Kanan A. MAMMADLI

ON SOLVABILITY OF A MIXED PROBLEM FOR A HYPERBOLIC SYSTEM WHEN THE CORRESPONDING SPECTRAL PROBLEM IS ALMOST REGULAR

Abstract

One-dimensional mixed problem for a hyperbolic system under boundary conditions regular by Birkhoff, has been solved in [1] by the Fourier method. The case when the spectral problem is almost regular, is investigated here by Rasulov's residue method [2], and the fact that real parts of its eigen-values may infinitely increase, is of principle.

Let $Q_T = \{(t,x):\ 0 < t < T,\ 0 \le x \le 1\}$, $Q'_T = \{(t,x):\ 0 \le t < T,\ 0 \le x \le 1\}$. Let us consider the mixed problem

$$L\left(x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u(t,x) \equiv -\frac{\partial u}{\partial t} + A_1(x)\frac{\partial u}{\partial x} + A_0(x) u = 0, \quad (t,x) \in Q_T, \qquad (1)$$

$$u(0,x) = \varphi(x), \quad 0 \le x \le 1 \qquad (2)$$

$$lu \equiv \alpha u(t,0) + \beta u(t,1) = 0, \quad 0 < t < T, \qquad (3)$$

where $A_j(x)$, α, β are n-th order square matrices, $u(t,x)$ and $\varphi(x)$ are columns of height n.

We shall call the vector-function $u(t,x) \in C^1(Q_T) \cap C(Q'_T)$ satisfying the equalities (1)-(3) in the usual sense, a classical solution of this problem.

It is assumed that the following conditions are satisfied.

1°. At $x \in [0,1]$ the characteristic numbers (functions) $\theta_j(x)$ $\left(j = \overline{1,n}\right)$ of the matrix $A_1(x)$ are real, various, and non-zero.

2°. At some m, $A_j(x) \in C^{m+j}[0,1]$ $(j = 0,1)$ and the spectral problem

$$L\left(x, \lambda, \frac{d}{dx}\right) y(x,\lambda),\ ly = 0 \qquad (4)$$

is almost regular [3],[4] of the order p $(0 \le p \le m-1)$.

Under these conditions we can divide λ-plane into finite number of sectors in each of which as $|\lambda| \to \infty$ the characteristic determinant $\Delta(\lambda)$ of the Green matrix $G(x,\xi,\lambda)$ of the spectral problem (4) is represented in the following form [5]:

$$\Delta(\lambda) = \sum_{j=0}^{n_0} \lambda^{\nu_j} \left[æ_j + O\left(\frac{1}{\lambda}\right)\right] e^{\lambda \Omega_j}, \qquad (5)$$

where n_0 is a natural number, ν_j $(-p \le \nu_j \le 0)$ is an integer, $0 \ne æ_j$ is a complex number, Ω_j $\left(j = \overline{0,n_0}\right)$ are real, different

$$\Omega_j \in \left\{0,\ \sum_{k=1}^{r} \int_0^1 \theta_{l_k}(\xi)\, d\xi,\ 1 \le r \le n,\ \ 1 \le l_k \le n \right\}.$$

Moreover, we can introduce such numeration of the functions θ_j that it were

$$\theta_1(x) < \theta_2(x) < \ \ldots \ < \theta_{n_1}(x) < 0 < \theta_{n_1+1} < \ \ldots \ < \theta_n(x).$$

$1 < n_1 < n$ and under such numeration

$$\Omega_0 = \sum_{j=0}^{n_1} \int_0^1 \theta_j(x)\,dx, \quad \Omega_{n_0} = \sum_{j=n_1+1}^{n} \int_0^1 \theta_j(x)\,dx.$$

