho;\frac{3}{2}}(y); \quad i = 1, ..., n. \tag{50}$$

Now it suffices to take into account that

$$\sum_{i=1}^{n}\left(1+\frac{R}{\rho}\right)^{\alpha_i} \le \sum_{i=1}^{n}\left[1+\left(\frac{R}{\rho}\right)^{\alpha_i}\right] \le n\left(1+\sum_{i=1}^{n}\left(\frac{R}{\rho}\right)^{\alpha_i}\right),$$

and the required estimate (48) follows from (49)-(50).

Corollary 2. *If conditions (1)-(2) are fulfilled and $y \neq 0$, then for $x \in \mathcal{E}_{d|y|_\alpha;1}(y)$; $x \neq y$ for fundamental solution $G(x,y)$ the following estimate holds*

$$G(x,y) \ge C_{20}(\gamma, \alpha, n)\frac{\left(|x-y|_\alpha\right)^{2-n-\frac{<\alpha>}{2}}}{1+\sum\limits_{i=1}^{n}\left(\frac{|y|_\alpha}{|x-y|_\alpha}\right)^{\alpha_i}}. \tag{51}$$

If $y = 0$, then estimate (51) holds for all $x \neq 0$. Here $d = \frac{1}{n 2^{\frac{2}{2+\alpha}}}$.

To prove it at first we show that if $y \neq 0$, then $y \notin \mathcal{E}_{d|y|_\alpha;2}(0)$. In fact, since

$$|y|_\alpha = \sum_{i=1}^{n}|y_i|^{\frac{2}{2+\alpha_i}}, \tag{52}$$

then there exists i_0, $1 \le i_0 \le n$ such that

$$|y_{i_0}|^{\frac{2}{2+\alpha_{i_0}}} \ge \frac{|y|_\alpha}{n}.$$

Thus,

$$\frac{y_{i_0}^2}{\left(|y|_\alpha\right)^{\alpha_{i_0}}} \ge \frac{\left(|y|_\alpha\right)^2}{n^{2+\alpha_{i_0}}}.$$

By that

$$\sum_{i=1}^{n}\frac{y_i^2}{\left(d|y|_\alpha\right)^{\alpha_i}} \ge \frac{y_{i_0}^2}{\left(d|y|_\alpha\right)^{\alpha_{i_0}}} \ge \frac{\left(d|y|_\alpha\right)^2}{(dn)^{2+\alpha_{i_0}}} = \frac{4\left(d|y|_\alpha\right)^2}{\left(2^{\frac{2}{2+\alpha_{i_0}}}dn\right)^{2+\alpha_{i_0}}}.$$

Now it suffices to note that $2^{\frac{2}{2+\alpha_{i_0}}}dn \le 2^{\frac{2}{2+\alpha^-}}dn = 1$, and the required statement is proved. On the other hand, from (52) it follows that for all i, $1 \le i \le n$

$$|y_i|^{\frac{2}{2+\alpha_i}} \le |y|_\alpha,$$

i.e.

$$\sum_{i=1}^{n}\frac{y_i^2}{\left(|y|_\alpha\right)^{\alpha_i}} \le n\left(|y|_\alpha\right)^2.$$

Thus, we showed that $y \in \mathcal{E}_{|y|_\alpha;\sqrt{n}}(0)$ if $y \neq 0$.

Let now for $y \neq 0$ $x \in \mathcal{E}_{d|y|_\alpha;1}(y)$ and $x \neq y$. Denote $|x-y|_\alpha$ by ρ. It is clear that there exists $i_1,\ 1 \le i_1 \le n$ such that

$$|x_{i_1} - y_{i_1}|^{\frac{2}{2+\alpha_{i_1}}} \ge \frac{\rho}{n}.$$

Hence,

$$\sum_{i=1}^{n} \frac{(x_i - y_i)^2}{\rho_{\alpha_i}} \ge \frac{(x_i - y_{i_1})^2}{\rho^{\alpha_1}} \ge \frac{\rho^2}{n^{2+\alpha_{i_1}}} \ge \frac{\rho^2}{n^{2+\alpha^+}}.$$

Thus, $x \notin \mathcal{E}_{\rho;d_1}(y)$, where $d_1 = \frac{1}{n^{1+\frac{\alpha^+}{2}}}$. Analogously one can show that $x \in \mathcal{E}_{\rho;\sqrt{n}}(y)$. Now required estimate (51) for $y \neq 0$ follows from (27) and corollary 1 to lemma 7. If $y = 0$, then (51) immediately follows from (27) and lemma 7.

Let $F(x,y)$ be a positive function defined in $\mathbb{E}_n \times \mathbb{E}_n$ continuous for $x \neq y$, moreover, $\lim\limits_{x\to y} F(x,y) = \infty$ (condition (A)).

Let further $E \subset \mathbb{E}_n$ be some compact set. We call measure μ on E $[F]$-admissible if supp $\mu \subset E$ and

$$V_\mu^E(x) = \int\limits_E F(x,y)\, d\mu(y) \le 1 \quad \text{for} \quad x \in \operatorname{supp} \mu.$$

Quantity $\sup \mu(E) = cap_{[F]}(E)$, where exact upper bound is taken over all $[F]$-admissible measures, is called $[F]$-capacity of the compact set E.

