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ESTIMATION OF GREEN FUNCTION OF THE FIRST BOUNDARY VALUE PROBLEM FOR THE SECOND ORDER NONUNIFORMLY DEGENERATED DIVERGENT ELLIPTIC EQUATIONS

Abstract

The Dirichlet problem for a class of the second order divergent structure elliptic equations allowing nonuniform degeneration at the boundary point of the domain is considered. The estimations of Green function of the given problem are established at some neighbourhood of this point.

Introduction Let $\mathbb{E}_n$ be an n-dimensional Euclidean space of the points $x = (x_1, ..., x_n)$, $n \geq 3$, D be a bounded domain in $\mathbb{E}_n$ with the boundary $\partial D, 0 \in \partial D$. Consider the following equation in D

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0 \tag{1}$$

in assumption that $\|a_{ij}(x)\|$ is a real symmetric matrix with elements measurable in D, where for almost all $x \in D$ and any $\zeta \in \mathbb{E}_n$ the condition

$$\gamma \sum_{i=1}^{n} \lambda_i(x) \zeta_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \zeta_i \zeta_j \leq \gamma^{-1} \sum_{i=1}^{n} \lambda_i(x) \zeta_1^2 \tag{2}$$

is fulfilled. Here $\gamma \in (0,1]$ is a constant $\lambda_i(x) = g_i(\rho(x,0)), \rho(x,0) = \sum_{i=1}^{n} \omega_i(|x_i|)$, $g_i(t) = \frac{\left(\omega_i^{-1}(t)\right)^2}{t^2}$; $i = 1, ..., n$. At that $\omega_i(t)$ are strongly monotonically increasing functions for $t \in [0, diamD]$, $\omega_i^{-1}(t)$ are the functions inverse to $\omega_i(t)$, besides for $i = 1, .., n$ and sufficiently small t

$$d\omega_i(t) \leq \omega_i(2t) \leq 2\omega_i(t), \tag{3}$$

$$\left(\frac{\omega_i(t)}{t} \right)^{q-1} \int_0^{\omega_i^{-1}(t)} \left(\frac{\omega_i(\tau)}{\tau} \right)^q d\tau \leq A\, t, \tag{4}$$

where $d > 1$, $q > n$ and $A > 0$ are some constants.

The goal of our paper is to obtain an estimation of the Green function of the first boundary value problem for the equation (1) in the neighbourhood of the boundary point 0. Note that the analogous result for the second order nondegenerate elliptic equation with uniform degeneration in [2]-[3]. The estimations established in this paper may be used for obtaining Wiener type criterion of regularity of boundary point with respect to Dirichlet problem for the equation (1).

1. Some known results

Denote $p \in (1,\infty)$ by $W^1_{p,\lambda}(D)$ Banach space of the functions $u(x)$ given on D with the finite norm

$$\|u\|_{W^1_{p,\lambda}(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n (\lambda_i(x))^{p/2} |u_i|^p\right) dx\right)^{1/p},$$

and let $\overset{\circ}{W}{}^1_{p,\lambda}(D)$ be a subspace of $W^1_{p,\lambda}(D)$, dense set in which is a set of all the functions from $C_0^\infty(D)$. Here for $i = 1,...,n$ $u_i = \frac{\partial u}{\partial x_i}$. We shall denote by $W^{-1}_{p,\lambda}(D)$ a space adjoint to $W^1_{p,\lambda}(D)$ and by $M_0(D)$ - a space of functions satisfying uniform condition in D and vanishing near ∂D.

Let $\omega(x)$ be a positive and finite function almost everywhere in D. Denote by $L_{p,\omega}(D)$ a Banach space of the functions $u(x)$ given on D with the finite norm

$$\|u\|_{L_{p,\omega}(D)} = \left(\int_D [\omega(x)]^{p/2} |u|^p dx\right)^{1/p}.$$

We shall denote by $\sum$ a ball of sufficiently large radius, containing $\bar{D}$ strongly inside itself.

Definition 1. *We say that the function $u(x) \in \overset{\circ}{W}{}^1_{p,\lambda}(\sum)$ is nonnegative on the set E in the sense of $\mathcal{L}_p(\sum)$ (or almost everywhere) if $mes\{[x : u(x) < 0] \cap E\} = 0$.*

Definition 2. *We say that the function $u(x) \in \overset{\circ}{W}{}^1_{p,\lambda}(\sum)$ is nonnegative on the set E in the sense of $W^1_{p,\lambda}(\sum)$ if there exists a sequence of the functions $u_m(x) \in M_0(\sum)$ such that $u_m(x) \geq 0$ on E and $\lim\limits_{m\to\infty} \|u_m - u\|_{W^1_{p,\lambda}(\sum)} = 0$.*

It is clear that if $u(x) \geq 0$ on E in the sense of $\overset{\circ}{W}{}^1_{p,\lambda}(\sum)$ then $u(x) \geq 0$ almost everywhere on E. Inversely, if $u(x) \geq 0$ almost everywhere on E in the sense of $\overset{\circ}{W}{}^1_{p,\lambda}(\sum)$ on any subset $E' \subset E$ such that $dist(E', \partial E) > 0$.

Let $u(x) \in \mathcal{L}_p(\sum)$, k be a positive number, the function

$$\{u(x)\}^k = \begin{cases} u(x), & if \quad u(x) \leq k \\ k, & if \quad u(x) > k \end{cases}$$

is called k-truncation of the function $u(x)$.

By analogy we can determine other inequalities in the sense of $\overset{\circ}{W}{}^1_{p,\lambda}(\sum)$, too. For example, $u(x) \geq const$, $u(x) \leq 0$, $v(x) \leq u(x)$, $u(x) = const$ on E.

We cite some simple assertions without proof

Lemma 1. *If $u(x) \geq 1$ on E in the sense of $\overset{\circ}{W}{}^1_{p,\lambda}(\sum)$, then $\{u(x)\}^1 = 1$ on E in the same sense.*

Lemma 2. *The set of functions $u(x) \in \overset{\circ}{W}{}^1_{p,\lambda}(\sum)$ such that $u(x) \geq 1$ on E in the sense of $\overset{\circ}{W}{}^1_{p,\lambda}(\sum)$ represents a convex closed set.*

Definition 3. *We say that the function* $u(x) \in W^1_{p,\lambda}(D)$ *is nonnegative on* ∂D *in the sense of* $W^1_{p,\lambda}(D)$ *if there exists a sequence of the functions* $u_m(x) \in C^1(\bar{D})$ *such that* $u_m(x) \geq 0$ *on* ∂D *and* $\lim\limits_{m\to\infty} \|u_m - u\|_{W^1_{p,\lambda}(D)} = 0$.

Lemma 3. *Let* $u(x) \in W^1_{p,\lambda}(D)$ *and* $u(x) \leq 0$ *on* ∂D *in the sense of* $W^1_{p,\lambda}$. *Then for any* $k > 0$

$$\left(u(x) - \{u(x)\}^k\right) \in \overset{\circ}{W}{}^1_{p,\lambda}(D).$$

Definition 4. *Let* $f^0(x) \in L_2(D)$, $f^i(x) \in \mathcal{L}^{-1}_{2,\lambda_i}(D)$; $i = 1, ..., n$ *be given functions,* $\Phi(x) \in W^1_{2,\lambda}(D)$. *The function* $u(x)$ *is called a generalized solution in* D *of the equation*

$$Lu = \sum_{i,j=1}^{n} (d_{ij}(x) u_j)_i = f^0(x) + \sum_{i=1}^{n} \left(f^i(x)\right)_i \tag{5}$$

if for any $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$, *the integral identity*

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx = \int_D \left(-f^0(x)\varphi + \sum_{i=1}^{n} f^i(x)\varphi_i\right) dx \tag{6}$$

is satisfied.

