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# NEUMANN PROBLEM FOR SECOND ORDER NONDIVERGENT ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

**Abstract**

The second boundary value problem for second order elliptic equations of nondivergent structure with, generally speaking, discontinuous coefficients is considered. It is assumed that the matrix of the principal part of equations satisfies Cordes condition. Strong (almost everywhere) solvability of this problem is established in the corresponding Sobolev space.

**Introduction.** Let $\mathbf{E}_n$ be $n$-dimensional Euclidean space of points $x=(x_1,...,x_n)$, $D$ be a bounded domain in $\mathbf{E}_n$ with boundary $\partial D\in C^2$. Consider in $D$ Neumann problem

$$\mathcal{L}u=\sum_{i,j=1}^{n}a_{ij}(x)\,u_{x_ix_j}+\sum_{i=1}^{n}b_i(x)\,u_{x_i}=f(x)\,,\quad x\in D \tag{1}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial D}=0\ , \tag{2}$$

where $\frac{\partial}{\partial n}$ is a derivative in the direction of outward normal to $\partial D$.

We assume that the coefficients of operator $\mathcal{L}$ are real, measurable in $D$ functions and, moreover, $\|a_{ij}(x)\|$ is symmetric matrix, at that for almost all $x\in D$ and for any $\xi\in\mathbf{E}_n$ condition

$$\gamma|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x)\,\xi_i\xi_j\le\gamma^{-1}|\xi|^2 \tag{3}$$

is satisfied with the constant $\gamma\in(0,1]$. It also assumed that

$$\sigma=\sup_{x\in D}\frac{\sum\limits_{i,j=1}^{n}a_{ij}^2(x)}{\left[\sum\limits_{i=1}^{n}a_{ii}(x)\right]^2}<\frac{1}{n-1}\ , \tag{4}$$

$$b_i(x)\in L_s(D)\,;\quad i=1,...,n, \tag{5}$$

where $s=n$ for $n>2$, $s=2+v$ with some positive $v$ for $n=2$. Condition (4) is called Cordes condition and it is understood to within equivalence and nondegenerate linear transformation in the following sense: equation (1) can be replaced by equivalent equation $\mathcal{M}u=F$, domain $D$ can be covered by a finite number of subdomains $D^{(1)},...,D^{(M)}$ so that in each $D^{(i)}$ there exists nondegenerate transformation of coordinates at which image of operator $\mathcal{M}$ satisfies condition of the form (4) in image of $D^{(i)}$; $i=1,...,M$. The aim of the present paper is to prove strong (almost everywhere) solvability of boundary value problem (1)-(2) in special subspace of space $W_2^2(D)$ for any $f(x)\in L_2(D)$. Note that questions of classical solvability of Neumann problem for elliptic equations with smooth coefficients were studied in papers of Schechter [1]-[3], Lopatinsky [4], Shapiro [5] and Agmon, Douglis and Nirenberg [6]. As to strong solvability of the above mentioned problem for equations with continuous coefficients, we note in this connection Ladyzhenskaya's papers [7]-[8] and monograph [9]. We also mention papers [10]-[13] in which Dirichlet problem for nondivergence elliptic and parabolic equations of the second order under Cordes condition has been investigated. At last we note papers [14]-[15] in which under the condition similar to (4), strong solvability of mixed boundary value problem is proved for a class of second order nonlinear parabolic equations of nondivergent structure.

**1. Some auxiliary estimates.** At first, we agree in some denotations and definitions. We denote by $u_i$ and $u_{ij}$ derivatives $u_{x_i}$ and $u_{x_i x_j}$ respectively; $i,j = 1,...,n$. Let for $x^0 \in \mathbf{E}_n$, $R > 0$ $\quad B_R\left(x^0\right)$ be a ball $\left\{x : |x - x^0| < R\right\}$. Let further $W_2^1\left(D\right)$ and $W_2^2\left(D\right)$ be Banach spaces of functions $u\left(x\right)$ given on $D$ with finite norms

$$\|u\|_{W_2^1(D)} = \left(\int\limits_D \left(u^2 + \sum_{i=1}^n u_i^2\right) dx\right)^{\frac{1}{2}}$$

and

$$\|u\|_{W_2^2(D)} = \left(\int\limits_D \left(u^2 + \sum_{i=1}^n u_i^2 + \sum_{i,j=1}^n u_{ij}^2\right) dx\right)^{\frac{1}{2}}$$

respectively.

Let give meaning to condition (2) for functions from $W_2^2\left(D\right)$. We will say that function $u\left(x\right) \in W_2^2\left(D\right)$ belongs to subspace $\overset{\vee}{W}{}_2^2\left(D\right)$, if for any function $\eta\left(x\right) \in W_2^1\left(D\right)$ the following equality holds

$$\int\limits_D \eta \Delta u dx = -\int\limits_D \sum_{i=1}^n \eta_i u_i dx \ ,$$

where $\Delta$ is Laplace operator.

In other words, $\overset{\vee}{W}{}_2^2\left(D\right)$ is subspace $W_2^2\left(D\right)$ in which the set of all functions $u\left(x\right) \in C^\infty\left(\bar{D}\right)$ for which $\frac{\partial u}{\partial n}\big|_{\partial D} = 0$ is dense set. Function $u\left(x\right) \in \overset{\vee}{W}{}_2^2\left(D\right)$ is called strong solution of second boundary value problem (1)-(2) if it satisfies equation (1) a.e. in $D$.

It is clear that boundary value problem (1)-(2) cannot be uniquely solvable in ordinary sense, because its solution (if it exists) is defined within a constant addend. Therefore, in order to give meaning to the notion of uniqueness of solution of problem (1-2), we impose additional condition solution

$$u_D = \oint\limits_D u dx = 0, \tag{6}$$

where

$$\oint_D u dx = \frac{1}{mesD} \int_D u dx.$$

Strong solution of second boundary value problem (1)-(2) satisfying condition (6), is called its normalized solution. Everywhere later on notation $C(\cdots)$ means that the positive constant $C$ depends only on contents of brackets.

**Lemma 1.** *Let $\bar{B}_R(x^0) \subset D$.Then for any function $u(x) \in C_0^\infty(B_R(x^0))$ the following equality holds*

$$\int_{B_R(x^0)} \sum_{i,j=1}^n u_{i\ j}^2 dx = \int_{B_R(x^0)} (\Delta u)^2 dx. \tag{7}$$

**Proof.** Everywhere later on for simplicity we will denote ball $B_R(x^0)$ simply by $B$. We have

$$\int_B (\Delta u)^2 dx \int_B \sum_{i,j=1}^n u_{ii} u_{jj} dx = -\int_B \sum_{i,j=1}^n u_i u_{jji}\ dx = \int_B \sum_{i,j=1}^n u_{ij}^2\ dx,$$

and the required equality (7) is proved.

Let $\delta = \sup\limits_{x \in D} (\sum\limits_{i,j=1}^n (a_{ij}(x) - \delta_{ij})^2)^{\frac{1}{2}}, \mathcal{L}' = \sum\limits_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$, where $\delta_{ij}-$ is a Cronecker's symbol.

**Lemma 2.** *Let $\bar{B}_R(x^0) \subset D$.That time if $\delta < 1$, then for any function $u(x) \in C_0^\infty \left(B_R\left(x^0\right)\right)$ estimate*

$$\int_{B_R(x^0)} \sum_{i,j=1}^n u_{ij}^2 dx \leq C_1(\delta) \int_{B_R(x^0)} \left(\mathcal{L}' u\right)^2 dx \tag{8}$$

*holds.*

**Proof.** According to lemma 1 we have

$$\left(\int_B \sum_{i,j=1}^n u_{ij}^2 dx\right)^{\frac{1}{2}} = \|\Delta u\|_{L_2(B)} \leq \|\mathcal{L}' u\|_{L_2(B)} + \|(\Delta - \mathcal{L}') u\|_{L_2(B)} \leq$$

$$\leq \|\mathcal{L}' u\|_{L_2(B)} + \delta \left(\int_B \sum_{i,j=1}^n u_{ij}^2 dx\right)^{\frac{1}{2}}.$$

Now it suffices to use condition $\delta < 1$, and the required estimate (8) is proved.