Theorem 6. *Let the coefficients of operator $\mathcal{L}$ satisfy conditions (1)-(2). Then for removability of the compact set $E \subset D$ with respect to the first boundary value problem for operator $\mathcal{L}$ in space $\mathcal{M}(D)$ it is sufficient that*

$$cap_{[F_1]}(E) = 0, \tag{53}$$

where

$$F_1(x,y) = \left[1 + \sum_{i=1}^{n} \left(\frac{|y|_\alpha}{|x-y|_\alpha}\right)^{\alpha_i}\right]^{-1} (|x-y|_\alpha)^{2-n-\frac{<\alpha>}{2}}.$$

Proof. We use the following statement proved in [7]. Let the function $F(x,y)$ satisfy condition (A), the compact set E have zero $[F]$-capacity, μ be nonzero measure concentrated on E. Then there exists a point x^0 $\in$supp μ such that $V_\mu^E(x^0) = \infty$. At that potential of measure μ cannot be bounded on any portion of supp μ, i.e. for arbitrary open set B, if E' =supp $\mu \cap B$, then potential $V_\mu^{E'}(x)$ is unbounded. In particular, if B is arbitrary neighbourhood of the point x^0, then $V_\mu^{E'}(x^0) = \infty$.

Let now condition (53) be fulfilled, μ be arbitrary nonzero measure, concentrated on E, x^0 $\in$supp μ be a point which corresponds to foregoing statement for $F = F_1$. At first suppose $x^0 \neq 0$. Then $|x^0|_\alpha = v > 0$. Let further B be so small neighbourhood of the point x^0, that if E' =supp $\mu \cap B$, then

$$\sup_{y\in E'} |y|_\alpha \le (1+)\,\varepsilon v, \qquad \inf_{y \in E'} |y|_\alpha \ge (1-\varepsilon)\, v,$$

where number $\varepsilon > 0$ will be chosen later. Consider ellipsoids $\mathcal{E}_{d|y|_\alpha;1}(y)$ for $y \in E'$. Now we choose ε so small that $x^0 \in \mathcal{E}_{d|y|_\alpha;1}(y)$ for all $y \in E'$. Then according to corollary 2 to lemma 7 we obtain

$$U_\mu^E\left(x^0\right)=\int\limits_E G\left(x^0,y\right)d\mu\left(y\right)\geq\int\limits_{E'} G\left(x^0,y\right)d\mu\left(y\right)\geq$$

$$\geq C_{20}\int\limits_{E'} F_1\left(x^0,y\right)d\mu\left(y\right)=C_{20}V_\mu^{E'}\left(x^0\right)=\infty.$$

Hence, it follows that no nonzero measure μ, concentrated on E, can be $\mathcal{L}$-admissible. Thus, $cap_{\mathcal{L}}\left(E\right)=0$, and the required statement follows from theorem 5.

Let now $x^0=0$. Then using the equality $G\left(x,y\right)=G\left(y,x\right)$ and corollary 2 to lemma 7, we conclude

$$U_\mu^E\left(0\right)=\int\limits_E G\left(0,y\right)d\mu\left(y\right)=\int\limits_E G\left(y,0\right)d\mu\left(y\right)\geq$$

$$\geq C_{20}\int\limits_E F_1\left(y,0\right)d\mu\left(y\right)=C_{20}\int\limits_E F_1\left(0,y\right)d\mu\left(y\right)=C_{20}V_\mu^E\left(0\right)=\infty.$$

The theorem is proved.

Remark. Let conditions of the present theorem be fulfilled, and the compact set $E\subset D$ be removable with respect to the first boundary value problem for the operator $\mathcal{L}$ in space $\mathcal{M}\left(D\right)$. Then $mes\left(E\right)=0$.

To prove it at first note that reasoning by the same way as in proof of estimate (51) one can show that at $x\in E_{d|y|_\alpha;1}\left(y\right),\ x\neq y\ \ (y\neq 0)$ and for any $x\neq y\ \ (y=0)$ estimate

$$G\left(x,y\right)\leq C_{21}\left(\gamma,\alpha,n\right)\left(|x-y|_\alpha\right)^{2-n-\frac{<\alpha>}{2}}\ . \tag{54}$$

holds. Further, analogously to theorem 6 it is shown that if the compact set E is removable, then according to (54) $cap_{[F_2]}\left(E\right)=0$, where $F_2\left(x,y\right)=\left(|x-y|_\alpha\right)^{2-n\frac{<\alpha>}{2}}$. Hence, if $mes\left(E\right)>0$, then there exists a point $x^1\in E$ such that $V^E\left(x^1\right)=\infty$, where

