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NEW PROOF OF GAROFALO-LANCONELLI REGULARITY CRITERION FOR THE SECOND ORDER PARABOLIC EQUATIONS OF DIVERGENT STRUCTURE II

Abstract

In the paper the first boundary value problem for divergent parabolic equations of the second order is considered. Simple proof of regularity criterion of boundary point of Wiener type established by Garofalo and Lanconelli is given for domains whose boundaries have a special symmetry in some neighbourhood of this point.

4^0. Sufficient condition of regularity

Let $\mathbb{E}_n$ and $\mathbb{R}_{n+1}$ be Euclidean spaces of points $x=(x_1,...,x_n)$ and $(x,t)=(x_1,...,x_n,t)$ respectively, D be a bounded domain in $\mathbb{R}_{n+1}$ with boundary ∂D and parabolic boundary $\Gamma(D)$. Consider in D the first boundary value problem

$$\mathcal{L}u=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)-\frac{\partial u}{\partial t}=0\ ;\quad (x,t)\in D, \tag{1}$$

$$u|_{\Gamma(D)}=\varphi\ ;\quad \varphi\in C(\Gamma(D)) \tag{2}$$

under the assumption that $\|a_{ij}(x,t)\|$ is a real symmetrical matrix, in addition, for all $(x,t)\in D$ and $\xi\in\mathbb{E}_n$ the following condition is fulfilled

$$\gamma|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\gamma^{-1}|\xi|^2\ ;\quad \gamma\in(0,1]-\text{constant}. \tag{3}$$

Besides, we will assume that the coefficients of the operator $\mathcal{L}$ are infinitely differentiable in $\bar{D}$ functions.

Let further $u_\varphi(x,t)$ be generalized by Perron solution of the first boundary value problem (1)-(2). Point $\left(x^0,t^0\right)\in\Gamma(D)$ is called regular with respect to the first boundary value problem (1)-(2) if for any function $\varphi(x,t)\in C(\Gamma(D))$ the limit equality

$$\lim_{\substack{(x,t)\to\left(x^0,t^0\right)\\(x,t)\in D}}u_\varphi(x,t)=\varphi\left(x^0,t^0\right)$$

holds. Garofalo and Lanconelli [1] have established the regularity criterion of boundary point which is an analogue of the correspondent Evans and Gariepy [2] criterion for the heat equation. The aim of the present paper is finding a new proof of Garofalo and Lanconelli criterion for domains whose boundaries have a special symmetry in some neighbourhood of the boundary point being investigated for regularity.

Thus, we will assume that $(0,0)\in\Gamma(D)$ and intersection of D with the layer $\mathcal{B}=\{(x,t):x\in\mathbb{E}_n,-b<t<0\}$ is represented in the form $\Big\{(x,t):|x|^2<-t\alpha(-t)\,,$

$-b < t < 0\}$, where b is sufficiently small positive number, $\alpha(z)$ is positive continuously differentiable and non-increasing function on $(0,b]$, moreover, there exists finite or infinite limit

$$\left.\begin{array}{r} d = \lim\limits_{z\to 0+} |\alpha'(z)|\, z \ln \dfrac{1}{z}, \\ \text{and, if } d=4, \quad \text{then} \quad \displaystyle\int_0^b \psi^+(z)\, dz < \infty \ . \end{array}\right\} \tag{4}$$

Here $\psi(z) = |\alpha'(z)| - \frac{4}{z \ln \frac{1}{z}}, \quad \psi^+(z) = \max\{\psi(z), 0\}$.

Besides, we will assume that

$$a_{ij}(0,0) = \delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j \ . \end{cases} \tag{5}$$

Let for $s > 0$ and $\beta > 0$

$$G_{s,\beta}(x,t) = \begin{cases} t^{-s} \exp\left[-\frac{|x|^2}{4\beta\, t}\right], & \text{if} \quad t > 0 \ , \\ 0 \quad , & \text{if} \qquad t \leq 0 \ . \end{cases}$$

For all natural m denote the set $\left\{(y,\tau) : e^{\frac{mn}{2}} \leq G_{\frac{n}{2},1}(y,-\tau) \leq e^{\frac{(m+1)n}{2}}\right\}$ by A_m, and let $H_m = A_m \backslash D$.

Main theorem. *If conditions (3)-(5) are fulfilled relative to the coefficients of operator $\mathcal{L}$ and domain D, then for regularity of the point $(0,0)$ with respect to the first boundary value problem (1)-(2) it is necessary and sufficient that*

$$\sum_{m=1}^{\infty} e^{\frac{mn}{2}} p(H_m) = \infty. \tag{6}$$

Here $p(H_m)$ is a heat capacity of compact H_m.

The present paper is continuation of paper [3].

In this connection we will use the results have been obtained in it preserving all the denotations and common numeration of items, theorems, lemmas and constants.

Lemma 5. *Let for $i \geq 3 \quad m_i = [5i \ln\ln i] + 1$. Then for sufficiently large i*

$$m_{i+1} - m_i \geq 4 \ln\ln m_i .$$

Proof. We have

$$m_{i+1} - m_i \geq 5(i+1)\ln\ln(i+1) - 5i\ln\ln i - 1 \geq 5\ln\ln i - 1 =$$

$$= 5\ln\left[\ln(6i\ln\ln i) - \ln(6\ln\ln i)\right] - 1 \geq 5\ln\left[\frac{1}{2}\ln(6i\ln\ln i)\right] - 1 \geq$$

$$\geq 4\ln\ln(6i\ln\ln i)\,,$$

if i is sufficiently large. Now it suffices to note that for large $i \quad 6i\ln\ln i \geq m_i$. The lemma is proved.

Corollary. *Let $u(x,t)$ be positive and bounded in D_{m_i} $\mathcal{L}-$ subparabolic function vanishing on $S(D_{m_i})$, in addition, the coefficients of operator $\mathcal{L}$ and domain D satisfy the conditions (3)-(5) and (12). Then for sufficiently large i*

$$\sup_{D_{m_i}} u \geq \left(1+\eta\, e^{m_i s^+_{m_i}} p^+_{m_i}\left(H^+_{m_i}\right)\right) \sup_{D_{m_{i+1}}} u. \tag{56}$$

The required statement follows from theorem 1 and lemma 5.