**Lemma 3.** *Let $\bar{B}_R\left(x^0\right) \subset D$. At that time if $\delta < 1$ and $R \leq 1$, then for any function $u(x) \in C_0^\infty \left(B_R\left(x^0\right)\right)$ estimate*

$$\|u\|_{W_2^2(B_R(x^0))} \leq C_2(\delta) \left\|\mathcal{L}' u\right\|_{L_2(B_R(x^0))} \tag{9}$$

*holds.*

**Proof.** Let $\Pi_R$ be a hypercube $\left\{x : |x_i - x_i^0| < R;\ i = 1, ..., n\right\}$. We continue function $u(x)$ by zero into $\Pi_R \backslash B$ and again denote the continued function by

$u(x)$. It is clear that $u(x) \in C_0^{\infty}(\Pi_R)$. Denote $(x_2, ..., x_n)$ by $x'$. Then for $x_1 \in \left(x_1^0 - R, x_1^0 + R\right)$ we have

$$u\left(x_1, x'\right) = \int_{x_1^0 - R}^{x_1} u_1\left(t, x'\right) dt \ .$$

Thus,

$$u^2\left(x_1, x'\right) \le 2R \int_{x_1^0 - R}^{x_1} u_1^2\left(t, x'\right) dt \le 2R \int_{x_1^0 - R}^{x_1^0 + R} u_1^2\left(t, x'\right) dt \ .$$

Therefore

$$\int_{\Pi_R} u^2 dx \le 4R^2 \int_{\Pi_R} u_1^2 dx \le 4R^2 \int_{\Pi_R} \sum_{i=1}^{n} u_i^2 dx \ .$$

Now it suffices to remember that $u(x) = 0$ for $x \in \Pi_R \backslash B$ and condition $R \le 1$. Then

$$\int_B u^2 dx \le 4 \int_B \sum_{i=1}^{n} u_i^2 dx \ . \tag{10}$$

Quite analogously we obtain

$$\int_B \sum_{i=1}^{n} u_i^2 dx \le 4 \int_B \sum_{i,j=1}^{n} u_{ij}^2 dx \ , \tag{11}$$

and required estimate (9) follows from (10), (11) and lemma 2.

**Lemma 4.** *If the coefficients of operator $\mathcal{L}'$ satisfy supposition (5), then condition (4) coincides with condition $\delta < 1$ to within equivalence.*

**Proof.** Let us devide both sides of equation (1) by function $\lambda \sum_{i=1}^{n} a_{ii}(x)$ where $\lambda = \sup_{x \in D} \sum_{i,j=1}^{n} a_{ij}^2(x)$. By virtue of (3) the obtained equation will be equivalent to equation (1). At that condition $\delta^2 < 1$ equivalent to condition $\delta < 1$ (because $\delta \ge 0$) will be written in the form

$$\frac{\sup_{x \in D} \sum_{i,j=1}^{n} a_{ij}^2(x)}{\lambda^2} - \frac{2}{\lambda} < 1 - n. \tag{12}$$

But it is clear that conditions (4) and (12) coincide. The lemma is proved.

For $\varphi(x) \in L_p(D)$, $p \in [1, \infty)$ quantity

$$æ_{\varphi;p}(\omega) = \sup_{e \in D, mes\ e \le \omega} \left( \int_e |\varphi|^p dx \right)^{\frac{1}{p}}$$

is called $AC$-modulus of function $\varphi$ of order $p$. Everywhere later on notation $C(\mathcal{L})$ means that the positive constant $C$ depends only on $\gamma, \sigma, v$,$æ_{b_1;s}$, ...,$æ_{b_n;s}$ and $\sum_{i=1}^{n} \|b_i\|_{L_s(D)}$.

**Lemma 5.** *Let coefficients of operator $\mathcal{L}$ satisfy conditions (3)-(5). There exists $R_0(\mathcal{L},n)$ such that if $\bar{B}_R(x^0)\subset D$ and $R\le R_0$, then for any function $u(x)\in C_0^\infty(B_R(x^0))$ estimate*

$$\|u\|_{W_2^2(B_R(x^0))}\le C_3(\gamma,\sigma)\|\mathcal{L}u\|_{L_2(B_R(x^0))} \tag{13}$$

*holds.*

**Proof.** We will use the following Sobolev imbedding theorem (see [9]): for any function $u(x)\in W_2^1(D)$ inequality

$$\|u\|_{L_q(D)}\le C_4(q,n)\|u\|_{W_2^1(D)} \tag{14}$$

holds. At that $q\in[1,\frac{2n}{n-2}]$ if $n>2$; $q\in[1,\infty)$ if $n=2$.

From (9) we have

$$\|u\|_{W_2^2(B)}\le C_2\|\mathcal{L}u\|_{L_2(B)}+C_2\sum_{i=1}^n\|b_iu_i\|_{L_2(B)}\ . \tag{15}$$

Let at first $n>2$. Assuming in (14) $u_i$ instead of $u$ $(i=1,...,n)$ and $q=\frac{2n}{n-2}$, we obtain

$$\sum_{i=1}^n\|b_iu_i\|_{L_2(B)}\le\sum_{i=1}^n\|b_i\|_{L_n(B)}\|u_i\|_{L_{\frac{2n}{n-2}}(B)}\le$$
$$\le C_4(n)\|u\|_{W_2^2(B)}\sum_{i=1}^n\|b_i\|_{L_n(B)}\le C_4(n)\|u\|_{W_2^2(B)}\sum_{i=1}^n æ_{b_i;n}(\omega_{R_0}), \tag{16}$$

where $\omega_{R_0}=\Omega_nR_0^n$, $\Omega_n=mesB_1(0)$.

If $n=2$, then we put $u_i$ instead of $u$ $(i=1,2)$ into (14) and $q=\frac{2(2+v)}{v}$. Then

$$\sum_{i=1}^2\|b_iu_i\|_{L_2(B)}\le\sum_{i=1}^2\|b_i\|_{L_{2+v}(B)}\|u_i\|_{L_{\frac{2(2+v)}{v}}(B)}\le C_4'(v)\|u\|_{W_2^2(B)}\times$$
$$\times\sum_{i=1}^2\|b_i\|_{L_2(B)}\le C_4'(v)\|u\|_{W_2^2(B)}\sum_{i=1}^2 æ_{b_i;2+v}(\omega_{R_0})\ . \tag{17}$$

Let

$$A(R_0)=\begin{cases}C_4(n)\sum_{i=1}^n æ_{b_i;n}(\omega_{R_0}), & if\quad n>2\\ C_4'(v)\sum_{i=1}^2 æ_{b_i;2+v}(\omega_{R_0}), & if\quad n=2\ .\end{cases}$$

From conditions (5) it follows that $\lim\limits_{R_0\to0}A(R_0)=0$. We choose and fix so small $R_0\le1$ that $A(R_0)\le\frac{1}{2C_2}$. Then the required estimate (13) follows from (15)-(17) with $C_3=2C_2$. The lemma is proved.