It is known [4], [6] that the function of the form (5) has a denumerable set of the zeros λ_N with the unique limit point $\lambda = \infty$ and these zeros admit the asymptotic representation

$$\lambda_N = aN + b\ln N + O(1) \quad (N \to \infty), \tag{6}$$

where b is a complex, a is a pure imaginary number. Outside the δ-neighbourhood of the points λ_N for the Green matrix (rather for its elements), the estimation

$$\left| \frac{\partial^k G(x,\xi,\lambda)}{\partial x^k} \right| \le C|\lambda|^{p+k} \quad (k = 0,1) \tag{7}$$

is valid. Besides, any vector-function $\varphi(x) \in D_p[0,1]$ is expanded into the uniformly convergent series

$$\varphi(x) = \sum_{N=1}^{\infty} \mathop{res}_{\lambda=\lambda_N} \int_0^1 G(x,\xi,\lambda)\,A_1^{-1}(\xi)\,\varphi(\xi)\,d\xi,$$

where

$$D_0[0,1] = \left\{ f(x) : f(x) \in C^1[0,1],\ lf(x) = 0 \right\}.$$

$$D_k[0,1] = \left\{ f(x) : \left[A_1(x)\frac{d}{dx} + A_0(x) \right]^j f(x) \in D_0[0,1],\ j = 0,1,\ldots,k \right\}.$$

It is also known that the formal solution of the problem (1)-(3) is represented in the following form

$$u(t,x) = \sum_{N=1}^{\infty} \mathop{res}_{\lambda=\lambda_N} e^{\lambda t} \int_0^1 G(x,\xi,\lambda)\,A_1^{-1}(\xi)\,\varphi(\xi)\,d\xi \tag{8}$$

In the present paper on the basis of the formula (8), we prove the existence of solution of the problem (1)-(3).

The following one is valid.

Lemma. *Let conditions* 1^0, 2^0 *be satisfied. Then there exists a number* $R_0 > 0$ *such that all zeros of* λ_N *of the function* $\Delta(\lambda)$ *are contained in the domain*

$$\Pi = \left\{ \lambda : -\frac{1}{2}R_0 + q_1 \ln|\lambda| < \operatorname{Re}\lambda < \frac{1}{2}R_0 + q_2 \ln|\lambda| \right\} \cup \left\{ \lambda : |\lambda| < R_0 \right\}, \tag{9}$$

where

$$q_1=\min_{1\le k\le n_0}\frac{\nu_k-\nu_0}{\Omega_0-\Omega_k},\quad q_2=\max_{1\le k\le n_0}\frac{\nu_k-\nu_{n_0}}{\Omega_{n_0}-\Omega_k}. \tag{10}$$

Proof. It is clear that all zeros of $\Delta(\lambda)$ having large modulus lie in the domains

$$\{\lambda:-h<(\nu_{k_i}-\nu_{j_i})\ln|\lambda|+\operatorname{Re}\lambda(\Omega_{k_i}-\Omega_{j_i})+h\},\quad \left(i=\overline{1,n_2}\right) \tag{11}$$

where $0<h$ is a sufficiently large fixed number, and $k_i,\ j_i\ \left(i=\overline{1,n_2}\right)$ are such ones from the numbers $0,1,...,n_0$ for which at

$$\{\lambda:\nu_{k_i}\ln|\lambda|+\operatorname{Re}\lambda\Omega_{k_i}=\nu_{j_i}\ln|\lambda|+\operatorname{Re}\lambda\Omega_{j_i}\}$$

the following inequalities are satisfied

$$\nu_{k_i}\ln|\lambda|+\operatorname{Re}\lambda\Omega_{k_i}=\nu_{j_i}\ln|\lambda|+\operatorname{Re}\lambda\Omega_{j_i}\ \ge\nu_l\ln|\lambda|+\operatorname{Re}\lambda\Omega_l \tag{12}$$