Theorem 2. *If the coefficients of operator $\mathcal{L}$ and domain D satisfy the conditions (3)-(5), then for regularity of the point $(0,0)$ with respect to the first boundary value problem (1)-(2) it is sufficient that*

$$\sum_{m=3}^{\infty} \frac{1}{\ln\ln m} e^{m s^+_m} p^+_m\left(H^+_m\right) = \infty. \tag{57}$$

Proof. According to [4] for the regularity of the point $(0,0)$ it suffices to show the following: for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that for any $\mathcal{L}$-subparabolic in D function $u(x,t) \leq 1$ vanishing on $\Gamma(D) \cap \{(x,t) : -\varepsilon_1 < t < 0\}$, for $(x,t) \in D \cap \{(x,t) : -\delta < t < 0\}$ the inequality $u(x,t) < \varepsilon_2$ is fulfilled. Let numbers ε_1, ε_2, function $u(x,t)$ be already given. Denote by $j \geq 3$ a natural number for which $z^+_{m_j} < \varepsilon_1$, and let the natural number $k > j+1$ be so that in D_{m_k} there exists a point $\left(x^0, t^0\right)$ at which

$$u\left(x^0, t^0\right) \geq \varepsilon_2. \tag{58}$$

Note that from (57) follows the fulfillment of condition (12). Applying successively inequality (56) and and using (58), we obtain

$$1 \geq M_{m_j} \geq \prod_{i=j}^{k-1} \left(1+\eta e^{m_i s^+_{m_i}} p^+_{m_i}\left(H^+_{m_i}\right)\right) \varepsilon_2,$$

i.e.

$$\sum_{i=j}^{k-1} \ln\left(1+\eta e^{m_i s^+_{m_i}} p^+_{m_i}\left(H^+_{m_i}\right)\right) \leq \ln\frac{1}{\varepsilon_2}. \tag{59}$$

On the other hand,

$$H^+_m \subset A^+_m \subset C^{-e^{-m},0}_{2\sqrt{\frac{\beta^+_m s^+_m}{e}e^{-\frac{m}{2}}}}(0) \subset C^{-\frac{3n}{e}e^{-m},0}_{\sqrt{\frac{3n}{e}}e^{-\frac{m}{2}}}(0),$$

if m is sufficiently large. Therefore, applying assertion 1, we conclude that for large m

$$p^+_m\left(H^+_m\right) \leq C_{21}(n)\, e^{-m s^+_m}.$$

Let $C_{22}(n) = \inf\limits_{t\in(0,C_{21}]} \dfrac{\ln(1+\eta t)}{t}$. Then from (59) it follows that for sufficiently large j

$$\sum_{i=j}^{k-1} e^{m_i s^+_{m_i}} p^+_{m_i}\left(H^+_{m_i}\right) \leq \frac{1}{C_{22}} \ln\frac{1}{\varepsilon_2}. \tag{60}$$

We will show now that (57) implies

$$\sum_{i=3}^{\infty} e^{m_i s^+_{m_i}} p^+_{m_i}\left(H^+_{m_i}\right)=\infty. \tag{61}$$

Without loss of generality, we can assume that sequence $\left\{e^{ms^+_m}p^+_m\left(H^+_m\right)\right\}$ monotonically decreasing tends to zero as $m\to\infty$. Therefore taking into account that for any continuous and monotonically decreasing on $[1,\infty)$, positive function $h(z)$ tending to zero as $z\to\infty$ for any natural $i>1$ the inequalities

$$\sum_{k=2}^{i} h(k)\le\int_1^i h(z)\,dz\le\sum_{k=1}^{i-1}h(k),$$

are true, we conclude

$$\sum_{i=3}^{q} e^{m_i s^+_{m_i}} p^+_{m_i}\left(H^+_{m_i}\right)\ge C_{23}(n)\int_3^{q+1} e^{m(t)s^+\left(e^{-m(t)}\right)}p^+_{\{m(t)\}}\left(H^+_{\{m(t)\}}\right)dt, \tag{62}$$

if natural number q is sufficiently large. Here $m(t)=5t\ln\ln t$, $H^+_{\{z\}}=$ $=\left\{(y,\tau):e^{zs^+\left(e^{-z}\right)}\le G^+_{(z)}(y,-\tau)\le e^{(z+1)s^+\left(e^{-z}\right)}\right\}\setminus D$, $p^+_{\{z\}}(E)$ is $\left(s^+\left(e^{-z}\right),\beta^+\left(e^{-z}\right)\right)-$ capacity of compact E.

But on the other hand

$$\int_3^{q+1} e^{m(t)s^+\left(e^{-m(t)}\right)}p^+_{\{m(t)\}}\left(H^+_{\{m(t)\}}\right)dt\ge\frac{1}{6}\int_{15\ln\ln 3}^{5(q+1)\ln\ln(q+1)} e^{zs^+\left(e^{-z}\right)}\times$$

$$\times p^+_{\{z\}}\left(H^+_{\{z\}}\right)\frac{dz}{\ln\ln z}\ge\frac{1}{6}\sum_{m=[15\ln\ln 3]+2}^{[5q\ln\ln q]}\frac{1}{\ln\ln m}e^{ms^+_m}p^+_m\left(H^+_m\right). \tag{63}$$

Now from (62)-(63) one can see that (57) implies the validity of (61). But by virtue of (61) inequality (60) can be satisfied only for $k\le l(\varepsilon_1,\varepsilon_2,n)$. Now it suffices to choose $\delta=z^+_{m_l+1}$, and the theorem is proved.

Lemma 6. *If*

$$\sum_{m=3}^{\infty}\frac{1}{\ln\ln m}e^{\frac{mn}{2}}p(H_m)=\infty, \tag{64}$$

then equality (57) holds.

Proof. Note that if in inequalities (7) and (8) $a_0=b_0=0$, then $s^+_m=\frac{n}{2}$, $\beta^+_m=1$ and $G^+_{[m]}(x,t)=G_{\frac{n}{2},1}(x,t)$. In this case denote the sequences $\{z^+_m\}$ and $\{\nu^+_m\}$ by $\{z_m\}$ and $\{\nu_m\}$, respectively. At that z_m is a root of equation

$$\gamma(z_m)=2n\ln\frac{e^{-m}}{z_m}, \tag{65}$$

situated on the interval $\left(\frac{e^{-m}}{m^{\lambda}}, e^{-m}\right)$. According to lemmas 3 and 4, for sufficiently large m

$$C_{24}(n)\left(z_m \ln \frac{1}{\nu_m}\right)^{\frac{n}{2}} \le p(H_m) \le C_{25}(n)\left(z_m \ln \frac{1}{\nu_m}\right)^{\frac{n}{2}},$$

and therefore (64) is equivalent to equality

$$\sum_{m=3}^{\infty} \frac{1}{\ln \ln m}\left(\nu_m \ln \frac{1}{\nu_m}\right)^{\frac{n}{2}} = \infty. \tag{66}$$

By analogy we conclude that (57) is equivalent to equality

$$\sum_{m=3}^{\infty} \frac{1}{\ln \ln m}\left(\nu_m^+\right)^{s_m^+}\left(\ln \frac{1}{\nu_m^+}\right)^{n-s_m^+} = \infty. \tag{67}$$