**Lemma 6.** *Let the coefficients of operator $\mathcal{L}$ satisfy conditions (3)-(5). If $\bar{B}_R(x^0)\subset D$ and $R\le R_0$, then for any function $u(x)\in C^\infty(\bar{B}_R(x^0))$ and arbitrary $\varepsilon>0$, estimate*

$$\|u\|_{W_2^2\left(B_{\frac{R}{2}}(x^0)\right)}\le C_3\|\mathcal{L}u\|_{L_2(B_R(x^0))}+$$
$$+\varepsilon\|u\|_{W_2^2(B_R(x^0))}+\frac{C_5(\mathcal{L},n)}{\varepsilon R^2}\|u\|_{L_2(B_R(x^0))} \tag{18}$$

*holds.*

**Proof.** Denote $B_{\frac{R}{2}}\left(x^0\right)$ by $B'$. We introduce function $\zeta(x)\in C_0^{\infty}(B)$ such that $\zeta(x)=1$ in $B'$, $0\le\zeta(x)\le 1$ and

$$|\xi_i|\le\frac{C_6}{R},\quad |\zeta_{ij}|\le\frac{C_6}{R^2};\quad i,j=1,...,n. \tag{19}$$

Applying lemma 5 to function $u(x)\,\zeta(x)$, we obtain

$$\|u\|_{W_2^2(B')}\le C_3\,\|\mathcal{L}(u\zeta)\|_{L_2(B)}\ . \tag{20}$$

But on the other hand,

$$\mathcal{L}(u\zeta)=\zeta\mathcal{L}u+u\mathcal{L}\zeta+\in\sum_{i,j=1}^{n}a_{ij}(x)\,u_i\zeta_j\ . \tag{21}$$

Besides, subject to (19)

$$|u\mathcal{L}\zeta|\le|u|\left(\sum_{i,j=1}^{n}|a_{ij}(x)|\ |\zeta_{ij}|+\sum_{i=1}^{n}|b_i(x)|\ |\zeta_i|\right)\le$$

$$\le\frac{C_6|u|}{R^2}\sum_{i,j=1}^{n}|a_{ij}(x)|+\frac{C_6|u|}{R}\sum_{i=1}^{n}|b_i(x)|\ .$$

Assuming in condition (3) $\zeta=(0,...,1,...,0)$, where 1 is on $i$-th place, we obtain

$$\gamma\le a_{ii}(x)\le\gamma^{-1};\quad i=1,...,n. \tag{22}$$

If we suppose in (3) $\zeta=(0,...,1,...,1,...,0)$, where 1 is on $i$-th and $j$-th places, then

$$2\gamma\le 2a_{ij}(x)+a_{ij}(x)+a_{jj}(x)\le 2\gamma^{-1}\ ,$$

or subject to (22),

$$|a_{ij}(x)|\le\gamma^{-1}-\gamma;\quad i,j=1,...,n;\ i\ne j \tag{23}$$

Thus, from (22) and (23) we conclude

$$|a_{ij}(x)|\le\gamma^{-1};\quad i,j=1,...,n\ .$$

Hence,

$$\|u\mathcal{L}\zeta\|_{L_2(B)}\le\frac{C_6\gamma^{-1}n^2}{R^2}\,\|u\|_{L_2(B)}+\frac{C_6}{R}\sum_{i=1}^{n}\|b_iu\|_{L_2(B)}\,. \tag{24}$$

We confine ourselves to the case $n>2$. Plane case is considered analogously. Applying inequality (14) at $q=\frac{2n}{n-2}$, we obtain

$$\begin{gathered}\|u\mathcal{L}\zeta\|_{L_2(B)}\le\frac{C_7(\gamma,n)}{R^2}\,\|u\|_{L_2(B)}+\frac{C_6}{R}\sum_{i=1}^{n}\|b_i\|_{L_n(B)}\,\|u\|_{L_{\frac{2n}{n-2}}(B)}\le\\ \le\frac{C_7}{R^2}\,\|u\|_{L_2(B)}+\frac{C_6C_4(n)}{R}J\,\|u\|_{W_2^1(B)}\ ,\end{gathered} \tag{25}$$

where $J=\sum\limits_{i=1}^{n}\|b_i\|_{L_n(D)}$. Further subject to (3) and (19) we have

$$2\left|\sum_{i,j=1}^{n}a_{ij}\left(x\right)u_i\zeta_j\right|\leq 2\left(\sum_{i,j=1}^{n}a_{ij}\left(x\right)u_iu_j\right)^{\frac{1}{2}}\left(\sum_{i,j=1}^{n}a_{ij}\left(x\right)\zeta_i\zeta_j\right)^{\frac{1}{2}}\leq\frac{2\gamma^{-1}nC_6}{R}\left(\sum_{i=1}^{n}u_i^2\right)^{\frac{1}{2}}.$$

Therefore

$$2\left\|\sum_{i,j=1}^{n}a_{ij}u_i\zeta_j\right\|_{L_2(B)}\leq\frac{2\gamma^{-1}nC_6}{R}\|u\|_{W_2^1(B)}\ . \tag{26}$$

Now from (20), (21), (25) and (26) we conclude

$$\|u\|_{W_2^2(B')}\leq C_3\|\mathcal{L}u\|_{L_2(B)}+\frac{C_8\left(\mathcal{L},n\right)}{R}\|u\|_{W_2^1(B)}+\frac{C_3C_7}{R^2}\|u\|_{W_2^1(B)}\,. \tag{27}$$

Now we use interpolation inequality (see [9]), according to which for any $\varepsilon'>0$

$$\|u\|_{W_2^1(B)}\leq\varepsilon'\|u\|_{W_2^2(B)}+\frac{C_9\left(n\right)}{\varepsilon'}\|u\|_{L_2(B)}\ . \tag{28}$$

Let us fix arbitrary $\varepsilon>0$. Without loss of generality one can assume that $\varepsilon\leq 1$. Then by means of inequality (28) at $\varepsilon'=\frac{\varepsilon R}{C_8}$ from (27) we obtain

$$\|u\|_{W_2^2(B')}\leq C_3\|\mathcal{L}u\|_{L_2(B)}+\varepsilon\|u\|_{W_2^2(B)}+\frac{C_8^2C_9+C_3C_7}{\varepsilon R^2}\|u\|_{L_2(B)}\ ,$$

and the required estimate (18) is proved.

Let for $\rho>0\quad D_\rho=\{x:x\in D,\ dist\,(x,\partial D)>\rho\}$ .

**Corollary.** *If conditions (3)-(5) are fulfilled for the coefficients of operator* $\mathcal{L}$ *then for any function* $u\left(x\right)\in C^\infty\left(\bar{D}\right)$ *for any* $\varepsilon>0$ *and sufficiently small* $\rho>0$ *estimate*

$$\|u\|_{W_2^2(D_\rho)}^2\leq C_{10}\left(\gamma,\sigma,n\right)\|\mathcal{L}u\|_{L_2(D)}^2+\varepsilon^2\|u\|_{W_2^2(D)}^2+\frac{C_{11}\left(\mathcal{L},n,\rho\right)}{\varepsilon^2}\|u\|_{L_2(D)}^2 \tag{29}$$

*holds.*

To prove (29) without loss of generality we assume that $\rho\leq R_0$. Let us cover $D_\rho$ by a countable system of balls $\left\{B_{\frac{\rho}{2}}\left(x^m\right)\right\}$, $x^m\in D_\rho$; $m=1,2,...$; of finite multiplicity $l\left(n\right)$. At that one can assume that the covering by system $\left\{B_\rho\left(x^m\right)\right\}$; $m=1,2,...$; also has finite multiplicity $l_1\left(n\right)$. According to (18) for any $\varepsilon'>0$ and for any natural $m$, the following inequality holds

$$\begin{gathered}\|u\|_{W_2^2\left(B_{\frac{\rho}{2}}(x^m)\right)}^2\leq 3C_3^2\|\mathcal{L}u\|_{L_2(B_\rho(x^m))}^2+\\ +3\left(\varepsilon'\right)^2\|u\|_{W_2^2(B_\rho(x^m))}^2+\tfrac{3C_5^2}{(\varepsilon')^2\rho^4}\|u\|_{L_2(B_\rho(x^m))}^2\,.\end{gathered} \tag{30}$$

We fix arbitrary $\varepsilon>0$ and suppose $\varepsilon'=\frac{\varepsilon}{\sqrt{3l_1}}$. Summing now inequalities (30) over all natural $m$, we arrive at the required estimate (29).