for any $l\in\{0,1,...,n_0\}$. We obtain form (11) that for large N

$$-\frac{\nu_{j_i}-\nu_{k_i}}{\Omega_{k_i}-\Omega_{j_i}}\ln|\lambda_N|-\frac{h}{\Omega_{k_i}-\Omega_{j_i}}<\operatorname{Re}\lambda_n<\frac{\nu_{j_i}-\nu_{k_i}}{\Omega_{k_i}-\Omega_{j_i}}\ln|\lambda_N|-\frac{h}{\Omega_{k_i}-\Omega_{j_i}} \tag{13}$$

is valid, and from (10) we have

$$\nu_{j_i}+\frac{\nu_{k_i}-\nu_{j_i}}{\Omega_{j_i}-\Omega_{k_i}}\Omega_{j_i}\ge\nu_l+\frac{\nu_{k_i}-\nu_{j_i}}{\Omega_{j_i}-\Omega_{k_i}}\ \Omega_l\quad\left(l=\overline{0,n_0}\right). \tag{14}$$

Assuming in (14) $l=0$, we find

$$\frac{\nu_{k_i}-\nu_0}{\Omega_0-\Omega_{k_i}}\ge\frac{\nu_{k_i}-\nu_{j_i}}{\Omega_{j_i}-\Omega_{k_i}},$$

and if we take $l=n_0$, then from (14) we obtain

$$\frac{\nu_{j_i}-\nu_{n_0}}{\Omega_{n_0}-\Omega_{j_i}}\ge\frac{\nu_{k_i}-\nu_{j_i}}{\Omega_{j_i}-\Omega_{k_i}}.$$

The last two inequalities subject to the notations (10) and inequalities (13), prove the validity of the assertion of the lemma.

Now, by Γ we denote the boundary of the domain Π consisting of the two disconnected lines

$$\Gamma=\Gamma^{(1)}\cup\Gamma^{(2)},\ \ \Gamma^{(j)}=\Big\{\lambda:\operatorname{Re}\lambda=\tfrac{(-1)^jR_0}{2}+q_j\ln|\lambda|,$$
$$|\operatorname{Im}\lambda|>\left[\tfrac{3}{4}R_0^2+(-1)^{j-1}q_jR_0\ln R_0-q_j^2\ln^2R_0\right]^{\frac12}\Big\}\cup$$
$$\cup\Big\{\lambda:|\lambda|=R_0,\quad|\operatorname{Im}\lambda|\le\left[\tfrac{3}{4}R_0^2+(-1)^{j-1}q_jR_0\ln R_0-q_j^2\ln^2R_0\right]^{\frac12},$$
$$(-1)^j\operatorname{Re}\lambda\ge R_0+(-1)^jq_j\ln R_0\Big\}\quad(j=1,2).$$

Let $\{r_\nu\}$ be a sequence of the positive numbers r_ν such that $r_{\nu+1}>r_\nu,\ \lim\limits_{\nu\to\infty}=\infty$ and the circles $\{\lambda:|\lambda|=r_\nu\}$ do not intersect with δ-neighbourhood of the points λ_N (by virtue of (6) , such sequence exists). We accept some more notations

$$\Gamma_\nu^{(j)}=\Gamma^{(j)}\cap\{\lambda:|\lambda|=r_\nu\},\ \gamma_\nu^{(j)}=\Big\{\lambda:|\lambda|=r_\nu,\ -\tfrac{R_0}{2}+q_1\ln r_\nu\le\operatorname{Re}\lambda\le$$
$$\le\tfrac{R_0}{2}+q_2\ln r_\nu,(-1)^j\operatorname{Im}\lambda<0\Big\},\quad(j=1,2)\ \ S_\nu=\Gamma_\nu^{(1)}\cup\gamma_\nu^{(1)}\cup\Gamma_\nu^{(2)}\cup\gamma_\nu^{(21)}.$$

After such spade work we can prove the following basic assertion of the given paper.