If besides

$$(u(x) - \Phi(x)) \in \overset{\circ}{W}{}^1_{2,\lambda}(D), \tag{7}$$

then the function $u(x)$ is called a general solution of the first boundary value problem (5), (7).

Definition 5. *The function* $u(x) \in W^{1,loc}_{2,\lambda}(D)$ *is a general local solution of the equation (1) in* D *if for any function* $v(x) \in C^1(\bar{D})$ *with compact support in* D, *the integral identity*

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_i v_j dx = 0 \tag{8}$$

is satisfied.

Definition 6. *The function* $u(x) \in W^1_{2,\lambda}(D)$ *is called* $\mathcal{L}$*-subelliptic in* D *if for any nonnegative function* $\varphi(x)$ *from* $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ *the inequality*

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx \leq 0$$

is satisfied.

If the function $u(x)$ is $\mathcal{L}$-subelliptic in D, then the function $u(x)$ is called L-superbolic in D.

Denote for $R > 0$, $k > 0$ and $x^0 \in \mathbb{E}_n$ by $\varepsilon_{R;K}(x^0)$ the ellipsoid $\left\{x : \sum\limits_{i=1}^{n} \frac{(x_i - x_i^0)^2}{\left(\omega_i^{-1}(R)\right)^2} < K^2\right\}$. Let further $\frac{1}{2} \leq \rho < \rho' \leq 1$, $\bar{\varepsilon}_{R,\rho'}(x^0) \subset D$ and $x^0 \in \varepsilon_{R;\frac{1}{4\sqrt{n}}}(0)$.

Lemma 4. *For any general local solution* $u(x)$ *of the equation (1) in* D*, the estimation*

$$\int\limits_{\varepsilon_{R;\rho}(x^0)} \sum_{i=1}^{n} \lambda_i(x)(u_i)^2 dx \le \frac{C_1(\gamma,\lambda,n)}{(\rho'-\rho)^2 R^2} \int\limits_{\varepsilon_{R;\rho'}(x^0)} u^2 dx \qquad (9)$$

is valid.

Here and further the notation $C(...)$ denotes a positive constant c depending only on the quantities appearing in parentheses. At that $\lambda=\lambda(x)=(\lambda_1(x),...,\lambda_n(x))$.

Proof. Let $\psi(x)\in C_0^{\infty}\left(\varepsilon_{R;\rho'}\left(x^0\right)\right),\ \ 0\le\psi(x)\le 1$ where $\psi(x)=1$ in $\varepsilon_{R;\rho}\left(x^0\right)$ and

$$|\psi_i| \le \frac{C_2(n)}{(\rho'-\rho)\,\omega_i^{-1}(R)}; \quad i=1,...,n. \qquad (10)$$

We assume $v(x)=u(x)\psi^2(x)$ in the integral identity (8).

We obtain

$$\int\limits_{\varepsilon_{R;\rho'}(x^0)} \sum_{i,j=1}^{n} a_{ij}(x)\,u_i\left(u_j\psi^2+2u\psi\psi_j\right)dx = 0\ .$$

Hence it follows that

$$\int\limits_{\varepsilon_{R;\rho'}(x^0)} \sum_{i,j=1}^{n} a_{ij}(x)\,u_iu_j\psi^2 dx = -2\int\limits_{\varepsilon_{R;\rho'}(x^0)} u\psi\sum_{i,j=1}^{n} a_{ij}(x)\,u_i\psi_j dx \le$$

$$\le 2\int\limits_{\varepsilon_{R;\rho'}(x^0)} |u|\psi\left(\sum_{i,j=1}^{n} a_{ij}(x)\,u_iu_j\right)^{1/2}\left(\sum_{i,j=1}^{n} a_{ij}(x)\,\psi_i\psi_j\right)^{1/2} dx\ .$$

Using now the inequality $2ab\le\varepsilon a^2+\frac{1}{\varepsilon}b^2$ at $\varepsilon=\frac{1}{2}$ valid for any $\varepsilon>0$, we conclude

$$\int\limits_{\varepsilon_{R;\rho'}(x^0)} \psi^2\sum_{i,j=1}^{n} a_{ij}(x)\,u_iu_j dx \le \frac{1}{2}\int\limits_{\varepsilon_{R;\rho'}(x^0)} \psi^2\sum_{i,j=1}^{n} a_{ij}(x)\,u_iu_j dx +$$
$$+2\int\limits_{\varepsilon_{R;\rho'}(x^0)} u^2\sum_{i,j=1}^{n} a_{ij}(x)\,\psi_i\psi_j dx\ .$$

Allowing for the condition (2) we obtain

$$\int\limits_{\varepsilon_{R;\rho}(x^0)} \sum_{i=1}^{n} \lambda_i(x)(u_i)^2 dx \le 4\gamma^{-2}\int\limits_{\varepsilon_{R;\rho'}(x^0)} u^2\sum_{i=1}^{n} \lambda_i(x)(\psi_i)^2 dx\ . \qquad (11)$$

If $x\in\varepsilon_{R;\rho'}\left(x^0\right)$, then for $i=1,...,n$

$$|x_i-x_i^0| < \rho'\omega_i^{-1}(R) \le \omega_i^{-1}(R), \quad \text{i.e.}$$

$$|x_i| \le |x_i^0| + |x_i-x_i^0| \le 2\omega_i^{-1}(R)\ .$$

Thus, we derive from the condition (3)

$$\rho\left(x,0\right)\leq\sum_{i=1}^{n}\omega_i\left(2\omega_i^{-1}\left(R\right)\right)\leq 2nR\ .$$

On the other hand the integral in the right hand side of the inequality (11) really taken over the set $\varepsilon_{R;\rho'}\left(x_0\right)\backslash\varepsilon_{R;\rho}\left(x^0\right)$. Hence, it follows that $x\notin\varepsilon_{R;\rho}\left(x^0\right)$, i.e. $x\notin\varepsilon_{R;\frac{1}{2}}\left(x^0\right)$. Then there will be found $i_0,\ 1\leq i_0\leq n$ such that

$$\left|x_{i_0}-x_{i_0}^0\right|\geq\frac{1}{2\sqrt{n}}\omega_{i_0}^{-1}\left(R\right)\ .$$

At the same time for any $i,\ 1\leq i\leq n$

$$\left|x_i^0\right|\leq\frac{1}{4\sqrt{n}}\omega_i^{-1}\left(R\right)\ .$$

Therefore

$$\left|x_{i_0}\right|\geq\left|x_{i_0}-x_{i_0}^0\right|-\left|x_{i_0}^0\right|\geq\frac{1}{4\sqrt{n}}\omega_{i_0}^{-1}\left(R\right)\ ,$$

and by that

$$\rho\left(x,0\right)\geq\omega_{i_0}\left(\frac{1}{4\sqrt{n}}\omega_{i_0}^{-1}\left(R\right)\right)\ .$$

Let now l be a least natural number for which $4^{l-2}\geq n$, i.e. $2^l\frac{1}{4\sqrt{n}}\geq 1$. We have from the condition (3)

$$\omega_{i_0}\left(\tfrac{1}{4\sqrt{n}}\omega_{i_0}^{-1}\left(R\right)\right)\geq 2^{-1}\omega_{i_0}\left(\tfrac{2}{4\sqrt{n}}\omega_{i_0}^{-1}\left(R\right)\right)\geq ...\geq$$

$$\geq 2^{-l}\omega_{i_0}\left(\tfrac{2^l}{4\sqrt{n}}\omega_{i_0}^{-1}\left(R\right)\right)\geq 2^{-l}R\ .$$

Thus

$$\rho\left(x,0\right)\geq C_3\left(n\right)R\ ,$$

and we obtain for $i=1,...,n$

$$\lambda_i\left(x\right)\leq\frac{\left(\omega_i^{-1}\left(2nR\right)\right)^2}{C_3^2R^2}\leq C_4\left(n,\lambda\right)\frac{\left(\omega_i^{-1}\left(R\right)\right)^2}{R^2}. \tag{12}$$

Using (10) and (12) in (11) we conclude

$$\int\limits_{\varepsilon_{R;\rho}\left(x^0\right)}\sum_{i=1}^{n}\lambda_i\left(x\right)\left(u_i\right)^2dx\leq\frac{4\gamma^{-2}nC_2^2C_4}{\left(\rho'-\rho\right)^2R^2}\int\limits_{\varepsilon_{R;\rho'}\left(x^0\right)}u^2dx\ ,$$

which coincides with the required estimation (9).