**2. Main coercive inequalities.** For $\rho > 0$ denote the set $D\backslash D_\rho$ by $D'_\rho$. Everywhere later on notation $C(\partial D)$ means that the positive constant $C$ depends only on smoothness of surface $\partial D$.

**Lemma 7.** *Let conditions (3)-(5) be satisfied for the coefficients of operator $\mathcal{L}$. Then there exists $\rho_1(\gamma,\sigma,n,\partial D)$ so that for any function $u(x)\in C^\infty(\bar{D})$, $\frac{\partial u}{\partial n}\big|_{\partial D} = 0$, and arbitrary $\varepsilon > 0$ the following estimate holds*

$$\|u\|^2_{W_2^2(D'_{\rho_1})} \le C_{12}(\gamma,\sigma,n,\partial D)\|\mathcal{L}u\|^2_{L_2(D)} + \varepsilon^2\|u\|^2_{W_2^2(D)} +$$

$$+\frac{C_{13}(\mathcal{L},n,\partial D, diamD)}{\varepsilon^2}\|u\|^2_{L_2(D)}. \tag{31}$$

**Proof.** If suffices to consider the case, when the coefficients of operator $\mathcal{L}$ are infinitely differentiable in $\bar{D}$. Let us fix arbitrary $x^0\in\partial D$ and $\varepsilon > 0$. There exists orthogonal transformation of coordinates $x\to y$ so that the tangent hyperplane to $\partial\tilde{D}$ at the point $y^0$ is perpendicular to $y_n$ axis. Here $\tilde{D}$ and $y^0$ are images of domain $D$ and point $x^0$ respectively. Denote by $\tilde{\mathcal{L}}$ image of operator $\mathcal{L}$. For simplicity we will assume that intersection of $\partial\tilde{D}$ with some ball $B_h = B_{h(y^0)}(y^0)$ is given by equation $y_n = \psi(y_1,...,y_{n-1})$, at that $\psi\in C^2$, and the part of $\tilde{D}$ adjacent to $\partial\tilde{D}\cap B_h$ is on the set $\{y : y_n > \psi(y_1,...,y_{n-1})\}$. Let for $i = 1,...,n$ $\lambda_i(x)$ be eigenvalues of matrix $A(x) = \|a_{ij}(x)\|$. Then $TrA(x) = \sum\limits_{i=1}^n a_{ii}(x) = \sum\limits_{i=1}^n \lambda_i(x)$, $\sum\limits_{i,j=1}^n a^2_{ii}(x) = TrA^2(x) = \sum\limits_{i=1}^n \bar{\lambda}_i(x)$, where $\bar{\lambda}_i(x)$ are eigenvalues of matrix $A^2(x)$; $i = 1,...,n$. It is clear that $\bar{\lambda}_i(x) = \lambda_i^2(x)$; $i = 1,...,n$. Hence, condition (4) may be written in the following equivalent form

$$\sigma = \sup_{x\in D}\frac{\sum\limits_{i=1}^n \lambda_i^2(x)}{\left[\sum\limits_{i=1}^n \lambda_i(x)\right]^2} < \frac{1}{n-1}.$$

Thus, if matrix $A(x)$ satisfies condition (4) with constant $\sigma$, then image of this matrix at orthogonal transformation also satisfies condition of the form (4) with the same constant $\sigma$ (and together with it satisfies condition (3) with constant $\gamma$). Besides conditions of the form (5) are fulfilled for the minor coefficients, moreover, their $AC$-moduli of order $s$ and sum of norms in $L_s\left(\tilde{D}\right)$ are majorized by quantities depending only on $AC$-moduli of order $s$ of functions $b_i(x)$ and constant $\sum\limits_{i=1}^n \|b_i\|_{L_s(D)}$ respectively. Let us make one more transformation of coordinates $y\to \upsilon$ in the following way: $\upsilon_1 = y_1,...,\upsilon_{n-1} = y_{n-1}, \upsilon_n = y_n - \psi(y_1,...,y_{n-1})$. Denote by $\bar{\mathcal{L}}$ image of operator $\tilde{\mathcal{L}}$ and by $\bar{a}_{ij}(\upsilon)$ the leading coefficients of operator $\bar{\mathcal{L}}$; $i,j = 1,...,n$. It is clear that $\bar{\mathcal{L}}$ is operator of the form $\mathcal{L}$, moreover, its minor coefficients satisfy conditions of the form (5) and their $AC$-moduli of order $s$ and sum of norms in $L_s\left(\check{D}\right)$are majorized by quantities depending only on $AC$-moduli of functions $b_i(x)$, constant $\sum\limits_{i=1}^n \|b_i\|_{L_s(D)}$ and smoothness of boundary $\partial D$. Here $\check{D}$ is the image of domain $\tilde{D}$. Besides, if $\tilde{a}_{ij}(y)$ are the leading coefficients of operator $\tilde{\mathcal{L}}$, then

$$\bar{a}_{ij}(\upsilon)=\sum_{k,l=1}^{n}\tilde{a}_{kl}(y)\frac{\partial\upsilon_i}{\partial y_k}\frac{\partial\upsilon_j}{\partial y_l};\quad i,j=1,...,n\ .$$

Thus,

$$\bar{a}_{ij}(\upsilon)=\begin{cases}\tilde{a}_{ij}(y)\,; & if\quad 1\le i,j\le n-1\\ \tilde{a}_{nj}(y)-\sum\limits_{k=1}^{n-1}\tilde{a}_{kj}(y)\,\psi_{y_k}; & if\quad i=n,\quad 1\le j\le n-1\\ \tilde{a}_{nn}(y)-\sum\limits_{k,l=1}^{n-1}\tilde{a}_{kl}(y)\,\psi_{y_k}\psi_{y_l}-2\sum\limits_{k=1}^{n-1}\tilde{a}_{kn}(y)\,\psi_{y_k}; & if\quad i=j=n\ .\end{cases}$$

Let further $\upsilon^0$ be image of a point $y^0$, $\bar{y}=(y_1,...,y_{n-1})$. Taking into account inequalities $\psi_{y_i}(\bar{y}^0)=0;\ i=1,...,n-1$, we conclude that there exists $v(\upsilon^0,\gamma,\sigma,n,\partial D)$ so small that if conditions (3) and (4) are fulfilled for matrix $\|\tilde{a}_{ij}(y)\|$ with constants $\gamma$ and $\sigma$, then the same conditions are fulfilled for matrix $\|\bar{a}_{ij}(y)\|$ with constants $\frac{\gamma}{2}$ and $\frac{\sigma+\frac{1}{n-1}}{2}$ respectively if $\upsilon\in\check{D}\cap B_{v(\upsilon^0)}(\upsilon^0)$. Here for brevity we denote $v(\upsilon^0,\gamma,\sigma,n,\partial D)$ by $v(\upsilon^0)$.