$$V^E\left(x\right)=\int\limits_E F_2\left(x,y\right)dy.$$

Moreover, if B' is an arbitrary neighbourhood of the point $x^1,\ E'=B'\cap E$, then the potential $V^{E'}\left(x\right)$ is unbounded on E'. We restrict ourselves to the case $x^1\neq 0$. We choose such a small neighbourhood B' of the point x^1, that for all $x\in E',\ y\in E'$ the inequalities $|x_i-y_i|\leq 1;\quad i=1,...,n$ are satisfied. For $x\in E'$ we have

$$V^{E'}\left(x\right)=\int\limits_{E'}\left(\sum_{i=1}^n|x_i-y_i|^{\frac{2}{2+\alpha_i}}\right)^{2-n-\frac{<\alpha>}{2}}dy\leq\int\limits_{E'}\left(\sum_{i=1}^n|x_i-y_i|\right)^{2-n-\frac{<\alpha>}{2}}\leq$$

$$\leq\int\limits_{E'}|x-y|^{2-n-\frac{<\alpha>}{2}}dy\leq\int\limits_{B''}|z|^{2-n-\frac{<\alpha>}{2}}dy,$$

where B'' is a ball with the radius $\sqrt{n}$ centered at the origin. Now it suffices to note that according to condition (2) $\frac{<\alpha>}{2}<\frac{n}{n-1}\leq\frac{3}{2}$, and the statement of the corollary is proved.

References

[1]. Carleson L. *Selected problems on exceptional sets.* // "D. van Nostrand Company", Toronto-London-Melbourne, 1967.

[2]. Littman W., Stampacchia G., Weinberger H.F. *Regular points for elliptic equations with discontinuous coefficients.* // Ann. Scuola Norm. Sup. Pisa, 1963, v.17, pp.45-79.

[3]. Egorov Yu.V. *On removable singularities in boundary conditions for differential equations.* // Vestnik MGU, ser.1, 1985, No6, pp.30-36. (Russian)

[4]. Chanillo S., Wheeden R.L. *Harnack's inequality and mean-value inequalitites for solutions of degenerate elliptic equations.* // Comm. Part. Diff. Equat., 1986, No11, pp.1111-1134.

[5]. Channillo S., Wheeden R.L. *Existence and estimates of Green's function for degenerated equations.* // Ann. Scuoula Norm. Sup. Pisa, 1988, v.15, pp.309-340.

[6]. Landis E.M. *To question on uniqueness of solution of the first boundary value problem for elliptic and parabolic equations of second order.* // Uspekhi Mat. Nauk, 1978, v.33, No3, p.15. (Russian)

[7]. Gaidenko S.V. *On exceptional boundary sets and uniqueness of solutions of the Dirichlet problem for elliptic equation of second order.* // Matem. Sbornik, 1980, v.111(153), pp.116-134. (Russian)

[8]. Mamedov I.T. *On exceptional sets of solutions of Gilbarg-Serrin equation in the space of Hölder functions.* // Izvestiya AN Azerb., ser. phys.-tech. mat. nauk, 1998, v.XVIII, No2, pp.46-51. (Russian)

[9]. Mamedov I.T., Mamedova V.A. *On exceptional sets of solutions of the second order elliptic equations in non-divergence from.* // Proceedings of Inst. Math. Mech. Nat. Acad. Sci. Azerb., 2002, v.XVII(XXV), pp.115-121.

[10]. Moiseev E.I. *On Neumann problem in piece-wise smooth domain.* // Diff. Uravneniya, 1971, v.VII, No9, pp.1655-1656. (Russian)

[11]. Moiseev E.I. *On exceptional and unexceptional boundary sets of Neumann problem.* // Diff. Uravneniya, 1973, v.IX, No5, pp.901-911. (Russian)

[12]. Guseynov S.T. *On solvability of Dirichlet's problem for non-uniformly degenerating divergent elliptic equations of second order.* // Trans. Acad. Sci. Azerb., ser. phys.-tech. math. sci., 1998, v.XVIII, No3-4, pp.41-47.

[13]. Mamedov I.T., Guseynov S.T. *Interior apriori estimate of the Hölder's norm for the solutions of non-uniformly degenerate second order elliptic equations.* // Trans. Acad. Sci. Azerb., ser. phys.-tech. math. sci, 1999, v.XIX, No1-2, pp.108-121.

[14]. Guseynnov S.T. *The Harnack's type inequality for solutions of non-uniformly degenerate second order elliptic equations.* // Doklady AN Azerb., 1999, v.LV, No3-4, pp.3-10 (Russian)

[15]. Guseynov S.T. *The regularity test of a boundary point for non-uniformly degenerate second order elliptic equations.* // Proceedings of Inst. Math. Mech. Acad. Sci. Azerb., 1999, v.XI(XIX), pp.49-61.

[16]. Schwartz L. *Theorie des Distributions.* // "Hermann", Paris, 1950 (v.1), 1951 (v.2).

[17]. Landkof N.S. *Foundations of modern potential theory.* // "Nauka", Moscow, 1966. (Russian)

lham T.Mamedov, Dagbeyi M.Guliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., Az1141, Baku, Azerbaijan.
Tel.: (99412)393924, (99412)394720.

E-mail: ilham@lan.ab.az

Received March 4, 2003; Revised June 3, 2003.
Translated by Azizova R.A.