The lemma is proved.

Theorem 1 [4]. *Any general local solution $u(x)$ in D is Hölder continuous in any strongly interior of the subdomain D' of the domain D. In other words there exists $\alpha(\gamma,\lambda,n)$ and $K(\gamma,\lambda,n,dist(D',\partial D))$ such that for any $x^1,x^2\in D'$*

$$|u(x^1)-u(x^2)|\le K\|u\|_{\mathcal{L}_2(D)}|x^1-x^2|^\alpha .$$

Note that for divergent structure second order uniformly elliptic equations, the apriori estimation of Hölder norm of solutions is established in [5]-[6] and for equations with uniform degeneration this is established in [7]

Theorem 2 [8]. (Harnack type inequality). *For any positive general local solution $u(x)$ of the equation (1) in D in any strongly interior of subdomain D' of the domain D the estimation*

$$\sup_{D'}u\le K_1(\gamma,\lambda,n,D',D)\inf_{D'}u$$

is valid.

Theorem 3 [9]. *The first boundary value problem (5), (7) is uniquely generally solvable in the space $W^1_{2,\lambda}(D)$ for any $\Phi(x)\in W^1_{2,\lambda}(D)$, $f^0(x)\in\mathcal{L}_2(D)$, $f^i(x)\in\mathcal{L}_{2,\lambda_i^{-1}}(D)$; $i=1,...,n$.*

Theorem 4. *There exists $P_0(\lambda,n)$ such that if $f^0(x)\in\mathcal{L}_p(D)$, $f^i(x)\in\mathcal{L}_{p,\lambda_i^{-1}}(D)$ $(i=1,...,n)$ and $p\ge p_0$, then the general solution of the first boundary value problem (6), (7) is Hölder continuous in each strongly interior of subdomain of the domain D.*

The formulated statement may be proved by the method of [4]

Theorem 5. *If $u(x)$ is a general solution of the first boundary value problem (5), (7) with $\Phi(x)=0$, $f^0(x)\in\mathcal{L}_p$, $f^i(x)\in\mathcal{L}_{p,\lambda_i^{-1}}$ $(i=1,...,n)$ and $p\ge p_0$,, then there exists $\alpha_0(\lambda,n,p)$ such that*

$$\sup_D|u|\le C_5(\gamma,\lambda,n,p)(mesD)^{\alpha_0}\left(\|f^0\|_{\mathcal{L}_p(D)}+\sum_{i=1}^n\|f^i\|_{L_{p,\lambda_i^{-1}}}\right).$$

This statement follows from the result of [9] and embedding theorem.

2. Generalized solution of a boundary function

Let $h(x)\in C(\partial D)$. We now determine a general solution of the first boundary value problem

$$\mathcal{L}u=0,\quad x\in D;\ u|_{\partial D}=h\ . \tag{13}$$

If $h(x)$ is a trace on ∂D of some function $H(x)\in W^1_{2,\lambda}(D)$, then according to theorem 3 and 1, the problem (13) has unique generalized solution $u(x)$ Hölder continuous in each strongly interior of subdomain of the domain D. It is clear that the solution $u(x)$ remains unchangeable if we take the function $H_1(x)$ instead of $H(x)$ for which $(H(x)-H_1(x))\in\mathring{W}^1_{2,\lambda}(D)$. Let us introduce the factor space $\tau^{1,2}_\lambda=W^1_{2,\lambda}(D)\backslash\mathring{W}^1_{2,\lambda}(D)$ with the norm $\|H\|_{\tau^{1,2}_\lambda}=\inf\limits_{\Phi-H\in\mathring{W}^1_{2,\lambda}(D)}\|\Phi\|_{W^1_{2,\lambda}(D)}$.

Thus, the solution $H(x) \in \tau_\lambda^{1,2}$ of the boundary value problem (13) corresponds to each function $u(x) \in W_{2,\lambda}^1(D)$. Denote on operator realizing this continuous linear mapping from $\tau_\lambda^{1,2}$ in $W_{2,\lambda}^1(D)$ by B, i.e. $u = BH$. According to the maximum principle, if the function $H(x)$ is bounded on ∂D in the sense of $W_{2,\lambda}^1(D)$, then

$$\inf_{\partial D} H \le \inf_D BH \le \sup_D BH \le \sup_{\partial D} H, \qquad (14)$$

where $\inf\limits_{\partial D} H \left(\sup\limits_{\partial D} H\right)$ denotes the exact lower (appear) bound of such numbers a for which $H(x) \ge a\,(H(x) \le a)$ on ∂D in the sense of $W_{2,\lambda}^1(D)$.

Let us introduce he following norm

$$< g >= \sup_{D'} \rho \left(\int_{D'} \sum_{i=1}^n \lambda_i(x) (g_i)^2 dx \right)^{1/2} + \sup_D |g| \ ,$$

where subdomain D' is such that $\bar{D}' \subset D$ and $\rho = dist(D', \partial D)$. Then it follows from lemma 4 and (14) that

$$< BH > \le C_6(\gamma, \lambda, n, D) \sup_{\partial D} |H| \ . \qquad (15)$$

Hence, we conclude that BH is a continuous linear mapping of the subset $B_\lambda^{1,2}$ of the functions from $\tau_\lambda^{1,2}$ bounded on ∂D in the sense of $W_{2,\lambda}^1(D)$ in space of functions with the finite norm $< u >$. On the other hand any function $h(x) \in C(\partial D)$ may be approximated in the norm $\sup\limits_{\partial D} |h|$ by the functions smooth in some neighbourhood $\bar{D}$. Therefore the set $B_\lambda^{1,2}$ is dense in a space of functions $h(x)$ continuous on ∂D with the norm $\sup\limits_{\partial D} |h|$. By that the mapping $u = Bh$ considered on a set of functions from $\tau_\lambda^{1,2}$ continuous on ∂D may be expanded on whole space $C(\partial D)$. We denote the obtained expansion by B.

Definition 7. *The function $u(x) = Bh(x)$ is called a generalized solution of the first boundary value problem (13).*

It is clear that the function $u(x)$ is a generalized local solution of the equation (1) in D? Hölder continuous in each strongly interior of subdomain of the domain D. Besides, for it the estimations of the forms (14) and (15) are valid.

Definition 8. *The point $y \in \partial D$ is called a regular point with respect to the first boundary value problem (13), if at any function $h(x) \in C(\partial D)$ for the generalized solution $u = Bh$ the limit equality*

$$\lim_{\substack{x \to y \\ x \in D}} u(x) = h(y) \qquad (16)$$

is valid.