At first we will show that from

$$\sum_{m=3}^{\infty} \frac{1}{\ln \ln m}\left(\nu_m^+ \ln \frac{1}{\nu_m^+}\right)^{\frac{n}{2}} = \infty \tag{68}$$

equality (67) follows. It is clear that for sufficiently large m

$$\left(\nu_m^+\right)^{\frac{n}{2}-s_m^+} \le e. \tag{69}$$

In fact, (69) is equivalent to inequality

$$\left(s_m^+ - \frac{n}{2}\right)\ln \frac{1}{\nu_m^+} \le 1,$$

which is fulfilled by virtue of the fact that

$$\left(s_m^+ - \frac{n}{2}\right)\ln \frac{1}{\nu_m^+} = a_2^+ e^{-m} \ln \frac{1}{\nu_m} \le a_2^+ \lambda e^{-m} \ln m \le 1$$

for large m. Quite analogously we obtain that

$$\left(\ln \frac{1}{\nu_m}\right)^{s_m^+ - \frac{n}{2}} \le e, \tag{70}$$

if m is sufficiently large. From (69)-(70) we derive for large m

$$\left(\nu_m^+ \ln \frac{1}{\nu_m^+}\right)^{\frac{n}{2}} \le \left(\nu_m^+\right)^{s_m^+}\left(\ln \frac{1}{\nu_m^+}\right)^{n-s_m^+}\left(\nu_m^+\right)^{\frac{n}{2}-s_m^+}\left(\ln \frac{1}{\nu_m^+}\right)^{s_m^+-\frac{n}{2}} \le$$

$$\le e^2\left(\nu_m^+\right)^{s_m^+}\left(\ln \frac{1}{\nu_m^+}\right)^{n-s_m^+}.$$

Thus, (68) implies the validity of equality (67). In order to complete the proof of the lemma it suffices to show that (66) implies equality (68).

Note that

$$4\beta_m^+ s_m^+ = 4\left[\frac{n}{2} + \left(a_2^+ - \frac{n}{2}a_1^+\right)e^{-m} - a_1^+ a_2^+ e^{-2m}\right] \geq 2n,$$

if m is sufficiently large. Therefore

$$\nu_m^+ = \exp\left[-\frac{1}{4\beta_m^+ s_m^+}\alpha\left(z_m^+\right)\right] \geq \exp\left[-\frac{\alpha\left(z_m^+\right)}{2n}\right]. \tag{71}$$

Now suppose that for some sufficiently large $z_m^+ < z_m$. According to lemma 2 and the note in the beginning of the proof of the present lemma from (64) it follows that constant d in condition (4) does not exceed 4. Thus, for small z $|\alpha'(z)| \leq \frac{5}{z\ln\frac{1}{z}}$ and therefore, subject to (65) and (73)

$$\frac{\nu_m^+}{\nu_m} = \frac{z_m^+}{z_m} \geq \exp\left[-\frac{\alpha\left(z_m^+\right) - \alpha\left(z_m\right)}{2n}\right] \geq \exp\left[-\frac{5}{2n}\int_{z_m^+}^{z_m} \frac{dt}{t\ln\frac{1}{t}}\right] \geq$$

$$\geq \exp\left[-\frac{1}{2}\int_{z_m^+}^{z_m}\frac{dt}{t}\right] = \left(\frac{z_m^+}{z_m}\right)^{\frac{1}{2}}, \text{i.e. } \frac{z_m^+}{z_m} \geq 1,$$

if m is sufficiently large. The last inequality contradicts to our assumption. We conclude that for sufficiently large m $\nu_m^+ \geq \nu_m$. If we note that function $z\ln\frac{1}{z}$ increases for $z \in \left(0, e^{-1}\right)$, we obtain for large m

$$\left(\nu_m \ln\frac{1}{\nu_m}\right)^{\frac{n}{2}} \leq \left(\nu_m^+ \ln\frac{1}{\nu_m^+}\right)^{\frac{n}{2}}.$$

Thus, (66) implies the validity of equality (68). The lemma is proved.

Theorem 3. *Let conditions (3)-(5) be satisfied for the coefficients of operator* $\mathcal{L}$ *and domain* D. *Then (6) is a sufficient regularity condition of point* $(0,0)$ *with respect to the first boundary value problem (1)-(2).*

Proof. It suffices to show that for the considered class of domains series in conditions (6) and (64) converge or diverge simultaneously. This is equivalent to the proposition on the fact that conditions (66) and

$$\sum_{m=1}^{\infty}\left(\nu_m \ln\frac{1}{\nu_m}\right)^{\frac{n}{2}} = \infty, \tag{72}$$

either satisfied simultaneously or are not satisfied simultaneously. If in condition (4) $d > 4$, then, as it was shown above, the both series converge. Let $d < 4$. Then there exists a constant $d' < 4$ such that for small z $\alpha(z) \leq d' \ln\ln\frac{1}{z}$. For large m we have

$$\nu_m = \exp\left[-\frac{\alpha\left(z_m\right)}{2n}\right] \geq \exp\left[-\frac{d'}{2n}\ln\ln\frac{1}{z_m}\right] \geq \exp\left[-\frac{d'}{2n}\ln\left(m + \lambda\ln m\right)\right] \geq$$

$$\geq \exp\left[-\frac{d'}{2n}\ln\left(2m\right)\right] = C_{26}\left(n, d'\right) m^{-\frac{d'}{2n}}.$$

By this

$$\left(\nu_m \ln \frac{1}{\nu_m}\right)^{\frac{n}{2}} \geq C_{27}\left(n, d'\right) m^{-\frac{d'}{4}} (\ln m)^{\frac{n}{2}},$$

and both conditions (66) and (72) are satisfied.

Let now $d = 4$. Then for sufficiently small z

$$\left|\alpha'(z)\right| \leq \frac{4}{z \ln \frac{1}{z}} + \psi^+(z),$$

and therefore

$$\alpha(z) \leq \alpha(b) + 4\int_z^b \frac{dt}{t \ln \frac{1}{t}} + \int_z^b \psi^+(t)\, dt \leq \alpha(b) - 4 \ln\ln \frac{1}{b} +$$

$$+\int_0^b \psi^+(t)\, dt + 4\ln\ln\frac{1}{z} \leq C_{28}(\alpha, b) + 4\ln\ln\frac{1}{z}.$$

Hence, for large m

$$\nu_m \geq \exp\left[-\frac{C_{28}}{2n}\right] \exp\left[-\frac{2}{n}\ln\ln\frac{1}{z_m}\right] = C_{29}(\alpha, b, n)\left(\ln\frac{1}{z_m}\right)^{-\frac{2}{n}} \geq$$

$$\geq C_{29}(m + \lambda \ln m)^{-\frac{2}{n}} \geq 2^{-\frac{2}{n}} C_{29} m^{-\frac{2}{n}}.$$

Thus,

$$\left(\nu_m \ln \frac{1}{\nu_m}\right)^{\frac{n}{2}} \geq C_{30}(\alpha, b, n)\, m^{-1} (\ln m)^{\frac{n}{2}},$$

which implies the fulfillment of both conditions (66) and (72). The theorem is proved.