Let $\bar{u}(\upsilon)$ be image of function $u(x)$ after transformations $x\to y$ and $y\to\upsilon$. It is clear that $\check{D}\cap B_{v(\upsilon^0)}(\upsilon^0)$ is a semiball $B^+_{v(\upsilon^0)}=\{\upsilon:|\upsilon-\upsilon^0|<v(\upsilon^0),\upsilon_n>0\}$. We continue function $\bar{u}(\upsilon)$ and the coefficients of operator $\bar{\mathcal{L}}$ in an even way through the hyperplane $\upsilon_n=0$ into semiball $B^-_{v(\upsilon^0)}(\upsilon^0)=B_{v(\upsilon^0)}(\upsilon^0)\backslash\bar{B}^+_{v(\upsilon^0)}(\upsilon^0)$ and denote continued function and operator again by $\bar{u}(\upsilon)$ and $\bar{\mathcal{L}}$ respectively. According to the hypothesis $\bar{u}_{\upsilon_n}|_{\upsilon_n=0}=0$ and therefore $\bar{u}(\upsilon)\in W_2^2(B_{v(\upsilon^0)})$. According to lemma 6, for any $\varepsilon'>0$

$$\|\bar{u}\|^2_{W_2^2\left(B_{\frac{v(\upsilon^0)}{2}}(\upsilon^0)\right)}\le 3C_3^2\left\|\bar{\mathcal{L}}\bar{u}\right\|^2_{L_2\left(B_{v(\upsilon^0)}(\upsilon^0)\right)}+3(\varepsilon')^2\|\bar{u}\|^2_{W_2^2\left(B_{v(\upsilon^0)}(\upsilon^0)\right)}+$$

$$+\frac{3C_5^2}{(\varepsilon')^2v^4(\upsilon^0)}\|\bar{u}\|^2_{L_2\left(B_{v(\upsilon^0)}(\upsilon^0)\right)}\ .$$

Taking into account the character of continuation of function $\bar{u}(\upsilon)$ and the coefficiewnts of operator $\bar{\mathcal{L}}$, we conclude

$$\|\bar{u}\|^2_{W_2^2\left(B^+_{\frac{v(\upsilon^0)}{2}}(\upsilon^0)\right)}\le 3C_3^2\left\|\bar{\mathcal{L}}\bar{u}\right\|^2_{L_2\left(B^+_{v(\upsilon^0)}(\upsilon^0)\right)}+3(\varepsilon')^2\|\bar{u}\|^2_{W_2^2\left(B^+_{v(\upsilon^0)}(\upsilon^0)\right)}+$$

$$+\frac{3C_5^2}{(\varepsilon')^2v^4(\upsilon^0)}\|\bar{u}\|^2_{L_2\left(B^+_{v(\upsilon^0)}(\upsilon^0)\right)}\ ,$$

where $B^+_{\frac{v(\upsilon^0)}{2}}(\upsilon^0)=\left\{\upsilon:|\upsilon-\upsilon^0|<\frac{v(\upsilon^0)}{2},\ \upsilon_n>0\right\}$. Returning to the coordinates $x$, we obtain

$$\|u\|^2_{W_2^2(\check{B}'(x^0))}\le C_{14}(\gamma,\sigma,n,\partial D)\|\mathcal{L}u\|^2_{L_2(\check{B}(x^0))}+C_{15}(n,\partial D)(\varepsilon')^2\times$$
$$\times\|u\|^2_{W_2^2(\check{B}(x^0))}+\frac{C_{16}(\mathcal{L},n,\partial D)}{(\varepsilon')^2v^4(\upsilon^0)}\|u\|^2_{L_2(\check{B}(x^0))}\ ,\qquad (32)$$

where $\check{B}'\left(x^0\right)$ and $\check{B}\left(x^0\right)$ are inverse images of hemispheres $B^{+}_{\frac{v(v^0)}{2}}\left(v^0\right)$ and $B^{+}_{v(v^0)}\left(v^0\right)$ respectively. By definition of the boundary of class $C^2$ (see [9]) there exists $\rho_1\left(\gamma,\sigma,n,\partial D\right)$ such that $D'_{\rho_1}$ can be covered by a finite number $N$ of domains of the form $\check{B}'\left(x^i\right)$; $i=0,...,N-1$. At that $N=N\left(\gamma,\sigma,n,\partial D\right)$. Denoting $\min\limits_{i=0,...,N-1}\left\{v\left(v^i\right)\right\}$ by $v^0$, copying out inequalities of the form (32) for domains $\check{B}'\left(x^i\right)$ and $\check{B}\left(x^i\right)$ respectively (replacing $\left(v\left(v^i\right)\right)$ by $v^0$), summing them over $i$ from 0 to $N-1$ and choosing $\varepsilon'=\frac{\varepsilon}{\sqrt{C_{15}N}}$ we arrive at the requred estimate (31). The lemma is proved.

**Remark.** It is clear that $v^0=\beta\left(n,\partial D\right)diamD$.

**Theorem 1.** *If conditions (3)-(5) are satisfied for the coefficients of operator $\mathcal{L}$, then for any functions $u\left(x\right)\in\check{W}^2_2\left(D\right)$ the following estimate holds*

$$\left\|u\right\|_{W^2_2(D)}\leq C_{17}\left(\gamma,\sigma,n,\partial D\right)\left\|\mathcal{L}u\right\|_{L_2(D)}+C_{18}\left(\mathcal{L},n,\partial D,diam\ D\right)\left\|u\right\|_{L_2(D)}. \quad (33)$$

**Proof.** If sufficient to consider the case $u\left(x\right)\in C^{\infty}\left(\bar{D}\right)$, $\left.\frac{\partial u}{\partial n}\right|_{\partial D}=0$. Let us sum inequalities (29) (at $\rho=\rho_1$) and (31). We obtain

$$\left\|u\right\|^2_{W^2_2(D)}\leq\left(C_{17}+C_{12}\right)\left\|\mathcal{L}u\right\|^2_{L_2(D)}+2\varepsilon^2\left\|u\right\|^2_{W^2_2(D)}+\frac{C_{11}+C_{13}}{\varepsilon^2}\left\|u\right\|^2_{L_2(D)}.$$

Not it suffices to choose $\varepsilon=\frac{1}{2}$, and the required estimate (33) is proved.

**Theorem 2.** *Let conditions (3)-(5) be fulfilled for the coefficients of operator $\mathcal{L}$. Then there exists $d\left(\mathcal{L},n,\partial D\right)$ so that if $mesD\leq d$, then for any function $u\left(x\right)\in\check{W}^2_2\left(D\right)$, $u_D=0$, the following estimate holds*

$$\left\|u\right\|_{W^2_2(D)}\leq C_{19}\left(\gamma,\sigma,n,\partial D\right)\left\|\mathcal{L}u\right\|_{L_2(D)}. \quad (34)$$

**Proof.** It suffices to consider the case $u\left(x\right)\in C^{\infty}\left(\bar{D}\right)$, $\left.\frac{\partial u}{\partial n}\right|_{\partial D}=0$, $u_D=0$. At first let us prove an auxiliary fact: for any function from the above mentioned class, inequality

$$\int\limits_D\left(\Delta u\right)^2dx=\int\limits_D\sum_{i,j=1}^n u^2_{ij}dx \quad (35)$$

is satisfied. In fact,

$$\int\limits_D\left(\Delta u\right)^2dx=\int\limits_D\sum_{i,j=1}^n u_{ii}u_{jj}dx=-\int\limits_D\sum_{i,j=1}^n u_{iij}u_jdx+\int\limits_{\partial D}\sum_{i,j=1}^n u_{ii}u_j\cos\left(n,x_j\right)ds\ .$$

On the other hand,

$$\int\limits_{\partial D}\sum_{i,j=1}^n u_{ii}u_j\cos\left(n,x_j\right)ds=\int\limits_{\partial D}\sum_{i=1}^n u_{ii}\sum_{j=1}^n u_j\cos\left(n,x_j\right)ds=\int\limits_{\partial D}\Delta u\frac{\partial u}{\partial n}ds=0\ .$$