By analogy with [1] we may show that if the equality (16) is satisfied for any continuous function from $\tau_\lambda^{1,2}$, then the y is a regular point.

Definition 9. *The function $v_y(x)$ being a generalized solution of the equation (1) in D is called barrier at the point $y \in \partial D$ if*

i) for any $\rho > 0$ there exists $m > 0$ such that $v_y(x) > m$ in the sense of $W^1_{2,\lambda}(D)$ on the set $\{x : x \in \partial D,\ |x-y| > \rho\}$;

ii) $v_y(x)$ is continuous at the point y where $v_y(y) = 0$.

It is clear that the point $y \in \partial D$ is regular with respect to the first boundary value problem (13) iff the barrier $v_y(x)$ exists in it.

Denote for $x^0 \in \mathbb{E}_n$ and $R > 0$ the ball $\{x : |x - x^0| < R\}$ by $Q_R(x^0)$.

Lemma 5. *Let $D' \subset D$, $y \in \partial D' \cap \partial D$ besides for some $\rho > 0$*

$$\partial D \cap Q_\rho(y) = \partial D' \cap Q_\rho(y). \tag{17}$$

Then for regularity of the point y for domain D it is necessary and sufficient that it be regular for the domain D'.

Proof. Let the point y is regular for the domain D'. We show that this point is regular for the domain D. Consider the function $v_y(x) = B(|x-y|)$ being a general solution of the Dirichlet problem for the domain D with the boundary function $h(x) = |x-y|$. It is enough to show that this function is barrier with respect to D. It follows from the maximum principle that $v_y(x) \le C_7$ for $x \in D$. Then there exists a constant $C_8 > 0$ such that $v_y(x) \le C_8|x-y|$ for $x \in \partial D' \backslash (\partial D \cap \partial D')$. Let now $v'_y(x)$ be a generalized solution of the Dirichlet problem for the domain D' with the boundary function $h(x) = |x-y|$. According to the maximum principle $v_y(x) \le C_9 v'_y(x)$ for $x \in D'$ with some positive constant C_9, since $v'_y(x) = |x-y|,\ x \in \partial D'$. Taking into account that $v_y(x) \ge 0$ for $x \in D'$ we conclude that

$$\lim_{\substack{x \to y \\ x \in D'}} v_y(x) = 0\ .$$

Now it is enough to use (17), we obtain

$$\lim_{\substack{x \to y \\ x \in D}} v_y(x) = 0\ ,$$

i.e. $v_y(x)$ is a barrier with respect to D. Now assume that the point y is regular for the domain D. We determine the following set for $\rho > 0$

$$E_\rho = (\Sigma \backslash D) \cap \bar{Q}_\rho(y)\,, \quad D_\rho = \Sigma \backslash E_\rho,$$

moreover we assume ρ to be small that $\bar{Q}_\rho(y) \subset \Sigma$. It is clear that $\partial D_\rho \cap Q_\rho(y) = \partial D \cap Q_\rho(y)$ and $D \subset D_\rho$. According to the first part of the proof from the regularity of the point y for the domain D its regularity follows for the domain D_ρ. Consider now the generalized solution $u^\rho(x)$ of the first boundary value problem for the domain D_ρ with the boundary function $h(x)$ equal to the unit on ∂E_ρ and to zero on $\partial\Sigma$. By the assumption the function $u^\rho(x)$ is continuous at the point y, i.e.

$$\lim_{\substack{x \to y \\ x \in D_\rho}} u^\rho(x) = 1\ .$$

Let now $\omega_y(x) = \sum\limits_{k=2}^{\infty} 2^{-k}\left(1 - u^{\frac{\rho}{k}}(x)\right)$. According to the maximum principle $0 \leq u^{\frac{\rho}{k}}(x) \leq 1$ for $x \in D\rho$. Therefore the series determining $\omega_y(x)$ converges uniformly in D_ρ, i.e., the function is continuous at each strongly interior of subdomain of the domain D. We conclude from the continuity of the functions $u^{\frac{\rho}{k}}(x)$ at the point y for all the natural numbers k that $\omega_y(x)$ is continuous at the point y. Thus

$$\lim_{\substack{x\to y\\ x\in D_\rho}} \omega_y(x) = 0, \quad \text{i.e.} \quad \lim_{\substack{x\to y\\ x\in D'}} \omega_y(x) = 0\ .$$

It is clear that the function $\omega_y(x)$ is a generalized solution of the equation (1) in D_ρ. Further we note that the strong maimum principle according to which for each natural number k there exists a positive number a_k such that

$$u^{\frac{g}{k}}(x) \leq 1 - a_k \quad \text{for} \quad x \in \partial D' \backslash Q_{\frac{2\rho}{k}}(y)$$

follows from the Harnack type inequality.

Denote $2^{-k}a_k$ by m_k. Then for $x \in \partial D' \backslash Q_{\frac{2\rho}{k}}(y)$ $\quad 1 - u^{\frac{\rho}{k}}(x) \geq 2^k m_k$. We now fix an arbitrary natural number $k_0 \geq 2$. Then for $x \in \partial D' \backslash Q_{\frac{2\rho}{k_0}}(y)$, we have

$$\omega_y(x) \geq 2^{-k_0}\left(1 - u^{\frac{\rho}{k_0}}(x)\right) \geq m_k,$$

i.e. the function $\omega_y(x)$ is the barrier with respect to the domain D. The lemma is proved.

Remark. It is clear that for the regularity of the point y, it is necessary and sufficient that the functions $u^\rho(x)$ be continuous at the point y for sufficiently small ρ

3. L **- capacity**

Let $E \subset \Sigma$ be some compact, $U = \{\varphi(x) : \varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(\Sigma),\ \varphi(x) \geq 1$ on E in the sense of $\overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)\}$. Consider the following functional for $\varphi(x) \in U$

$$D_{\mathcal{L}}(\varphi) = \int_\Sigma \sum_{i,j=1}^n a_{ij}(x)\,\varphi_i\varphi_j dx\ .$$

The number $\inf\limits_{\varphi\in U} D_L(\varphi)$ is called $\mathcal{L}$-capacity of the compact E with respect to the ball Σ and is denoted by $cap_L^{(\Sigma)}(E)$. $\mathcal{L}$-capacity of the compact E with respect to $\mathbb{E}_n$ is called simply its $\mathcal{L}$-capacity and is denoted by $cap_L(E)$.

Theorem 6. *There exists the unique function* $u(x) \in U$ *such that* $D_{\mathcal{L}}(u) = cap_{\mathcal{L}}^{(\Sigma)}(E)$. *By that* $u(x) = 1$ *on* E *in the sense of* $\overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$.

The proof is led in the following scheme: the existence and uniqueness of the extremum function $u(x)$ follows from the convexity and completeness of the set U, from the convexity of the functional $D_{\mathcal{L}}(\varphi)$ and from the fact that $\overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$ is a Hilbert space. Further, since $u(x) \geq 1$ on E in the sense of $\overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$, then $\{u(x)\}^1 \in U$,, where

$$D_{\mathcal{L}}\left(\{u\}^1\right) \leq D_{\mathcal{L}}(u)\ .$$

Hence, we conclude that $D_{\mathcal{L}}\left(\{u\}^1\right) = D_{\mathcal{L}}(u)$ and by virtue of the inequalities of the extremum function $\{u\}^1 = u$, i.e. $u(x) = 1$ on E in sense of $\overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$.