Remark. As one can see from the proof, in fact, for sufficiently large m

$$\nu_m \geq C_{31}(\alpha, b, n)\, m^{-\frac{2}{n}} \exp\left[-\frac{1}{2n}\int_{z_m}^b \psi^+(t)\, dt\right] \geq$$

$$\geq C_{31} m^{-\frac{2}{n}} \exp\left[-\frac{1}{2n}\int_{e^{-m}/m^\lambda}^b \psi^+(t)\, dt\right] = C_{31}\exp\left[-\frac{1}{2n}\int_{e^{-m}/m^\lambda}^{e^{-m}} \psi^+(t)\, dt\right] \times$$

$$\times m^{-\frac{2}{n}} \exp\left[-\frac{1}{2n}\int_{e^{-m}}^b \psi^+(t)\, dt\right]. \tag{73}$$

On the other hand, for small z $\psi^+(z) \leq \frac{1}{z \ln \frac{1}{z}}$. Therefore

$$\int_{e^{-m}/m^\lambda}^{e^{-m}} \psi^+(t)\, dt \leq \int_{e^{-m}/m^\lambda}^{e^{-m}} \frac{dt}{t \ln \frac{1}{t}} = \ln\ln\left(e^m m^\lambda\right) - \ln\ln e^m =$$

$$= \ln \frac{m + \lambda \ln m}{m} \leq 2$$

if m is sufficiently large. Thus, from (73) we obtain

$$\nu_m \geq C_{32} (\alpha, b, n)\, m^{-\frac{2}{n}} \exp \left[-\frac{1}{2n} \int_{e^{-m}}^{b} \psi^+ (t)\, dt \right],$$

i.e.

$$\nu_m \ln \frac{1}{\nu_m} \geq C_{32} m^{-\frac{2}{n}} \exp \left[-\frac{1}{2n} \int_{e^{-m}}^{b} \psi^+ (t)\, dt \right] \times$$

$$\times \ln \left[m^{\frac{2}{n}} \exp \left[\frac{1}{2n} \int_{e^{-m}}^{b} \psi^+ (t)\, dt \right] \right] \geq \frac{2C_{32}}{n} m^{-\frac{2}{n}} \times$$

$$\times \exp \left[-\frac{1}{2n} \int_{e^{-m}}^{b} \psi^+ (t)\, dt \right] \ln m.$$

Hence,

$$\left(\nu_m \ln \frac{1}{\nu_m} \right)^{\frac{n}{2}} \geq C_{33} (\alpha, b, n)\, m^{-1} \exp \left[-\frac{1}{4} \exp \int_{e^{-m}}^{b} \psi^+ (t)\, dt \right] (\ln m)^{\frac{n}{2}}, \qquad (74)$$

if m is sufficiently large. Now we will impose a weaker condition on function $\psi^+ (z)$ than in (4), exactly, suppose that for sufficiently small z

$$\frac{1}{\ln \ln \ln \frac{1}{z}} \int_{z}^{b} \psi^+ (t)\, dt \leq 2 (n+2). \qquad (75)$$

Then for large m

$$\exp \left[-\frac{1}{4} \int_{e^{-m}}^{b} \psi^+ (t)\, dt \right] \geq \exp \left[-\frac{n+2}{2} \ln \ln m \right] = (\ln m)^{-\frac{n}{2}-1},$$

which together with (74) implies

$$\left(\nu_m \ln \frac{1}{\nu_m} \right)^{\frac{n}{2}} \geq C_{33} (m \ln m)^{-1}.$$

Thus, both conditions (66) and (72) are fulfilled, i.e. theorem 3 is also valid in the case when function $\psi^+ (z)$ in (4) satisfies condition (75).

5^0. Necessary condition of regularity

Everywhere in the course of this item it is assumed that condition (4) is fulfilled without any additional restrictions imposed on function $\psi^+ (z)$.

At first we will prove rough necessary regularity condition in order to throw out the class of domains D for which point $(0,0)$ is irregular with respect to the first boundary value problem (1)-(2), and at that condition (6) is not fulfilled.

Lemma 7. *Let the coefficients of operator $\mathcal{L}$ satisfy the condition (3), $\bar{s} = \frac{\gamma^2 n}{3}$, $\bar{\beta} = \gamma^{-1}$, $G_{\bar{s},\bar{\beta}}(x,t) = G^0(x,t)$. Then there exists $r'(\gamma, n, b_0)$ such that if (y,τ) is an arbitrary fixed point from $\mathbb{C}_{r'}$, then function $G^0(x-y, t-\tau)$ is $\mathcal{L}$-superparabolic for $(x,t) \in (D \cap \mathbb{C}_{r'}) \setminus \{(y,\tau)\}$.*

Proof. It suffices to show that $\mathcal{L}_{(x,t)} G^0(x-y, t-\tau) \le 0$ for $(x,t) \in (D \cap \mathbb{C}_{r'}) \setminus \{(y,\tau)\}$. We restrict ourselves to the case $t > \tau$. Subject to (3) and (8) we have

$$I'(x,t) = \frac{\mathcal{L}G^0}{G^0} \cdot (t-\tau) = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{(x_i - y_i)(x_j - y_j)}{4\bar{\beta}^2 (t-\tau)} - \frac{1}{2\bar{\beta}} \sum_{i=1}^{n} a_{ii}(x,t) -$$

$$-\frac{1}{2\bar{\beta}} \sum_{i=1}^{n} b_i(x,t)(x_i - y_i) + \bar{s} - \frac{|x-y|^2}{4\bar{\beta}(t-\tau)} \le \frac{|x-y|^2}{4\bar{\beta}(t-\tau)} \left(\frac{\gamma^{-1}}{\bar{\beta}} - 1 \right) -$$

$$-\frac{\gamma n}{2\bar{\beta}} + \bar{s} + \frac{b_0 \sqrt{n}}{2\bar{\beta}} |x-y| \le \frac{\gamma^2 n}{3} + \gamma b_0 \sqrt{n} r' - \frac{\gamma^2 n}{2} =$$

$$= -\frac{\gamma^2 n}{6} \left(1 - \frac{\gamma^{-1} b_0}{\sqrt{n}} r' \right). \tag{76}$$

We choose and fix r' so that $b_0 r' \le \gamma \sqrt{n}$. Then from (76) it follows that $I'(x,t) \le 0$. The lemma is proved.

Let for natural m $A_m^0 = \left\{ (y,\tau) : e^{m\bar{s}} \le G^0(y, -\tau) \le e^{(m+1)\bar{s}} \right\}$, $H_m^0 = A_m^0 \backslash D$. Denote parabolic $(\bar{s}, \bar{\beta})$ – capacity of compact E by $\bar{p}(E)$.

Theorem 4. *If the coefficients of operator $\mathcal{L}$ satisfy the condition (3), then for regularity of point $(0,0)$ with respect to the first to the first boundary value problem (1)-(2) it is necessary that*

$$\sum_{m=1}^{\infty} e^{m\bar{s}} \bar{p}\left(H_m^0\right) = \infty. \tag{77}$$

Proof. Let condition (77) be violated. Denote by $m \ge \ln \frac{1}{r'}$ the least natural number for which

$$\sum_{m=m_1}^{\infty} e^{m\bar{s}} \bar{p}\left(H_m^0\right) \le \frac{e^{-\bar{s}}}{8}. \tag{78}$$