Theorem. *Let conditions* $1^0, 2^0$ *be satisfied and* $\varphi(x) \in D_{k(p)}[0,1]$, *where*

$$k(p) = \begin{cases} p+1+[q_2T+1]-\left[\frac{[q_2T]}{q_2T}\right], & if \quad q_2>0, \\ p+2, & if \quad q_2=0, \\ p+1, & if \quad q_2>0, \end{cases} \tag{15}$$

and $[\alpha]$ *is integer part of the number* α. *Then the problem (1)-(3) has the classical solution* $u(t,x)$ *represented by the formula (8).*

Proof. At first we note that by virtue of the known properties of the Green matrix, under the condition $\varphi(x) \in D_{k(p)}[0,1]$ we can represent the formula (8) in the following form

$$u(t,x) = \sum_{N=1}^{\infty} \underset{\lambda=\lambda_N}{res}\, \lambda^{-k(p)-1} e^{\lambda t} \int_0^1 G(x,\xi,\lambda) A_1^{-1}(\xi) \left[A_1(\xi)\frac{d}{d\xi} + A_0(\xi)\right]^{k(p)+1} \varphi(\xi)\, d\xi\ .$$

Then it is clear that for validity of carry of the operations $L\left(x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$, $t \to +0$, and l under the sign of series (8), it is sufficient to prove the uniform convergence of the sequences

$$\left\{J_\nu^{(0)}(t,x)\right\},\ \left\{J_\nu^{(1)}(t,x)\right\}, \tag{16}$$

respectively, in $\overline{Q}_{T_1}$ and in $\{(t,x): T_0 \le t \le T_1,\ 0 \le x \le 1\}$, for any $T_0, T_1 \ (0 < T_0 < T_1 < T)$, where

$$J_\nu^{(j)}(t,x) = \frac{1}{2\pi i} \int_{S_\nu} \lambda^{j-k(p)-1} e^{\lambda t} d\lambda \int_0^1 G(x,\xi,\lambda) A_1^{-1}(\xi) \left[A_1(\xi)\frac{d}{d\xi} + A_0(\xi)\right]^{k(p)+1}$$

$$\varphi(\xi)\, d\xi\, (j=0,1)\,.$$

For the investigation of these sequences we note that

$$J_{\nu l}^{(j)}(t,x) \equiv J_{\nu+l}^{(j)}(t,x) - J_\nu^{(j)}(t,x) = \frac{1}{2\pi i} \sum_{\mu=1}^{2} \int_{S_{\nu l}^{(\mu)}} \lambda^{j-k(p)-1} e^{\lambda t} d\lambda \times$$

$$\times \int_0^1 G(x,\xi,\lambda) A_1^{-1}(\xi) \left[A_1(\xi)\frac{d}{d\xi} + A_0(\xi)\right]^{k(p)+1} \varphi(\xi)\, d\xi, \tag{17}$$

where l is a natural number, and

$$S_{\nu,l}^{(1)} = a_{1\nu} a_{1\nu+l} \widehat{b_{1\nu+l}} b_{1\nu} a_{1\nu}, \quad S_{\nu,l}^{(2)} = a_{2\nu} a_{2\nu+l} \widehat{b_{2\nu+l}} b_{2\nu} a_{2\nu}$$

are closed contours (see Fig.1 which corresponds to the case $q_1 < 0,\ q_2 > 0$).

Because of the proved lemma, by fulfilling conditions$1^0, 2^0$ and at $\lambda \in S_{\nu,l}^{(\mu)}$, sufficiently large ν in the case $t \in [0, T_1]$ we have the estimation

$$\left| e^{\lambda t} \right| = e^{t \operatorname{Re} \lambda} \leq \begin{cases} Ce^{T_1 q_2 \ln|\lambda|}, & if \quad q_2 > 0, \\ C, \quad if \quad q_2 \leq 0, \end{cases} \tag{18}$$

and in the case $t \in [T_0, T_1]$ we have the estimation

$$\left| e^{\lambda t} \right| = e^{t \operatorname{Re} \lambda} \leq \begin{cases} Ce^{T_1 q_2 \ln|\lambda|}, & if \quad q_2 > 0, \\ C, \quad if \quad q_2 = 0, \\ Ce^{T_0 q_2 \ln|\lambda|}, & if \quad q_2 < 0, \end{cases}$$