Definition 10. *The function $u(x)$ giving the minimum of the functional $D_{\mathcal{L}}(\varphi)$ on the set U is called capacity potential of the compact E.*

Lemma 6. *Let $u(x)$ be capacity potential of the compact E. Then if $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$ and $\varphi(x) \geq 0$ on E in the sense of $\overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$, then*

$$\int_{\Sigma} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \varphi_j dx \geq 0. \tag{18}$$

Proof. We fix arbitrary $\varepsilon > 0$ and consider the function $v(x) = u(x) + \varepsilon\varphi(x)$. It is clear that $v(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$ and $v(x) \geq 1$ on E in the sense of $\overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$. Hence, it follows that $v(x) \in U$ and therefore

$$\int\limits_{\Sigma} \sum_{i,j=1}^{n} a_{ij}(x)(u_i + \varepsilon\varphi_i)\left(u_j + \varepsilon\varphi_j\right) dx \geq \int\limits_{\Sigma} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i u_j dx \ ,$$

i.e.

$$2\int\limits_{\Sigma} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \varphi_i dx + \varepsilon \int\limits_{\Sigma} \sum_{i,j=1}^{n} a_{ij}(x)\, \varphi_i \varphi_j dx \geq 0 \ .$$

Now it is enough to tend ε to zero and the required inequality (18) is proved.

It follows from lemma 6 that the capacity potential $u(x)$ of the compact E is $\mathcal{L}$-supeelliptic function in Σ, equal to 1 on E in the sense of $\overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$. Besides, since in (18) instead of $\varphi(x)$ we can take any function from $C^1\left(\bar{\Sigma}\right)$ with compact support in $\Sigma \backslash E$, then $u(x)$ is a generalized solution in $\Sigma \backslash E$ of the first boundary value problem (13) with $h(x) = 0$ on $\partial\Sigma$ and $h(x) = 1$ on ∂E.

Remark. It is clear that the functions $u^{\rho}(x)$ introduced in the previous item are capacity potentials of the sets E_{ρ}.

4. $\mathcal{L}_1$-solutions

Definition 11. *Let μ be measure of bounded variation on Σ. The function $u(x) \in \mathcal{L}_1(\Sigma)$ is called $\mathcal{L}_1$-solution of the equation $\mathcal{L}u = -\mu$ equal to zero on ∂D, if for any function $\varphi(x) \in \mathring{W}^1_{2,\lambda}(\Sigma) \cap C(\bar{\Sigma})$ such that $\mathcal{L}\varphi \in C(\Sigma)$, integral identity*

$$\int_\Sigma u\mathcal{L}\varphi dx = -\int_\Sigma \varphi d\mu$$

is satisfied.

According to theorem 3 there exists linear continuous operator G form $W^{-1}_{2,\lambda}(\Sigma)$ in $\mathring{W}^1_{2,\lambda}(\Sigma)$ such that for any functional $T \in W^{-1}_{2,\lambda}(\Sigma)$, the function $u = G(T)$ is the unique generalized solution from $\mathring{W}^1_{2,\lambda}(\Sigma)$ of the equation

$$Lu = T\ .$$

The operator G is called a Green operator.

It is clear that $u(x)$ is $\mathcal{L}_1$-solution of the equation $\mathcal{L}u = -\mu$ in S, vanishing on $\partial\Sigma$ iff for any function $\psi(x) \in C(\bar{\Sigma})$, the identity

$$\int_\Sigma u\psi dx = -\int_\Sigma G(\psi)\, d\mu \tag{19}$$

is fulfilled.

Remark. Let $u(x) \in \mathring{W}^1_{2,\lambda}(\Sigma)$ and for any function $\varphi(x) \in \mathring{W}^1_{2,\lambda}(\Sigma) \cap C(\bar{\Sigma})$ such that $\mathcal{L}\varphi \in C(\bar{\Sigma})$ the integral identity

$$\int_\Sigma \sum_{i,j=1}^n a_{ij}(x)\, u_i\varphi_j dx = \int_\Sigma \varphi d\mu \tag{20}$$

is fulfilled.

Then the function $u(x)$ is $\mathcal{L}_1$-solution of the equation $\mathcal{L}u = -\mu$, equal to zero on $\partial\Sigma$. Really, assume $\varphi = G(\psi)$, where $\psi \in C(\bar{\Sigma})$. We have according to (20)

$$\begin{gathered}\int_\Sigma u\psi dx = \int_\Sigma u\mathcal{L}\varphi dx = \int_\Sigma u \sum_{i,j=1}^n \left(a_{ij}(x)\varphi_j\right)_i dx = \\ = -\int_\Sigma \sum_{i,j=1}^n a_{ij}(x)\, u_i\varphi_j dx = -\int_\Sigma \varphi d\mu = -\int_\Sigma G(\psi)\, d\mu,\end{gathered} \tag{21}$$

and the required statement follows from (19).

On the other hand

$$\left|\int_\Sigma \varphi d\mu\right| \le \int_\Sigma \left|\sum_{i,j=1}^n a_{ij}(x)\, u_i\varphi_j\right| dx \le \int_\Sigma \left(\sum_{i,j=1}^n a_{ij}(x)\, u_i u_j\right)^{1/2} \times$$

$$\times \left(\sum_{i,j=1}^n a_{ij}(x)\, \varphi_i\varphi_j\right)^{1/2} dx \le \left(\int_\Sigma \sum_{i,j=1}^n a_{ij}(x)\, u_i u_j dx\right)^{1/2} \times$$

$$\times \left(\int_\Sigma \sum_{i,j=1}^n a_{ij}(x)\, \varphi_i\varphi_j dx\right)^{1/2} \le \gamma^{-1}\left(\int_\Sigma \sum_{i=1}^n \lambda_i(x)\, u_i^2 dx\right)^{1/2} \times$$

$$\times \left(\int_\Sigma \sum_{i=1}^n \lambda_i(x)\, \varphi_i^2 dx\right)^{1/2} \le \gamma^{-1}\|u\|_{W^1_{2,\lambda}(\Sigma)}\|\varphi\|_{W^1_{2,\lambda}(\Sigma)} = C_{10}(\gamma,u)\|\varphi\|_{W^1_{2,\lambda}(\Sigma)}$$

follows from (21).

Thus $\mu \in \overset{\circ}{W}{}^{-1}_{2,\lambda}(\Sigma)$. By that the following theorem is proved.

Theorem 7. *$\mathcal{L}_1$ - solution $u(x)$ of the equation $\mathcal{L}u = -\mu$ in Σ vanishing on $\partial\Sigma$, belongs to the space $\overset{\circ}{W}{}^{-1}_{2,\lambda}(\Sigma)$ iff $\mu \in \overset{\circ}{W}{}^{-1}_{2,\lambda}(\Sigma)$. We now note not specifying it we assume the coefficients of the operator $\mathcal{L}$ be extended in $\mathbb{E}_n\backslash D$ in the following form: $a_{ij}(x) = \delta_{ij}\lambda_i(x)$ for $x \in \mathbb{E}_n\backslash D$* and $i,j = 1,...,n$.. *Here δ_{ij} is a Kronecker's symbol. Let further $f^{(h)}(x)$ be Friedrichs averaging of the function $f(x)$ with the parameter $h > 0$ and*

$$\mathcal{L}_n = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^{(h)}(x)\frac{\partial}{\partial x_j}\right) .$$

Theorem 8. *Let μ be measure of bounded variation on Σ, $u_h(x)$ be $\mathcal{L}_1$-solution of the equation $\mathcal{L}_h u_h = -\mu$ in Σ, vanishing on $\partial\Sigma$. Then there exist $P_1(\lambda,n) > 1$ and $q_1(\lambda,n) > 1$ such that the sequence $\{u_h(x)\}$ converges to $\mathcal{L}_1$-solution $u(x)$ of the equation $\mathcal{L}u = -\mu$ in Σ, vanishing on $\partial\Sigma$, weakly in $\overset{\circ}{W}{}^1_{p,\lambda}(\Sigma)$ and strongly in $\mathcal{L}_q(\Sigma)$, if only $1 \le p < p_1$, $1 \le q < q_1$..*

This theorem may be proved by the method of [9].