We choose continuous boundary function $\varphi(x,t)$ of the first boundary value problem (1)-(2) in the following way: $\varphi(0,0) = 1$, $\varphi(x,t) = 0$ at $t \le \inf_{(y,\tau) \in H_{m_1}^0} \{\tau\}, 0 \le \varphi(x,t) \le 1$. According to continuity of parabolic $(\bar{s}, \bar{\beta})$-capacity as function of set, and proposition 2, for any $\varepsilon_m > 0$ there exists step domain $Q_m \supset H_m^0$ with sufficiently smooth boundaries if bases of the cylinders forming them and measure μ_m on $\bar{Q}_m$ such that

$$U_m(x,t) = \int_{\bar{Q}_m} G^0(x-y, t-\tau)\, d\mu_m(y,\tau),$$

then

$$U_m|_{\Gamma^-(Q_m)} = 1, \quad \mu_m\left(\bar{Q}_m\right) = \bar{p}\left(\bar{Q}_m\right) \le \bar{p}\left(H_m^0\right) + \varepsilon_m. \tag{79}$$

Moreover we can assume that

$$Q_m \subset \left\{(y,\tau) : \frac{1}{2}e^{m\bar{s}} \le G^0(y,-\tau) \le 2e^{(m+1)\bar{s}}\right\}. \tag{80}$$

Suppose $\varepsilon_m = \frac{e^{-\bar{s}}}{8}\left(2e^{\bar{s}}\right)^{-m}$. According to lemma 7 function $U(x,t) = \sum\limits_{m=m_1}^{\infty} U_m(x,t)$ is $\mathcal{L}$-superparabolic in

$$D^1 = (D \cap \mathbb{C}_{r'}) \setminus \overline{\bigcup_{m=m_1}^{\infty} Q_m}.$$

Besides, by virtue of the choice of function $\varphi(x,t)$ and (79)

$$u_\varphi(x,t) \le U(x,t) \quad \text{for} \quad (x,t) \in \Gamma\left(D^1\right).$$

According to maximum principle $u_\varphi(x,t) \le U(x,t)$ in D^1. Therefore taking into account (78)-(80), we obtain

$$\overline{\lim_{\substack{(x,t)\to(0,0)\\(x,t)\in D^1}}} u_\varphi(x,t) \le U(0,0) \le 2\sum_{m=m_1}^{\infty} e^{(m+1)\bar{s}}\mu_m\left(\bar{Q}_m\right) \le$$

$$\le 2e^{\bar{s}}\sum_{m=m_1}^{\infty} e^{m\bar{s}}\bar{p}\left(H_m^0\right) + 2e^{\bar{s}}\sum_{m=m_1}^{\infty} e^{m\bar{s}}\varepsilon_m \le \frac{1}{4} + \frac{1}{4}\sum_{m=m_1}^{\infty} 2^{-m} \le \frac{1}{2}.$$

Thus, point $(0,0)$ is irregular. The theorem is proved.

Lemma 8. *Let the coefficients of operator $\mathcal{L}$ and domain D satisfy the conditions (3)-(4). Then if in condition (4) $d > 4\gamma^{-1}$, then point $(0,0)$ is irregular with respect to the first boundary value problem (1)-(2) and at that condition (6) is not fulfilled.*

Proof. The second part of the statement of lemma follows from lemma 2 (which is to be applied for $a_0 = b_0 = 0$), since $4\gamma^{-1} \ge 4$. Let us prove the first part restricting ourselves to the case of finite d. Denote for natural m the module of time coordinate of intersection points of $\Gamma(D)$ with the level surface $G^0(y,-\tau) = e^{m\bar{s}}$ by $\bar{z}_m$. Obviously, $\bar{z}_m$ is a root of equation

$$\alpha(\bar{z}_m) = 4\bar{\beta}\bar{s}\ln\frac{e^{-m}}{\bar{z}_m}. \tag{81}$$

We will show that equation (81) for sufficiently large m really has a root, lying on the interval $\left(\frac{e^{-m}}{m^{\lambda_1}}, e^{-m}\right)$, where $\lambda_1 = \frac{3(d+3)}{4\gamma n}$. Remembering the proof of lemma 2, we conclude that for this it suffices to verify in validity of inequality

$$\alpha\left(\frac{e^{-m}}{m^{\lambda_1}}\right) < 4\bar{\beta}\bar{s}\lambda_1\ln m.$$

Taking into account that

$$\lim_{z\to 0+}\frac{\alpha(z)}{\ln\ln\frac{1}{z}}=\lim_{z\to 0+}\left|\alpha'(z)\right| z\ln\frac{1}{z}=d, \tag{82}$$

we obtain that for sufficiently small z $\alpha(z)\le(d+1)\ln\ln\frac{1}{z}$. On the other hand $4\bar\beta\bar s=\frac{4\gamma n}{3}$. Therefore

$$\alpha\left(\frac{e^{-m}}{m^{\lambda_1}}\right)-4\bar\beta\bar s\ \lambda_1\ln m\ge(d+1)\ln(m+\lambda_1\ln m)-\frac{4\gamma n}{3}\lambda_1\ln m\le$$

$$\le\ln m\left(d+2-\frac{4\gamma n}{3}\lambda_1\right)=-\ln m<0,$$

if m is sufficiently large.

Thus, the existence of root of equation (81) on the given interval is proved. If there are more than one root, then denote by $\bar z_m$ exact lower bound of these roots. Further for sufficiently large m we have

$$H_m^0\subset C^{-4\bar\beta\bar s\bar z_m\ln\frac{e^{-m}}{\bar z_m}}_{\sqrt{4\bar\beta\bar s\bar z_m\ln\frac{e^{-m}}{\bar z_m}}}(0),$$

i.e. according to proposition 1

$$\bar p\left(H_m^0\right)\le C_{34}(\gamma,n)\left(\bar z\ln\frac{e^{-m}}{\bar z_m}\right)^{\bar s}.$$

Now in order to prove the statement of lemma, by virtue of theorem 4, it suffices to show that

$$\sum_{m=1}^{\infty}\left(\bar\nu_m\ln\frac{1}{\bar\nu_m}\right)^{\bar s}<\infty, \tag{83}$$

where $\bar\nu_m=e^m\bar z_m$. (82) implies the existence of a constant $d_0>4\gamma^{-1}$ such that for sufficiently small z $\alpha(z)\ge d_0\ln\ln\frac{1}{z}$. Therefore using (81), we obtain

$$\bar\nu_m=\exp\left[-\frac{1}{4\bar\beta\bar s}\alpha(\bar z_m)\right]\le\exp\left[-\frac{d_0}{4\bar\beta\bar s}\ln\ln\frac{1}{\bar z_m}\right]\le$$

$$\le\exp\left[-\frac{d_0}{4\bar\beta\bar s}\ln m\right]=m^{-\frac{d_0}{4\bar\beta}}.$$

Thus,

$$\left(\bar\nu_m\ln\frac{1}{\bar\nu_m}\right)^{\bar s}\le C_{35}(\alpha,\gamma,n)\,m^{-\frac{d_0}{4\bar\beta}}(\ln m)^{\bar s},$$

if m is sufficiently large. Now it suffices to remember that $\bar\beta=\gamma^{-1}$, and the required inequality (83) and together with it the lemma are proved.