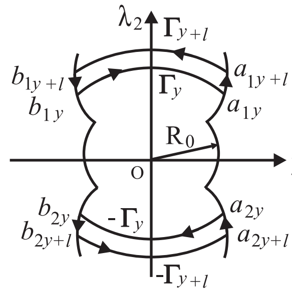

Fig.1.

Subject to these estimations and estimation (7), under the conditions of the theorem for the vector-function, under the sign of contour integral in (16) we obtain the inequality (in the sense of its fulfillment for each element of the given vector-function)

$$\left| \lambda^{-k(p)} e^{\lambda t} \int_0^1 G(x, \xi, \lambda) A_1^{-1}(\xi) \left[A_1(\xi) \frac{d}{d\xi} + A_0(\xi) \right]^{k(p)+1} \varphi(\xi) \, d\xi \right| \leq$$
$$\leq \begin{cases} C |\lambda|^{p-k(p)+T_1 q_2}, & if \quad q_2 > 0, \\ C |\lambda|^{p-k(p)}, & if \quad q_2 = 0, \\ C |\lambda|^{p-k(p)+T_0 q_2}, & if \quad q_2 < 0, \end{cases}$$

in the case $t \in [T_0, T_1]$, and the inequality

$$\left| \lambda^{-k(p)-1} e^{\lambda t} \int_0^1 G(x, \xi, \lambda) A_1^{-1}(\xi) \left[A_1(\xi) \frac{d}{d\xi} + A_0(\xi) \right]^{k(p)+1} \varphi(\xi) \, d\xi \right| \leq$$
$$\leq \begin{cases} C |\lambda|^{p-k(p)-1+T_1 q_2}, & if \quad q_2 > 0, \\ C |\lambda|^{p-k(p)-1}, & if \quad q_2 \leq 0, \end{cases}$$

in the case $t \in [0, T_1]$.

It follows from these estimations and notation (15) that in all the cases ($q_2 > 0$, $q_2 = 0$, $q_2 < 0$) the inequality

$$\left| J_{\nu l}^{(j)}(t, x) \right| \leq C \sum_{\mu=1}^{2} \int_{S_{\nu,l}^{(\mu)}} |\lambda|^q \; |d\lambda| \tag{19}$$

holds for $T_0 \le t \le T_1,\ 0 \le x \le 1$, and

$$\left| J_{\nu l}^{(0)}(t,x) \right| \le C \sum_{\mu=1}^{2} \int\limits_{S_{\nu,l}^{(\mu)}} |\lambda|^q \ |d\lambda| \tag{20}$$

holds for $0 \le t \le T_1,\ 0 \le x \le 1$, where $q < -1$. And from the inequalities (18), (19), subject to the structures of the contours $S_{\nu,l}^{(\mu)}$ we easily obtain that

$$J_{\nu l}^{(j)}(t,x) \to 0,\ J_{\nu l}^{(0)}(t,x) \to 0$$

as $\nu \to \infty$, uniformly with respect to (t,x) in corresponding sets. The latter means that the sequences (16) are uniformly convergent correspondingly at $0 \le t \le T_1,\ 0 \le x \le 1$ and $T_0 \le t \le T_1,\ 0 \le x \le 1$, for any $T_0, T_1 \ \ (0 < T_0 < T_1 < T)$.

The theorem is proved.

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Kanan A. Mamedli
Baku State University.
23, Z.I.Khalilov str., AZ1148, Baku. Azerbaijan.
Tel.: (99412) 39-03-93 (off).
E-mail: itkanan@azercell.com

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