5. Green function

Definition 12. *Let y be an arbitrary fixed point Σ. The function $g(x,y)$ being $\mathcal{L}_1$-solution of the equation $\mathcal{L}g = -\delta(x-y)$ in Σ, vanishing on $\partial\Sigma$ is called Green function of the operator $\mathcal{L}$ in Σ.*

Here $\delta(x)$ is generalized Dirac function.

It follows from the given definition that if $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$ is a generalized solution of the equation $\mathcal{L}\varphi = -\psi$ in Σ where $\psi \in C(\bar\Sigma)$, then it may be represented in the form of

$$\varphi(y) = \int_\Sigma g(x,y)\,\psi(x)\,dx\ . \tag{22}$$

Theorem 9. *For any measure μ of the bounded variation on Σ, the function*

$$u(x) = \int_{\Sigma} g(x,y)\, d\mu(y) \tag{23}$$

is finite everywhere in Σ and is $\mathcal{L}_1$-solution of the equation $\mathcal{L}u = -\mu$ in Σ, vanishing on $\partial\Sigma$.

Proof. Let $\psi(x) \geq 0$, $\psi(x) \in C\left(\bar{\Sigma}\right)$ and $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(\Sigma)$ be a generalized solution of the equation $\mathcal{L}\varphi = -\psi$ in Σ. Then $\varphi(x) \in C\left(\bar{\Sigma}\right)$, since $\partial\Sigma$ is a smooth surface and the boundary function is equal to zero. According to the maximum principle $\varphi(x) \geq 0$ in Σ. Besides, for it, the representation (22) is valid. thus, there exists the integral

$$I = \int_{\Sigma} \varphi(y)\, d\mu(y) = \int_{\Sigma} d\mu(y) \int_{\Sigma} g(x,y)\, \psi(x)\, dx\ .$$

By the Fubini theorem subject to (23) we obtain

$$I = \int_{\Sigma} \psi(x)\, dx \int_{\Sigma} g(x,y)\, d\mu(y) = \int_{\Sigma} u(x)\, \psi(x)\, dx\ .$$

By the same taken

$$\int_{\Sigma} u\psi dx = \int_{\Sigma} \varphi d\mu = \int_{\Sigma} G(\psi)\, d\mu\ ,$$

i.e. according to (19) the function $u(x)$ is $\mathcal{L}_1$-solution of the equation $\mathcal{L}\mu = -\mu$ in Σ, vanishing on $\partial\Sigma$.

The theorem has been proved.

Theorem 10. *Let μ and υ be measures of bounded variation on Σ and the functions $u(x)$ and $\upsilon(x)$ be $\mathcal{L}_1$-solutions of the equations $Lu = -\mu$ and $\mathcal{L}\upsilon = -\upsilon$ in Σ respectively, vanishing on $\partial\Sigma$. Then*

$$\int_{\Sigma} u d\upsilon = \int_{\Sigma} \upsilon d\mu = \int\int_{\Sigma\times\Sigma} g(x,y)\, d\mu(x)\, d\upsilon(y)\ .$$

This assertion follows from the Fubini theorem. Let now $u(x)$ is capacity potential of the compact $E \subset \Sigma$. Then by lemma 6 for any function $\varphi(x) \in C_0^{\infty}(\Sigma)$, $\varphi(x) \geq 0$ on E, the inequality

$$\int_{\Sigma} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \varphi_j dx \geq 0$$

is valid.

Consider the functional $J(\varphi) = \int_{\Sigma} \sum\limits_{i,j=1}^{n} a_{ij}(x)\, u_i \varphi_j dx$. According to the Schwartz theorem there exists measure μ of bounded variation on Σ (with the respect on Σ) such that

$$J(\varphi) = \int_{\Sigma} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \varphi_j dx = \int_{\Sigma} \varphi d\mu. \tag{24}$$

But $u(x)=1$ on E in sense of $\overset{\circ}{W}{}^{1}_{2,\lambda}(\Sigma)$. Therefore, support of measure is on ∂E.

The measure μ is called capacity distribution of the compact E.

It is clear that the capacity potential $u(x)$ of the compact E is $\mathcal{L}_1$-solution of the equation $\mathcal{L}u=-\mu$ in Σ vanishing on $\partial\Sigma$. Therefore

$$u(x)=\int_{\Sigma} g(x,y)\,d\mu(y) \ .$$

On the other hand, since $u(x)=1$ on E in the sense of $\overset{\circ}{W}{}^{1}_{2,\lambda}(\Sigma)$, then there exists the sequence $\{\varphi_m(x)\}$ such that $\varphi_m(x)\in M_0(\Sigma)$, $\varphi_m(x)=1$ on E and $\|\varphi_m-u\|_{W^1_{2,\lambda}(\Sigma)}\to 0$ as $m\to\infty$. We rewrite the equality (24) by substituting the function $\varphi(x)$ instead of $\varphi_m(x)$. We have subject to inclusion supp$\mu\subset E$

$$\int_{\not\Sigma}\sum_{i,j=1}^{n} a_{ij}(x)\,u_i\,(\varphi_m)_j\,dx=\int_{\Sigma}\varphi_m d\mu=\int_{E}\varphi_m d\mu=\int_{E} d\mu=\mu(E)\,. \tag{25}$$

Passing to the limit as $m\to\infty$ in (25) and using that

$$\int_{\Sigma}\sum_{i,j=1}^{n} a_{ij}(x)\,u_i u_j dx = cap_{\mathcal{L}}^{(\Sigma)}(E)\,,$$

we pass to the equality

$$cap_{\mathcal{L}}^{(\Sigma)}(E)=\mu(E)\,. \tag{26}$$

6. Estimations of the Green function

For the further reasoning, we assume $\Pi_{R;\beta}(y)=\left\{x:|x_i-y_i|\le\beta\,\omega_i^{-1}(R)\,;\ i=1,..,n\right\}$ where $y\in\mathbb{E}_n$, $\beta>0$, $R>0$. Let $y\in\Pi_{R;\frac{1}{4}}(0)$ where we suppose k to be so small that $\Pi_{R;2}(0)\subset\Sigma$. Without losing generality we shall assume that $a_{ij}(x)\in C^{\infty}\left(\bar{\Sigma}\right)$ (the general case is obtained with the help of limit passage). Then it is known that $\lim\limits_{x\to y} g(x,y)=+\infty$ and besides, the function $g(x,y)$ is continuous at $x\ne y$. Hence, it follows that if a is some positive number, then y is an interior point the set $J_a=\{x:g(x,y)\ge a\}$. Consider the capacity potential $u(x)$ of this compact. It is clear that

$$u(x)=\int_{\Sigma} g(z,x)\,d\nu_a(z)\,,$$

where ν_a is capacity distribution of J_a. At that the function $u(x)$ is continuous at interior points of J_a since supp$\nu_a\subset\partial J_a$. Thus

$$1=\int_{\partial J_a} g(z,y)\ \ d\nu_a(z)\,.$$

On the other hand for $z \in \partial J_a$ $g(z,y) = a$. Therefore, subject to (28) we obtain

$$cap_{\mathcal{L}}^{(\Sigma)}(J_a) = \frac{1}{a}. \tag{27}$$

Let now $\alpha = \inf\limits_{x\in\partial\Pi_{R;1}} (y)$. It follows from the maximum principle that $\Pi_{R;1}(y) \subset J_a$ and from (27) we conclude

$$cap_{\mathcal{L}}^{(\Sigma)}(\Pi_{R;1}(y)) \le cap_{\mathcal{L}}^{(\Sigma)}(J_a) = \frac{1}{a} = \frac{1}{\inf\limits_{x\in\partial\Pi_{R;1}(y)} g(x,y)}. \tag{28}$$