Corollary. *If in condition (4) $d\le 4\gamma^{-1}$, then for sufficiently small z $\alpha(z)\le z^{-\frac{1}{2}}$.*

In fact,

$$\left|\alpha'(z)\right| z^{\frac{3}{2}} = \left|\alpha'(z)\right| z \ln\frac{1}{z}\cdot\frac{z^{\frac{1}{2}}}{\ln\frac{1}{z}} \to 0 \quad \text{as} \quad z\to 0+ .$$

Now for sufficiently small z $\quad |\alpha'(z)| \le \frac{z^{-\frac{3}{2}}}{4}$, i.e.

$$\alpha(z) = \alpha(b) - \int_z^b \alpha'(t)\,dt \le \alpha(b) + \frac{1}{2}z^{-\frac{1}{2}} - \frac{1}{2}b^{-\frac{1}{2}} \le z^{-\frac{1}{2}}.$$

Everywhere below not specifying it we will assume that constant d in condition (4), does not exceed $4\gamma^{-1}$, since in the contrary case according to lemma 8, condition (6) is necessary for regularity of point $(0,0)$ with respect to the first boundary value problem (1)-(2).

Lemma 9. *If the coefficients of operator $\mathcal{L}$ and domain D satisfy the conditions (3)-(5), then there exist constants $r_1(a_0,n)$, $a_1^-(a_0,n)$ and $a_2^-(a_0,b_0,n)$ such that if $r\in(0,r_1]$, $\beta^-(r) = 1 + a_1^- r^{\frac{1}{2}}$, $s^-(r) = \frac{n}{2} - a_2^- r^{\frac{1}{2}}$, (y,τ) is arbitrary fixed point from $\mathbb{C}_r$, then function $G_{s^-(r),\beta^-(r)}(x-y,t-\tau)$ is $\mathcal{L}$-superparabolic for $(x,t)\in (D\cap\mathbb{C}_{r_1})\setminus\{(y,\tau)\}$.*

Proof. We restrict ourselves to the case $r<r_1$. Denote function $G_{s^-(r),\beta^-(r)}$ by $G^-_{(r)}$. Function $G^-_{(r)}(x-y,t-\tau)$ vanishes at $t\le\tau$. Therefore it suffices to consider case $(x,t)\in\left(D\cap\mathbb{C}'_{r_1}\right)\setminus\{(y,\tau)\}$, where $\mathbb{C}'_{r_1} = B_{r_1}(0)\times\left(-r^2,0\right)$. Let us show that for mentioned (x,t) inequality $\mathcal{L}_{(x,t)}G^-_{(r)}(x-y,t-\tau)\le 0$ is fulfilled. We have

$$I''(x,t) \le \frac{\mathcal{L}G^-_{(r)}}{G^-_{(r)}}(t-\tau) \le \sum_{i,j=1}^{n}\left[a_{ij}(x,t)-\delta_{ij}\right]\frac{(x_i-y_i)(x_j-y_j)}{4\left(\beta^-\right)^2(t-\tau)} +$$

$$+\frac{|x-y|^2}{4\left(\beta^-\right)^2(t-\tau)} - \frac{1}{2\beta^-}\sum_{i=1}^{n}\left[a_{ii}(x,t)-1\right] - \frac{n}{2\beta^-} +$$

$$+\frac{b_0\sqrt{n}\,|x-y|}{2\beta^-} + s^- - \frac{|x-y|^2}{4\beta^-(t-\tau)}. \tag{84}$$

Assuming $r_1\le 1$ using (7) together with the corollary to lemma 8, we obtain

$$|a_{ij}(x,t)-\delta_{ij}| \le a_0(|x|+|t|) \le a_0\left(\sqrt{|t|\,\alpha(|t|)}+|t|\right) \le$$

$$\le 2a_0|t|^{\frac{1}{4}} \le 2ar^{\frac{1}{2}}, \qquad i,j=1,...,n, \tag{85}$$

$$|x-y| \le |x|+|y| \le |t|^{\frac{1}{4}} + r \le 2r^{\frac{1}{2}}. \tag{86}$$

Taking into account (85)-(86) in (84), we conclude

$$I''(x,t) \le \frac{|x-y|^2}{4\beta^-(t-\tau)}\left[\frac{1}{\beta^-} + \frac{2na_0r^{\frac{1}{2}}}{\beta^-} - 1\right] + s^- - \frac{n}{2\beta^-} +$$

$$+\frac{b_0\sqrt{n}r^{\frac{1}{2}}}{\beta^-}+\frac{na_0r^{\frac{1}{2}}}{\beta^-}.$$

Suppose $a_1^-=a_1^+=2na_0$ and choose $\beta^-(r)=1+a_1^-r^{\frac{1}{2}}$. Then

$$I''(x,t)\le s^--\frac{n}{2}+\frac{\left[n(n+1)a_0+b_0\sqrt{n}\right]\ r^{\frac{1}{2}}}{1+2na_0r^{\frac{1}{2}}}\le$$

$$\le s^--\frac{n}{2}+\left[n(n+1)a_0+b_0\sqrt{n}\right]r^{\frac{1}{2}}.$$

Now we fix such $r_1\le 1$ that $\left[(n+1)a_0+\dfrac{b_0}{\sqrt{n}}\right]r_1^{\frac{1}{2}}\le\dfrac{1}{3}$. Now it suffices to choose $s^-(r)=\frac{n}{2}-\left[n(n+1)a_0+b_0\sqrt{n}\right]r^{\frac{1}{2}}=\dfrac{n}{2}-a_2^-r^{\frac{1}{2}}$ and the lemma is proved.

Remark. It is clear that $a_2^->\frac{n}{2}a_1^-$, if $a_0>0$.

Let $s_m^-=s^-(e^{-m})$, $\beta_m^-=\beta^-(e^{-m})$, $G_{(e^{-m})}^-=G_{[m]}^-$, $A_m^-=$ $=\left\{(y,\tau):e^{ms_m^-}\le G_{[m]}^-(y,-\tau)\le e^{(m+1)s_m^-}\right\}$, $H_m^-=A_m^-\backslash D$. Denote parabolic (s_m^-,β_m^-)-capacity of compact E by $p_m^-(E)$.

Theorem 5. *If the coefficients of operator $\mathcal{L}$ and domain D satisfy the conditions (3)-(5), then for regularity of point* $(0,0)$ *with respect to the first boundary value problem (1)-(2) it is necessary that*

$$\sum_{m=1}^{\infty}e^{ms_m^-}p_m^-(H_m^-)=\infty. \tag{87}$$

Proof within obvious details coincides with the proof of theorem 4.