Therefore,

$$\begin{gathered}\int\limits_D\left(\Delta u\right)^2dx=-\int\limits_D\sum_{i,j=1}^n u_{iij}u_jdx=\\ \int\limits_D\sum_{i,j=1}^n u^2_{ij}dx-\int\limits_{\partial D}\sum_{i,j=1}^n u_{ij}u_j\cos\left(n,x_j\right)ds\ .\end{gathered} \quad (36)$$

Let us fix point $x^0 \in \partial D$ and consider expression $I\left(x^0\right) = \sum\limits_{i,j=1}^{n} u_{ij} u_j \cos\left(n, x_i\right)\Big|_{x=x^0}$.
We have

$$I\left(x^0\right) = \frac{1}{2}\sum_{i,j=1}^{n}\left(u_j^2\right)_i \cos\left(n, x_i\right) = \frac{1}{2}\sum_{j=1}^{n}\frac{\partial}{\partial n}\left(u_j^2\right) = \sum_{j=1}^{n} u_j \frac{\partial u_j}{\partial n}.$$

There exists orthogonal matrix $C = \|c_{kl}\|$ so that we can pass from coordinates $(x_1, ..., x_n)$ to the local system of coordinates

$$y_k = \sum_{l=1}^{n} c_{kl}\left(x_l - x_l^0\right) \;\; ; \quad k = 1, ..., n;$$

in which direction $y_n$ coincides with the direction of outward normal at the point $y = 0$. We have

$$u_j = \sum_{k=1}^{n}\frac{\partial u}{\partial y_k}c_{kj}, \quad \frac{\partial u_j}{\partial n} = \frac{\partial u_j}{\partial y_n} = \sum_{l=1}^{n}\frac{\partial^2 u}{\partial y_n \partial y_l}c_{lj}; \; j = 1, ..., n.$$

Thus,

$$I\left(x^0\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial u}{\partial y_k}c_{kj}\sum_{l=1}^{n}\frac{\partial^2 u}{\partial y_n \partial y_l}c_{lj} = \sum_{k,l=1}^{n} b_{kl}\frac{\partial u}{\partial y_k}\,\frac{\partial^2 u}{\partial y_n \partial y_l},$$

where $b_{kl} = \sum\limits_{j=1}^{n} c_{kj}c_{lj}$. Let $C^{-1} = \|\bar{c}_{kl}\|$ and $C^* = \|c_{kl}^*\|$ be inverse and adjoint matrices to the matrix $C$ respectively. Taking into account that $C^{-1} = C^*$, we obtain

$$b_{kl} = \sum_{j=1}^{n} c_{kl}c_{jl}^* = \sum_{j=1}^{n} c_{kj}\bar{c}_{jl} = \delta_{kl}; \quad k, l = 1, ..., n.$$

Therefore

$$I\left(x^0\right) = \sum_{k=1}^{n}\frac{\partial u}{\partial y_k}\frac{\partial^2 u}{\partial y_n \partial y_k}\;.$$

Now we use boundary condition

$$\frac{\partial u}{\partial y_n}\left(y_1, ..., y_{n-1}, \psi\left(y_1, ..., y_{n-1}\right)\right) = 0\;. \tag{37}$$

Let us differentiate identity (37) with respect to $y_i$; $i = 1, ..., n$. Taking into account that $\frac{\partial \psi}{\partial y_i} = 0$ at the point with coordinates $y_i = 0$; $i = 1, ..., n$; we obtain

$$\left.\frac{\partial^2 u}{\partial y_n \partial y_k}\right|_{y=0} = 0\;; \quad k = 1, ..., n-1. \tag{38}$$

From (37) and (38) we conclude that $I\left(x^0\right) = 0$. Taking into account the last equality in (36), we arrive at (35).

We will assume that condition (4) is fulfilled in the form $\delta < 1$ (see lemma 4). Then, according to (35)

$$\|\Delta u\|_{L_2(D)} \leq \|(\Delta - \mathcal{L}')\, u\|_{L_2(D)} + \|\mathcal{L}' u\|_{L_2(D)} \leq$$

$$\leq \delta \left( \int\limits_D \sum_{i,j=1}^n u_{ij}^2 dx \right)^{\frac{1}{2}} + \|\mathcal{L}' u\|_{L_2(D)} = \delta \,\|\cdot u\|_{L_2(D)} + \|\mathcal{L}' u\|_{L_2(D)} \ .$$

Thus,

$$\|\cdot u\|_{L_2(D)} \leq C_{20}\,(\delta) \left\|\mathcal{L}' u\right\|_{L_2(D)} \ . \tag{39}$$

We will use the following variant of Poincare type inequality proved in [16]: if $\partial D \in C^2$, then for any function $u\,(x) \in W_2^1\,(D)$, the following estimate holds

$$\left( \oint\limits_D |u - u_D|^{2k} dx \right)^{\frac{1}{2k}} \leq C_{21} diam D \left( \oint\limits_D \sum_{i=1}^n u_i^2 dx \right)^{\frac{1}{2}} , \tag{40}$$

where constants $k > 1$ and $C_{21} > 0$ depend only on $n$ and smoothness of boundary $\partial D$. Subject to (40), we have

$$\|u\|_{L_2(D)} = \|u - u_D\|_{L_2(D)} \leq \left( \int\limits_D |u - u_D|^{2k} dx \right)^{\frac{1}{2k}} (mes D)^{\frac{k-1}{2k}} =$$

$$= (mes D)^{\frac{1}{2}} \left( \oint\limits_D |u - u_D|^{2k} dx \right)^{\frac{1}{2k}} \leq C_{21} diam D\, (mes D)^{\frac{1}{2}} \left( \oint\limits_D \sum_{i=1}^n u_i^2 dx \right)^{\frac{1}{2}} = \tag{41}$$

$$= C_{21} diam D \left( \int\limits_D \sum_{i=1}^n u_i^2 dx \right)^{\frac{1}{2}} \ .$$

On the other hand,

$$J_1 = -\int\limits_D u \Delta u dx = -\int\limits_D \sum_{i=1}^n u \ u_{ii} dx = \int\limits_D \sum_{i=1}^n u_i^2 dx - \int\limits_{\partial D} \sum_{i=1}^n u u_i \cos\,(n, x_i)\, ds =$$

$$= \int\limits_D \sum_{i=1}^n u_i^2 dx - \int\limits_{\partial D} u \frac{\partial u}{\partial n} ds = \int\limits_D \sum_{i=1}^n u_i^2 dx \ .$$

Besides, for any $\varepsilon > 0$ subject to (41) we obtain

$$J_1 \leq \frac{\varepsilon}{2} \int\limits_D u^2 dx + \frac{1}{2\varepsilon} \int\limits_D (\Delta u)^2 dx \leq \frac{\varepsilon C_{21}^2 (diam D)^2}{2} \int\limits_D \sum_{i=1}^n u_i^2 dx + \frac{1}{2\varepsilon} \int\limits_D (\Delta u)^2 dx \ .$$

Assuming $\varepsilon = \frac{1}{C_{21}^2 (diam D)^2}$, we conclude

$$\left( \int\limits_D \sum_{i=1}^n u_i^2 dx \right)^{\frac{1}{2}} \leq C_{21} diam D \, \|\Delta u\|_{L_2(D)} \, . \tag{42}$$

Combining (41) and (42), we arrive at estimate

$$\|u\|_{L_2(D)} \le C_{21}^2 (diamD)^2 \|\Delta u\|_{L_2(D)} . \tag{43}$$

Now remembering the proofs of lemmas 6 and 7 and also the corollary to lemma 7, we obtain that constant $C_{18}$ in inequality (33) has the form $C_{18} = \frac{C_{22}(\mathcal{L},n,\partial D)}{(diamD)^2}$. Therefore the following inequality follows from (43) subject to (39) and (16)

$$C_{18} \|u\|_{L_2(D)} \le C_{23} (\mathcal{L}, n, \partial D) \|\mathcal{L}u\|_{L_2(D)} \le C_{23} \|\mathcal{L}u\|_{L_2(D)} +$$
$$+C_{23} \sum_{i=1}^{n} \|b_i u_j\|_{L_2(D)} \le C_{23} \|\mathcal{L}u\|_{L_2(D)} + C_{23} C_4 \|u\|_{W_2^2(D)} \sum_{i=1}^{n} æ_{b_i;n} \left(\omega^0\right) , \tag{44}$$

where $\omega^0 = mesD$ (we have confined ourselves to the case $n > 2$). Let us choose $d$ so small that $\sum\limits_{i=1}^{n} æ_{b_i;n} (d) \le \frac{1}{2C_{23}C_4}$. Then required estimate (34) follows from (33) and (44). The theorem is proved.