Analogously, if $b = \sup\limits_{x\in\partial\Pi_{R;1}(y)} g(x,y)$, then $\Pi_{R;1}(y) \supset J_b$ and therefore

$$cap_{\mathcal{L}}^{(\Sigma)}(\Pi_{R;1}(y)) \ge cap_{\mathcal{L}}^{(\Sigma)}(J_b) = \frac{1}{b} = \frac{1}{\sup\limits_{x\in\partial\Pi_{R;1}(y)} g(x,y)}. \tag{29}$$

It follows from (28)-(29) that

$$\inf_{x\in\partial\Pi_{R;1}(y)} g(x,y) \le \left(cap_{\mathcal{L}}^{(\Sigma)}(\Pi_{R;1}(y))\right)^{-1} \le \sup_{x\in\partial\Pi_{R;1}(y)} g(x,y). \tag{30}$$

But according to the Harnack type inequality

$$\sup_{x\in\partial\Pi_{R;1}(y)} g(x,y) \le C_{11}(\gamma,\lambda,n) \inf_{x\in\partial\Pi_{R;1}(y)} g(x,y). \tag{31}$$

But we obtain from (30)-(31) for $x \in \partial\Pi_{R;1}(y)$

$$C_{11}^{-1}\left[cap_{\mathcal{L}}^{(\Sigma)}(\Pi_{R;1}(y))\right]^{-1} \le g(x,y) \le C_{11}\left[cap_{\mathcal{L}}^{(\Sigma)}(\Pi_{R;1}(y))\right]^{-1}. \tag{32}$$

Thus, the following theorem has been proved.

Theorem 1.1. *If with respect to the coefficient of the operator $\mathcal{L}$, the conditions (2)-(4) $y \in \Pi_{R;\frac{1}{4}}(0)$, $x \in \partial\Pi_{R;1}(y)$ are satisfied, then for sufficiently small R, for the Green function $g(x,y)$ the estimations (32) are valid.*

Denote the model operator

$$\mathcal{L}_0 = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\lambda_i(x)\frac{\partial}{\partial x_i}\right)$$

by $\mathcal{L}_0$.

According to the condition (2) for any compact $E \subset \Sigma$

$$\gamma cap_{\mathcal{L}_0}^{(\Sigma)}(E) \le cap_{\mathcal{L}}^{(\Sigma)}(E) \le \gamma^{-1} cap_{\mathcal{L}_0}^{(\Sigma)}(E). \tag{33}$$

If we assume $\Sigma = \mathbb{E}_n$ and denote the function $g(x,y)$ by $G(x,y)$ (in this case, as it is clear that $G(x,y)$ is a fundamental solution of the equation (1), then we derive the following one from (32)-(33).

Corollary. *If the conditions of theorem 11 are satisfied, then there exists a constant* $C_{12}(\gamma,\lambda,n)$ *such that for sufficiently small* R

$$C_{12}^{-1}\left[cap\left(\Pi_{R;1}(y)\right)\right]^{-1}\le G(x,y)\le C_{12}\left[cap\left(\Pi_{R;1}(y)\right)\right]^{-1}, \qquad (34)$$

where $cap(E)=cap_{\mathcal{L}_0}(E)$.

Lemma 7. *Let* $y\in\Pi_{R;\frac{1}{4}}(0)$*. Then for sufficiently small* R *the estimation*

$$cap\left(\Pi_{R;1}(y)\right)\le C_{13}(\lambda,n)\,R^{-2}\prod_{i=1}^{n}\omega_i^{-1}(R) \qquad (35)$$

is valid.

Proof. Introduce the function $f^i(t)$ be the following form: $f^i(t)=1$, if, $|t-y_i|\le\le\omega_i^{-1}(R)$; $f^i(t)=0$ if $|t-y_i|\ge 2\omega_i^{-1}(R)$; $0\le f^i(t)\le 1$; $f^i(t)\in C^\infty(\mathbb{E}_1)$ and for$i=1,...,n$

$$\left|\frac{df^i(t)}{dt}\right|\le\frac{C_{14}}{\omega_i^{-1}(R)}.$$

Let further $u(x)=\prod_{i=1}^{n}f^i(x_i)$. It is clear that $u(x)=1$ for $x\in\Pi_{R;1}(y)$, $u(x)=0$ for $x\notin\Pi_{R;2}(y)$, $u(x)\in\overset{\circ}{W}{}^1_{2,\lambda}(\mathbb{E}_n)$, where for $i=1,...,n$

$$|u_i|\le\frac{C_{14}}{\omega_i^{-1}(R)}. \qquad (36)$$

We have subject to (36)

$$cap\left(\Pi_{R;1}(y)\right)\le D_{\mathcal{L}_0}(u)=\int\limits_{\Pi_{R;2}(y)}\sum_{i=1}^{n}\lambda_i(x)\,u_i^2dx=$$

$$=\int\limits_{\Pi_{R;2}(y)\backslash\Pi_{R;1}(y)}\sum_{i=1}^{n}\lambda_i(x)\,u_i^2dx\le C_{14}^2\int\limits_{\Pi_{R;2}(y)\backslash\Pi_{R;1}(y)}\sum_{i=1}^{n}\frac{\lambda_i(x)}{\left(\omega_i^{-1}(R)\right)^2}dx.$$

On the other hand acting in the same way as in conclusion we obtain that for $x\in\Pi_{R;2}(y)\backslash\Pi_{R;1}(y)$ and $i=1,...,n$

$$\lambda_i(x)\le C_{15}(\lambda,n)\,\frac{\left(\omega_i^{-1}(R)\right)^2}{R^2}.$$

Hence, it follows that

$$cap\left(\Pi_{R;1}(y)\right)\le C_{16}(\lambda,n)\,R^{-2}mes\left(\Pi_{R;2}(y)\backslash\Pi_{R;1}(y)\right)\le$$

$$\le C_{16}R^{-2}mes\left(\Pi_{R;2}(y)\right)=2^nC_{16}R^{-2}\prod_{i=1}^{n}\omega_i^{-1}(R).$$

By the same taken the required estimation (35) has been proved.

Lemma 8. *Let $y \in \Pi_{R;\frac{1}{4}}(0)$. Then for sufficiently small R the estimation*

$$cap\left(\Pi_{R;1}(y)\right) \geq C_{17}(\lambda, n) R^{-2} \prod_{i=1}^{n} \omega_i^{-1}(R) \tag{37}$$

is valid.