For natural m denote by z_m^- the module of time coordinate of intersection points of $\Gamma(D)$ with the level surface $G_{[m]}^-(y,-\tau)=e^{ms_m^-}$. It is clear that z_m^- is a root of equation

$$\alpha(z_m^-)=4\beta_m^-s_m^-\ln\frac{e^{-m}}{z_m^-}. \tag{88}$$

By the same way as in the proof of lemma 8 one can show that for sufficiently large m equation (88) really has a root on the interval $\left(\dfrac{e^{-m}}{m^{\lambda_1}},e^{-m}\right)$. If there are more than one root, then denote by z_m^- the exact lower bound of these roots. From (88) we have

$$\frac{z_m^-}{z_{m+1}^-}=e\cdot\exp\left[-\frac{1}{4\beta_m^-s_m^-}\alpha(z_m^-)+\frac{1}{4\beta_{m+1}^-s_{m+1}^-}\alpha(z_{m+1}^-)\right]=$$

$$=e\cdot\exp\left[-\frac{1}{4\beta_m^-s_m^-}\left(\alpha(z_m^-)-\alpha(z_{m+1}^-)\right)\right]\times \tag{89}$$

$$\times\exp\left[-\left(\frac{1}{4\beta_m^-s_m^-}-\frac{1}{4\beta_{m+1}^-s_{m+1}^-}\right)\alpha(z_{m+1}^-)\right].$$

On the other hand, taking into account that for sufficiently large m $\beta_m^- s_m^- \geq \frac{n}{3}$, we obtain

$$\frac{1}{4\beta_m^- s_m^-} - \frac{1}{4\beta_{m+1}^- s_{m+1}^-} = \frac{1}{4\beta_{m+1}^- s_{m+1}^-}\left(\frac{\beta_{m+1}^- s_{m+1}^-}{\beta_m^- s_m^-} - 1\right) \leq$$

$$\leq \frac{1}{4\cdot\frac{n}{3}}\left[\left(a_2^- - \frac{n}{2}a_1^-\right)\left(1 - e^{-\frac{1}{2}}\right)e^{-\frac{m}{2}} + a_1^- a_2^-\left(1 - e^{-1}\right)e^{-m}\right] \leq$$

$$\leq C_{36}\left(a_0, b_0, n\right)e^{-\frac{m}{2}}.$$

Further, since for sufficiently small z $\alpha(z) \leq (d+1)\ln\ln\frac{1}{z}$, then

$$\exp\left[-\left(\frac{1}{4\beta_m^- s_m^-} - \frac{1}{4\beta_{m+1}^- s_{m+1}^-}\right)\alpha\left(z_{m+1}^-\right)\right] \geq$$

$$\geq \exp\left[-C_{36}(d+1)e^{-\frac{m}{2}}\ln\ln\frac{1}{z_{m+1}^-}\right] \geq$$

$$\geq \exp\left[-C_{36}(d+1)e^{-\frac{m}{2}}\ln\left(m+1+\lambda_1\ln(m+1)\right)\right] \geq$$

$$\geq \exp\left[-C_{36}(d+1)e^{\frac{m}{2}}\ln(2m)\right] \geq e^{-\frac{1}{2}},$$

if m is sufficiently large. Thus, from (89) we conclude that for sufficiently large m

$$\frac{z_m^-}{z_{m+1}^-} \geq e^{\frac{1}{2}}\cdot\exp\left[-\frac{1}{4\beta_m^- s_m^-}\left(\alpha\left(z_m^-\right) - \alpha\left(z_{m+1}^-\right)\right)\right]. \tag{90}$$

If now we will suppose that for some sufficiently large m $z_{m+1}^- \geq z_m^-$, then from (90) we derive

$$\frac{z_m^-}{z_{m+1}^-} \geq e^{\frac{1}{2}}\cdot\exp\left[-\frac{3}{4n}(d+1)\int\limits_{z_m^-}^{z_{m+1}^-}\frac{dt}{t\ln\frac{1}{t}}\right] \geq$$

$$\geq e^{\frac{1}{2}}\exp\left[-\frac{1}{2}\int\limits_{z_m^-}^{z_{m+1}^-}\frac{dt}{t}\right] = e^{\frac{1}{2}}\left(\frac{z_m^-}{z_{m+1}^-}\right)^{\frac{1}{2}}, \text{ i.e. } \frac{z_m^-}{z_{m+1}^-} \geq e > 1.$$

The last inequality contradicts to our supposition. Thus, $z_{m+1}^- < z_m^-$ and from (90) we obtain

$$z_m^- \geq e^{\frac{1}{2}}z_{m+1}^-, \tag{91}$$

if m is sufficiently large.

On the other hand, since $\beta_m^- s_m^- \leq \beta_{m+1}^- s_{m+1}^-$, then from (89) we conclude

$$\frac{z_m^-}{z_{m+1}^-} \leq e\cdot\exp\left[\frac{1}{4\beta_m^- s_m^-}\left(\alpha\left(z_{m+1}^-\right) - \alpha\left(z_m^-\right)\right)\right] \leq e\left(\frac{z_m^-}{z_{m+1}^-}\right)^{\frac{1}{2}}$$

Thus, for large m

$$z_m^- \le e^2 z_{m+1}^- . \tag{92}$$

Denote $e^m z_m^-$ by ν_m^-. From (88) it follows that

$$\nu_m^- = \exp\left[-\frac{\alpha\left(z_m^-\right)}{4\beta_m^- s_m^-}\right].$$

From (91) we obtain that there exists nonnegative limit $\nu^0 = \lim\limits_{m\to\infty} \nu_m^-$.

Lemma 10. *If domain D satisfies the condition (4), then for sufficiently large m*

$$C_{37}(n)\left(z_m^- \ln\frac{1}{\nu_m^-}\right)^{s_m^-} \le p_m^-\left(H_m^-\right) \le C_{38}(n)\left(z_m^- \ln\frac{1}{\nu_m^-}\right)^{s_m^-}. \tag{93}$$

Proof. The upper estimate in (93) was actually proved in lemma 8. Let us prove the lower estimate restricting ourselves to the case $\nu_0 = 0$. Let $T_m = \left\{(y,\tau) : G_{[m]}^-(y,-\tau) = e^{ms_m^-}, \tau \ge -z_m^-\right\}$, $K_m(x,t) = \int\limits_{T_m} G_{[m]}^-(x-y,t-\tau)\, ds_{(y,\tau)}$. We pass from integration over surface T_m into integration over projection of T_m on hyperplane $\tau = -z_m^-$. Denoting by $\tilde{T}_m$, (x',t'), (ϑ,η) images of T_m and points (x,t), (y,τ) respectively, and reasoning in the same way as in the proof of lemma 4, we obtain

$$K_m(x,t) = \left(t'-\eta\right)^{-s_m^-} \int\limits_{\tilde{T}_m} \exp\left[-\frac{|x'-\vartheta|^2}{4\beta_m^-(t'-\eta)}\right]\sqrt{1+\left(\frac{d\eta}{d\vartheta}\right)^2}\, d\vartheta \le$$

$$\le C_{39}(n)\left(t'+z_m^-\right)^{-s_m^-} \int\limits_{\tilde{T}_m} \exp\left[-\frac{|x'-\vartheta|^2}{4\beta_m^-\left(t'+z_m^-\right)}\right] d\vartheta. \tag{94}$$

Two cases are possible : i) $t' < -z_m^- + z_m^- \ln\frac{1}{\nu_m^-}$; ii) $t'' \ge -z_m^- + z_m^- \ln\frac{1}{\nu_m^-}$. If case i) occurs, then from (94) we derive for sufficiently large m

$$K_m(x,t) \le \left(2\sqrt{\beta}\right)^n C_{39}\left(z_m^- \ln\frac{1}{\nu_m^-}\right)^{\frac{n}{2}-s_m^-}\left(\int\limits_{-\infty}^{\infty} e^{-\xi^2} d\xi\right)^n \le$$

$$\le C_{40}(n)\left(z_m^- \ln\frac{1}{\nu_m^-}\right)^{\frac{n}{2}-s_m^-}. \tag{95}$$