Let us demonstrate now one more approach to the obtaining of coercive inequality of the form (34) without any restriction on the quantity $mesD$. To this end we have to impose stronger conditions to the minor coefficients of operator $\mathcal{L}$ or, exactly, we assume that

$$b_i (x) \in L_{n+1} (D) ; \ i = 1, ..., n. \tag{5'}$$

**Theorem 3.** *If conditions (3), (4) and (5′) are fulfilled for the coefficients of operator* $\mathcal{L}$*, then at* $\mu > \mu_0 (\mathcal{L}, n, \partial D, diam\ D)$ *for any function* $u (x) \in \check{W}_2^2 (D)$ *estimate*

$$\|u\|_{W_2^2(D)} \le C_{24} (\gamma, \sigma, n, \mu, \partial D) \ \ \|\mathcal{L}u - \mu u\|_{L_2(D)} \tag{45}$$

*holds.*

**Proof.** Consider cylindrical domain $Q = D \times (0,T)$ in $(n + 1)$ dimensional Euclidean space of points $(x, t)$, where $T \in (0, T_0)$ and number $T_0$ will be chosen later. We define in $Q$ elliptic operator $\mathcal{M} = \mathcal{L} + \frac{\partial^2}{\partial t^2}$. It is clear that condition of the form (3) is fulfilled for operator $\mathcal{M}$ with constant $\gamma$. Besides, by virtue of (5') conditions of the form (5) are fulfilled for the case, when dimension of space is equal to $n + 1$. Since condition (4) is understood to within equivalence and nonsingular linear transformation, then without loss of generality it can be assumed that $\sum\limits_{i=1}^{n} a_{ii} (x) = n - 1$. Therefore, if (4) is satisfied, then

$$\sigma_{(\mathcal{M})} = \sup_{x\in D} \frac{\sum\limits_{i,j=1}^{n} a_{ij}^2 (x) + 1}{\left[\sum\limits_{i,j=1}^{n} a_{ii} (x) + 1\right]^2} = \frac{\sup\limits_{x\in D} \sum\limits_{i,j=1}^{n} a_{ij}^2 (x) + 1}{n^2} = \frac{\sigma (n-1)^2 + 1}{n^2} ,$$

at that $\frac{\sigma(n-1)^2+1}{n^2} < \frac{1}{n}$, i.e. $\sigma < \frac{1}{n-1}$.

Let $u (x) \in \check{W}_2^2 (D)$. Then $\upsilon (x, t) = u (x) \cos \frac{\pi t}{T} \in \check{W}_2^2 (Q)$. Applying estimate (33) to function $\upsilon (x, t)$ and taking into account that $\mathcal{M}\upsilon = \cos \frac{\pi t}{T} \left(\mathcal{L}u - \frac{\pi^2}{T^2} u\right)$, we obtain

$$\|u\|_{W_2^2(D)} \leq C_{17} \left\| \cos \frac{\pi t}{T} \left( \mathcal{L}u - \frac{\pi^2}{T^2} u \right) \right\|_{L_2(D)} + C_{18} \|\upsilon\|_{L_2(Q)} . \qquad (46)$$

On the other hand $\upsilon_{tt} = \frac{\pi^2}{T^2}\upsilon$, i.e. $\|\upsilon\|_{L_2(Q)} = \frac{T^2}{\pi^2} \|\upsilon_{tt}\|_{L_2(Q)} \leq \frac{T^2}{\pi^2} \|\upsilon\|_{W_2^2(Q)}$. Let us choose $T_0 = \frac{\pi}{\sqrt{C_{18}}}$ and suppose $\mu_0 = \frac{\pi^2}{T_0^2}$, $\mu = \frac{\pi^2}{T^2}$. Then at $T < T_0$, i.e. at $\mu > \mu_0$ from (46) we conclude

$$\|\upsilon\|_{W_2^2(Q)} \leq C_{25} (\gamma, \sigma, n, \mu, \partial D) \ \ \|\cos \sqrt{\mu} t \left( \mathcal{L}u - \mu u \right)\|_{L_2(Q)} . \qquad (47)$$

Now it suffices to take into consideration that

$$\|\upsilon\|_{W_2^2(Q)} \geq \|\cos \sqrt{\mu} t\|_{L_2\left(0, \frac{\pi}{\sqrt{\mu}}\right)} \|u\|_{W_2^2(D)} ,$$

$$\|\cos \sqrt{\mu} t \left( \mathcal{L}u - \mu u \right)\|_{L_2(Q)} = \|\cos \sqrt{\mu} t\|_{L_2\left(0, \frac{\pi}{\sqrt{\mu}}\right)} \|\mathcal{L}u - \mu u\|_{L_2(D)} ,$$

and the required estimate (45) follows from (47). The theorem is proved.

## 3. Strong solvability of the second boundary value problem

**Theorem 4.** *Let conditions (3)-(5) be satisfied for the coefficients of operator $\mathcal{L}$. Then if $mesD \leq d$, then second boundary value problem (1)-(2) has a unique normalized solution from space $\check{W}_2^2(D)$ for any $f(x) \in L_2(D)$. At that the following estimate holds for solution $u(x)$*

$$\|u\|_{W_2^2(D)} \leq C_{19} \|f\|_{L_2(D)} \ . \qquad (48)$$

**Proof.** We introduce a family of operators $\mathcal{L}_\tau = (1-\tau)\Delta + \tau\mathcal{L}$ for $\tau \in [0,1]$ and consider Neumann problem

$$L_\tau u = f(x), \quad x \in D; \quad \left. \frac{\partial u}{\partial n} \right|_{\partial D} = 0. \qquad (49)$$

Let us denote by $E$ the set of points $\tau$ of segment $[0,1]$, for which problem (49) has unique normalized solution for any $f(x) \in L_2(D)$. Let us show that $E$ is nonempty and simultaneously is open and closed with respect to segment $[0,1]$. Then $E = [0,1]$, and problem (49) is solvable at $\tau = 1$ when $\mathcal{L}_\tau = \mathcal{L}$. Nonemptiness of set $E$ follows from [8], where the solvability of problem (49) at $\tau = 0$ has been proved. Let us prove that set $E$ is open. To this end, at first, we verify in the following: conditions of the form (3) and (4) are fulfilled for operators $\mathcal{L}_\tau$ with constants independent of $\tau$, and, besides, conditions of the form (5) are satisfied, moreover, $AC$-moduli of order $s$ of the minor coefficients of operators $\mathcal{L}_\tau$ are majorized by the quantities dependent only on $AC$-moduli æ$_{b_1;s}$, ...,æ$_{b_n;s}$ respectively. It is obvious that we have to check only the fulfillment of condition of the form (4). Without loss of generality we assume that the trace of matrix $\|a_{ij}(x)\|$ is constant in $D$ and equal to 1. We have

$$\sigma^{(\tau)} = \sup_{x \in D} \frac{\sum\limits_{i,j=1}^{n} \left[ (1-\tau)^2 \delta_{ij} + 2\tau(1-\tau) a_{ij}(x) + \tau^2 a_{ij}^2(x) \right]}{(1-\tau)^2 n^2 + 2\tau(1-\tau)n + \tau^2} =$$

$$= \frac{(1-\tau)^2 n + 2\tau(1-\tau) + \tau^2\sigma}{(1-\tau)^2 n^2 + 2\tau(1-\tau)n + \tau^2} = \frac{1}{n} + \frac{\tau^2\left(\sigma - \frac{1}{n}\right)}{(1-\tau)^2 n^2 + 2\tau(1-\tau)n + \tau^2} \leq \frac{1}{n} + \frac{\tau^2\left(\sigma - \frac{1}{n}\right)}{\tau^2} = \sigma \ .$$