Proof. Let $\Pi'_R(y) = \left\{x : \frac{1}{2}\omega_i^{-1}(R) \leq |x_i - y_i| \leq \omega_i^{-1}(R);\ i = 1, ..., n\right\}$. It is clear that $\Pi'_R(y) \subset \Pi_{R;1}(y)$, therefore

$$cap\left(\Pi_{R;1}(y)\right) \geq cap\left(\Pi'_R(y)\right). \tag{38}$$

We make change of variables $z_i = \frac{x_i}{\omega_i^{-1}(R)};\ i = 1, ..., n.$

Thus the compact $\tilde{\Pi}'(\tilde{y}) = \left\{z : \frac{1}{2} \leq |z_i - \tilde{y}_i| \leq 1;\ i = 1, ..., n\right\}$ will be image of $\Pi'_R(y)$. Here $\tilde{y}$ is image of the point y. We have further for $x \in \Pi'_R(y)$ and $i = 1, ..., n$

$$\lambda_i(x) = \frac{\left(\omega_i^{-1}(\rho)\right)^2}{\rho^2} = \frac{\left[\omega_i^{-1}\left(\sum_{k=1}^{n}\omega_k(|x_k|)\right)\right]^2}{\left(\sum_{k=1}^{n}\omega_k(|x_k|)\right)^2} =$$

$$= \frac{\left[\omega_i^{-1}\left(\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)|z_k|\right)\right)\right]^2}{\left(\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)|z_k|\right)\right)^2}, \tag{39}$$

where $z \in \tilde{\Pi}'(\tilde{y})$. On the other hand, if $t \in (0,1)$ and natural number l be such that

$$2^{-l-1} \leq t < 2^{-l},$$

then for $i = 1, ..., n$, according to the condition (3)

$$\omega_i(t\ R) \geq \frac{1}{2}\omega_i(2tR) \geq 2^{-l-1}\omega_i\left(2^{l+1}t\ R\right) \geq 2^{-l-1}\omega_i(R) \geq \frac{t}{2}\omega_i(R).$$

It is clear that the analogous estimation holds if $t = 0$ and $t = 1$. Thus

$$\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)|z_k|\right) \geq \frac{1}{2}\sum_{k=1}^{n}|z_k|\,\omega_k\left(\omega_k^{-1}(R)\right) = \frac{R}{2}\sum_{k=1}^{n}|z_k|.$$

By the same taken for $i = 1, ..., n$

$$\left[\omega_i^{-1}\left(\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)|z_k|\right)\right)\right]^2 \geq \left[\omega_i^{-1}\left(\frac{R}{2}\sum_{k=1}^{n}|z_k|\right)\right]^2 \tag{40}$$

But for $k = 1, ..., n$ $\quad |z_k| \geq |z_k - \tilde{y}_k| - |\tilde{y}_k| \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$

Therefore, from (40) and from the condition (3) we conclude for $i=1,...,n$

$$\left[\omega_i^{-1}\left(\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)\,|z_k|\right)\right)\right]^2\geq\left[\omega_i^{-1}\left(\frac{Rn}{8}\right)\right]^2\geq C_{18}(\lambda,n)\left(\omega_i^{-1}(R)\right)^2. \quad (41)$$

Analogously, since for $k=1,...n \quad |z_k|\leq|\tilde{y}_k|+1\leq\frac{5}{4}$, we obtain

$$\left(\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)\,|z_k|\right)\right)^2\leq\left(\sum_{k=1}^{n}\omega_k\left(\frac{5}{4}\omega_k^{-1}(R)\right)\right)^2\leq$$

$$\leq 4\left(\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)\right)\right)^2=4n^2R^2. \quad (42)$$

Using (41) and (39) we pass to the estimation

$$\lambda_i(x)\geq C_{19}(\lambda,n)\,\frac{(\omega_i(R))^2}{R^2}, \quad (43)$$

if $x\in\Pi'_R(y)$ and $i=1,...,n.$

Let now $u(x)\in\mathring{W}^1_{2,\lambda}(\mathbb{E}_n)$ be an arbitrary function such that $u(x)=1$ on $\Pi'_R(y)$ in the sense of $\mathring{W}^1_{2,\lambda}(\mathbb{E}_n)$, and $\tilde{u}(z)$ be its image at transformation of coordinates made by us. We have subject to (43)

$$\int_{\mathbb{E}_n}\sum_{i=1}^{n}\lambda_i(x)\,u_i^2dx\geq C_{19}R^{-2}\int_{\mathbb{E}_n}\sum_{i=1}^{n}\left(\omega_i^{-1}(R)\right)^2u_i^2dx=$$

$$=C_{19}R^{-2}\int_{\mathbb{E}_n}\sum_{i=1}^{n}\left(\omega_i^{-1}(R)\right)^2\cdot\left(\frac{1}{\omega_i^{-1}(R)}\tilde{u}_{z_i}\right)^2\prod_{i=1}^{n}\omega_i^{-1}(R)\,dz=$$

$$=C_{19}R^{-2}\prod_{i=1}^{n}\omega_i^{-1}(R)\int_{\mathbb{E}_n}(\nabla\tilde{u})^2\,dz. \quad (44)$$

It follows from (44) that

$$cap\left(\Pi'_R(y)\right)\geq C_{19}R^{-2}\prod_{i=1}^{n}\omega_i^{-1}(R)\;\; p\left(\tilde{\Pi}'(y)\right),$$

where $p(E)$ is Wiener capacity of the compact E. But

$$p\left(\tilde{\Pi}'(\tilde{y})\right)=p\left(\left\{z:\frac{1}{2}\leq|z_i|\leq1;\; i=1,...,n\right\}\right)=C_{20}(n),$$

and thus

$$cap\left(\Pi'_R(y)\right)\geq C_{19}C_{20}R^{-2}\prod_{i=1}^{n}\omega_i^{-1}(R). \quad (45)$$

Now the required estimation (37) follows from (38) and (45).

Let now $\rho(x,y)=\sum_{i=1}^{n}\omega_i(|x_i-y_i|)$.

Theorem 12. *If with respect to the coefficients of the operator $\mathcal{L}$ the conditions (2)-(4), $x\in Q_\delta(0)$, $y\in Q_\delta(0)$, $x\neq y$ are satisfied, then for sufficiently small δ for the fundamental solution $G(x,y)$, the estimations*

$$C_{21}^{-1}(\gamma,\lambda,n)(\rho(x,y))^2\left(\prod_{i=1}^{n}\omega_i^{-1}(\rho(x,y))\right)^{-1}\leq$$

$$\leq G(x,y)\leq C_{21}(\rho(x,y))^2\left(\prod_{i=1}^{n}\omega_i^{-1}(\rho(x,y))\right)^{-1} \tag{46}$$

are valid.

Proof. Let x and y be located in the ball $Q_\delta(0)$ and $x\neq y$. Assume

$$R=\max_{1\leq i\leq n}\omega_i(|x_i-y_i|).$$

Then there i_0, $1\leq i_0\leq n$ exists such that $R=\omega_{i_0}(|x_{i_0}-y_{i_0}|)$. Thus

$$R\leq\sum_{i=1}^{n}\omega_i(|x_i-y_i|)=\rho(x,y). \tag{47}$$

On the hand side, since for $i\leq i\leq n$

$$\omega_i(|x_i-y_i|)\leq R,$$

then

$$\rho(x,y)=\sum_{i=1}^{n}\omega_i(|x_i-y_i|)\leq nR,\quad \text{i.e.}$$

$$R\geq\frac{1}{n}\rho(x,y). \tag{48}$$

According the corollary from theorem 11 and lemmas 7-1 that for $x\in\partial\Pi_{R;1}(y)$, the estimations

$$C_{22}^{-1}(\gamma,\lambda,n)\left[R^{-2}\prod_{i=1}^{n}\omega_i^{-1}(R)\right]^{-1}\leq G(x,y)\leq C_{22}\left[R^{-2}\prod_{i=1}^{n}\omega_i^{-1}(R)\right]^{-1} \tag{49}$$

are valid.

Now allowing for (47) and (48) in (49), we arrive at the required estimation (46) the theorem has been proved.

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