If ii) is satisfied, then $G_{[m]}^-(x'-\vartheta, t'+z_m) \le \left(z_m^- \ln\frac{1}{\nu_m^-}\right)^{-s_m^-}$. Therefore from (94) we conclude

$$K_m(x,t) \le C_{39}\left(z_m^- \ln\frac{1}{\nu_m^-}\right)^{-s_m^-} mes\left(\tilde{T}_m\right) = C_{39}\left(z_m^- \ln\frac{1}{\nu_m^-}\right)^{-s_m^-} \times$$

$$\times mes\left(B_{\sqrt{4\beta_m^- s_m^- z_m^- \ln\frac{1}{\nu_m'}}}(0)\right) \le C_{41}(n)\left(z_m^- \ln\frac{1}{\nu_m'}\right)^{\frac{n}{2}-s_m^-}. \tag{96}$$

From (95)-(96) it follows that measure, uniformly distributed on T_m with density $C_{42}^{-1}\left(z_m^- \ln \frac{1}{\nu_m'}\right)^{s_m^- - \frac{n}{2}}$ is (s_m^-, β_m^-)-admissible (with respect to H_m^-). Here $C_{42} = \max\{C_{40}, C_{41}\}$. By that

$$p_m^-\left(H_m^-\right) \geq C_{42}^{-1}\left(z_m^- \ln \frac{1}{\nu_m^-}\right)^{s_m^- - \frac{n}{2}} mes\left(T_m\right) \geq C_{42}^{-1}\left(z_m^- \ln \frac{1}{\nu_m^-}\right)^{s_m^- - \frac{n}{2}} \times$$

$$\times mes\left(\tilde{T}_m\right) = C_{42}^{-1}\left(z_m^- \ln \frac{1}{\nu_m^-}\right)^{s_m^- - \frac{n}{2}} \omega_n \left(4\beta_m^- s_m^- z_m^- \ln \frac{1}{\nu_m^-}\right)^{\frac{n}{2}} \geq$$

$$\geq C_{43}\left(n\right)\left(z_m^- \ln \frac{1}{\nu_m^-}\right)^{s_m^-},$$

if m is sufficiently large. The lemma is proved.

Theorem 6. *Let the coefficients of operator $\mathcal{L}$ and domain D satisfy the conditions (3)-(5). Then (6) is necessary regularity condition of point $(0,0)$ with respect to the first boundary value problem (1)-(2).*

Proof. At first note that according to lemma 10, condition (87) is equivalent to the following

$$\sum_{m=1}^{\infty}\left(\nu_m^- \ln \frac{1}{\nu_m^-}\right)^{s_m^-} = \infty.$$

Therefore subject to theorem 5, it suffices to show that

$$\sum_{m=1}^{\infty}\left(\nu_m \ln \frac{1}{\nu_m}\right)^{\frac{n}{2}} < \infty \tag{97}$$

implies

$$\sum_{m=1}^{\infty}\left(\nu_m^- \ln \frac{1}{\nu_m^-}\right)^{s_m^-} < \infty. \tag{98}$$

At first we show that

$$\sum_{m=1}^{\infty}\left(\nu_m^- \ln \frac{1}{\nu_m^-}\right)^{\frac{n}{2}} < \infty \tag{99}$$

implies (98). We again restrict ourselves to the case $\nu^0 = 0$. It is clear that for sufficiently large m

$$\left(\nu_m^-\right)^{s_m^- - \frac{n}{2}} \leq e. \tag{100}$$

In fact, (100) is equivalent to the inequality

$$\left(\frac{n}{2} - s_m^-\right) \ln \frac{1}{\nu_m^-} \leq 1$$

which is fulfilled by virtue of the fact that

$$\left(\frac{n}{2} - s_m^-\right) \ln \frac{1}{\nu_m^-} = a_2^- e^{-\frac{m}{2}} \ln \frac{1}{\nu_m^-} \leq a_2^- \lambda_1 e^{-\frac{m}{2}} \ln m \leq 1$$

for large m. We have

$$\left(\nu_m^- \ln \frac{1}{\nu_m^-}\right)^{s_m^-} = \left(\nu_m^- \ln \frac{1}{\nu_m^-}\right)^{\frac{n}{2}} \left(\nu_m^-\right)^{s_m^- - \frac{n}{2}} \left(\ln \frac{1}{\nu_m^-}\right)^{s_m^- - \frac{n}{2}} \le$$

$$\le \left(\nu_m^- \ln \frac{1}{\nu_m^-}\right)^{\frac{n}{2}} \left(\nu_m^-\right)^{s_m^- - \frac{n}{2}} \le e \left(\nu_m^- \ln \frac{1}{\nu_m^-}\right)^{\frac{n}{2}}.$$

Thus, (99) implies the validity of inequality (98). Now to complete the proof it suffices to show that (97) implies the validity of inequality (99).

Note that

$$4\beta_m^- s_m^- = 4\left[\frac{n}{2} - \left(a_2^- - \frac{n}{2}a_1^-\right)e^{-\frac{m}{2}} - a_1^- a_2^- e^{-m}\right] \le 2n.$$

Therefore

$$\nu_m^- = \exp\left[-\frac{1}{4\beta_m^- s_m^-}\alpha\left(z_m^-\right)\right] \le \exp\left[-\frac{\alpha\left(z_m^-\right)}{2n}\right].$$

Thus

$$\frac{\nu_m^-}{\nu_m} = \frac{z_m^-}{z_m} \le \exp\left[-\frac{1}{2n}\left(\alpha\left(z_m^-\right) - \alpha\left(z_m\right)\right)\right]. \tag{101}$$

Now if we suppose that for some sufficiently large m $z_m^- > z_m$ holds, then according to (101)

$$\frac{z_m^-}{z_m} \le \exp\left[\frac{1}{2n}\left(\alpha\left(z_m\right) - \alpha\left(z_m^-\right)\right)\right] \le \exp\left[\frac{d+1}{2n}\int_{z_m}^{z_m^-}\frac{dt}{t\ln\frac{1}{t}}\right] \le$$

$$\le \exp\left[\frac{1}{2}\int_{z_m}^{z_m^-}\frac{dt}{t}\right] = \left(\frac{z_m^-}{z_m}\right)^{\frac{1}{2}}, \quad \text{i.e.} \quad \frac{z_m^-}{z_m} \le 1.$$

The last inequality contradicts to our supposition. Thus, for large m $z_m^- \le z_m$ holds, i.e. $\nu_m^- \le \nu_m$. Therefore

$$\left(\nu_m^- \ln \frac{1}{\nu_m^-}\right)^{\frac{n}{2}} \le \left(\nu_m \ln \frac{1}{\nu_m}\right)^{\frac{n}{2}},$$

and (97) implies the validity of inequality (99). The theorem is proved.

By that, the proof of the main theorem is completed.

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