Let now $\tau_0 \in E$. Let us show that at $\varepsilon > 0$ chosen in the corresponding way, points $\tau \in [0,1]$ for which $|\tau - \tau_0| < \varepsilon$, belong to $E$. We represent boundary value problem (49) in the equivalent form

$$\mathcal{L}_{\tau_0} u = f(x) + (\mathcal{L}_{\tau_0} - \mathcal{L}_\tau) u, \quad x \in D; \quad u \in \check{W}_2^2(D), \quad u_D = 0. \tag{50}$$

Let $\omega(x) \in \check{W}_2^2(D), \quad \omega_D = 0$. Consider boundary value problem

$$\mathcal{L}_{\tau_0} u = f(x) + (\mathcal{L}_{\tau_0} - \mathcal{L}_\tau) \omega, \quad x \in D; \quad u \in \check{W}_2^2(D), \quad u_D = 0. \tag{51}$$

It is clear that $(\mathcal{L}_{\tau_0} - \mathcal{L}_\tau)\omega(x) \in L_2(D)$. According to our assumption, boundary value problem (51) has unique solution $u(x)$. In other words, operator $\mathcal{B}$ from the subspace of functions $\omega(x) \in \check{W}_2^2(D), \quad \omega_D = 0$ into itself is defined so that $u = \mathcal{B}\omega$. Let us show that one can choose so small $\varepsilon > 0$ that operator $\mathcal{B}$ is contraction. Let $\omega_{(i)}(x) \in \check{W}_2^2(D), \quad \left(\omega_{(i)}\right)_D = 0$; $u_{(i)} = \mathcal{B}\omega_{(i)}$; $i = 1,2$. We have

$$\mathcal{L}_{\tau_0}\left(u_{(1)} - u_{(2)}\right) = (\mathcal{L}_{\tau_0} - \mathcal{L}_\tau)\left(\omega_{(1)} - \omega_{(2)}\right), \quad \left(\omega_{(1)} - \omega_{(2)}\right) \in \check{W}_2^2(D),$$

$$\left(\omega_{(1)} - \omega_{(2)}\right)_D = 0 \ .$$

Taking into account that $\mathcal{L}_{\tau_0} - \mathcal{L}_\tau = (\tau_0 - \tau)(\mathcal{L} - \Delta)$, from theorem 2 we obtain

$$\left\|u_{(1)} - u_{(2)}\right\|_{W_2^2(D)} \le C_{19} |\tau_0 - \tau| \left\|(\mathcal{L} - \Delta)\left(\omega_{(1)} - \omega_{(2)}\right)\right\|_{L_2(D)} \tag{52}$$

Note that if $z \in W_2^2(D)$, then $\|(\mathcal{L} - \Delta) z\|_{L_2(D)} \le C_{26}(\mathcal{L}, n) \|z\|_{W_2^2(D)}$. Then from (52) we conclude

$$\left\|u_{(1)} - u_{(2)}\right\|_{W_2^2(D)} \le C_{19} C_{26} \varepsilon \left\|\omega_{(1)} - \omega_{(2)}\right\|_{W_2^2(D)} .$$

Now it suffices to choose $\varepsilon = \frac{1}{2C_{19}C_{26}}$, and hence it is proved that operator $\mathcal{B}$ is contraction. Hence, it has fixed point such that $u = \mathcal{B}u$. But at $\omega = u$ problem (51) coincides with problem (50), i.e. with problem (49). Thus, $E$ is an open set. Let us prove its closeness. Let $\tau_m \in E$; $m = 1, 2, ...$; $\lim\limits_{m \to \infty} \tau_m = \tau$. For natural $m$ denote by $u^{(m)}(x)$ the normalized solution of second boundary value problem

$$\mathcal{L}_{\tau_m} u^{(m)} = f(x), \ x \in D; \quad \left.\frac{\partial u^{(m)}}{\partial n}\right|_{\partial D} = 0.$$

According to theorem 2

$$\left\|u^{(m)}\right\|_{W_2^2(D)} \le C_{19} \|f\|_{L_2(D)} . \tag{53}$$

Weak compactness of sequence $\left\{u^{(m)}(x)\right\}$ in the subspace of functions from $\check{W}_2^2(D)$ with zeroth mean value over domain $D$ follows from (53). Thereby, there exist function $u(x) \in \check{W}_2^2(D), \ u_D = 0$ and subsequence of natural numbers $\{m_k\}, \ \lim\limits_{k \to \infty} m_k \to \infty$ so that for any $\varphi(x) \in C^\infty\left(\bar{D}\right)$ the following limiting equality holds

$$\lim_{k \to \infty} \left(\mathcal{L}_\tau u^{(m_k)}, \varphi\right) = (\mathcal{L}_\tau u, \varphi), \tag{54}$$

where $(g_1, g_2) = \int\limits_D g_1 g_2 dx$. On the other hand,

$$\left(\mathcal{L}_\tau u^{(m_k)}, \varphi\right) = \left(\left(\mathcal{L}_\tau - \mathcal{L}_{\tau_{m_k}}\right) u^{(m_k)}, \varphi\right) + \left(\mathcal{L}_{\tau_{m_k}} u^{(m_k)}, \varphi\right) =$$
$$= \left(\left(\mathcal{L}_\tau - \mathcal{L}_{\tau_{m_k}}\right) u^{(m_k)}, \varphi\right) + (f, \varphi) \ . \tag{55}$$

Besides

$$J_2(k) = \left|\left(\mathcal{L}_\tau - \mathcal{L}_{\tau_{m_k}}\right) u^{(m_k)}, \varphi\right| \le C_{26} \left|\tau - \tau_{m_k}\right| \ \left\|u^{(m_k)}\right\|_{W_2^2(D)} \|\varphi\|_{L_2(D)} \le$$
$$\le C_{26} C_{19} \left|\tau - \tau_{m_k}\right| \ \|f\|_{L_2(D)} \|\varphi\|_{L_2(D)} \ ,$$

i.e. $\lim\limits_{k\to\infty} J_2(k) = 0$. Thus, from (54) and (55) we conclude that $(\mathcal{L}_\tau u, \varphi) = (f, \varphi)$, i.e. $\mathcal{L}_\tau u = f(x)$ a.e. in $D$. Hence, $\tau \in E$ and the existence of normalized solution of problem (1)-(2) is shown. Its uniqueness and estimate (48) follow from inequality (34). The theorem is proved.

**Theorem 5.** *Let conditions (3), (4) and (5) be fulfilled for the coefficients of operator $\mathcal{L}$, and $\mu > \mu_0$. Then second boundary value problem*

$$\mathcal{L}u - \mu u = f(x), \ x \in D; \ \left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0$$

*is uniquely strongly solvable in $\check{W}_2^2(D)$ for any $f(x) \in L_2(D)$. At that for solution $u(x)$ the following estimate holds*

$$\|u\|_{W_2^2(D)} \le C_{24} \|f\|_{L_2(D)} \ .$$

This theorem is proved quite analogously to the previous one, just instead of inequality (34) it is necessary to use estimate (45).

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