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THE INVERSE PERIODIC PROBLEM FOR THE DIRAC OPERATOR

Abstract

In the paper the criterion that two sequences of real numbers are spectra of periodic and antiperiodic boundary problem generated by the Dirac equation is obtained.

The first result on the inverse periodic problem for the Hille operator is obtained in [1]. By the other approach, such a problem is completely solved in [2] (see also [3]). Here the asymptotic properties of special conformal mappings are generally used. By the method of this paper the characteristics of spectra of periodic and antiperiodic boundary problems generated by the Dirac equation are completely investigated in [4-5]. In the present paper it is used an other approach to the solution of the inverse problem (solvability conditions of inverse problem are essentially differ from the results of [5]).

We consider the boundary value problems generated on the segment $[0, \pi]$ by the Dirac equation

$$By'(x) + Q(x)\,y(x) = \lambda y(x) \tag{1}$$

and periodic boundary conditions

$$y(0) - y(\pi) = y'(0) - y'(\pi) = 0 \tag{2}$$

and also antiperiodic boundary conditions

$$y(0) + y(\pi) = y'(0) + y'(\pi) = 0 \tag{3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -\,p(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

and $p(x)$ and $q(x)$ are real periodic functions $(p(x+\pi) = p(x),\ q(x+\pi) = q(x))$ belonging to the space $L_2[0, \pi]$.

Theorem. *In order the sequence*

$$... < \mu_k^- \le \mu_k^+ < \mu_{k+1}^- \le \mu_{k+1}^+ < ... \quad (k = 0, \pm 1, \pm 2, ...)$$

consist of the spaectra $\{\mu_{2m}^\pm\}$ *and* $\{\mu_{2m+1}^\pm\}$ $(m = 0, \pm 1, \pm 2, ...)$ *of the boundary value problems of the form (1), (2) and (1), (3) respectively, it is necessary and sufficient that the following conditions are satisfied.*

1) the following asymptotic formula holds

$$\mu_k^\pm = k + \varepsilon_k^\pm, \quad \sum_{k=-\infty}^{\infty} \left(\varepsilon_k^\pm\right)^2 < \infty \tag{4}$$

2)

$$\triangle_a(m) - \triangle_p(m) = 4\,, \quad m = 0, \pm 1, \pm 2, ..., \tag{5}$$

where

$$\triangle_a(z) = 4\prod_{k=-\infty}^{\infty} \frac{\left(\mu_{2k-1}^- - z\right)\left(\mu_{2k-1}^+ - z\right)}{(2k-1)^2}, \tag{6}$$

$$\triangle_p(z) = -\pi^2\left(\mu_0^- - z\right)\left(\mu_0^+ - z\right)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\left(\mu_{2k}^- - z\right)\left(\mu_{2k}^+ - z\right)}{(2k)^2}. \tag{7}$$

Proof. Necessity. Denote by $c(\lambda,x) = \begin{pmatrix} c_1(\lambda,x) \\ c_2(\lambda,x) \end{pmatrix}$ and $s(\lambda,x) = \begin{pmatrix} s_1(\lambda,x) \\ s_2(\lambda,x) \end{pmatrix}$ the solutions of the equation (1) under the initial conditions $c(\lambda,x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $s(\lambda,x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then eigen values of periodic and antiperiodic problems coincide with zeros of the functions $\triangle_a(\lambda) = c_1(\lambda,\pi) + s_2(\lambda,\pi) + 2$ and $\triangle_a(\lambda) = c_1(\lambda,\pi) + s_2(\lambda,\pi) + 2$ respectively. The asymptotic formula (4) for these eigen values is derived in [4] (see also [3]). It follows from representations of the functions $c_1(\lambda,\pi)$ and $s_2(\lambda,\pi)$ [5] and asymptotic (4) that expansions on infinite production (6)-(7) hold. The validity of the equality (5) is obvious.

Sufficiency. It is known [6] that the functions $\triangle_p(z)$ and $\triangle_a(z)$ admit the representations

$$\triangle_p(z) = 2\cos\pi z - 2 + f_1(z), \quad \triangle_a(z) = 2\cos\pi z + 2 + f_2(z),$$

where

$$f_j(z) = \int_{-\pi}^{\pi} \widetilde{f}_j(t)\, e^{itz} dt, \quad \widetilde{f}_j(t) \in L_2[-\pi,\pi], \quad j = 1,2.$$

Then $\triangle_a(m) - \triangle_p(m) = 4 + f_2(m) - f_1(m)$. Allowing for (5) we find

$$f_2(m) - f_1(m) = \int_{-\pi}^{\pi} \left[\widetilde{f}_2(t) - \widetilde{f}_1(t)\right] e^{itz} dt = 0.$$

Hence by virtue of completeness of the system $\left\{e^{imt}\right\}$ in $L_2[-\pi,\pi]$ we obtain that $\widetilde{f}_2(t) - \widetilde{f}_1(t) = 0$ almost everywhere on $[-\pi,\pi]$.

Having denoted $\widetilde{f}(t) = \widetilde{f}_j(t) \quad (j = 1,2)$,

$$u_1(z) = 2\cos\pi z + \int_{-\pi}^{\pi} \widetilde{f}(t)\, e^{itz} dt \tag{8}$$

we have

$$\triangle_p(z) = u_1(z) - 2, \qquad \triangle_a(z) = u_1(z) + 2. \tag{9}$$

We take any point $\lambda_k \in \left[\mu_k^-, \mu_k^+\right]$. Then by virtue of (4)

$$\lambda_k = k + \delta_k, \quad \sum_{k=-\infty}^{\infty} \delta_k^2 < \infty \tag{10}$$

Therefore, using lemma 4 of [5] we find that the function $s_1(z) = \pi(\lambda_0 - z) \prod\limits_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\lambda_k - z}{k}$ has the form $s_1(z) = \sin\pi z + \int\limits_{-\pi}^{\pi} \varphi(t) e^{itz} dt,\ \varphi(t) \in L_2[-\pi,\pi]$. According to the formula (6), (7), (9) and the inequality $\mu_k^- \leq \lambda_k \leq \mu_k^+$ we obtain

$$u_1^2(\lambda_k) - 4 = \triangle_p(\lambda_k)\triangle_a(\lambda_k) = -4\pi^2 \left(\mu_0^- - \lambda_k\right)\left(\mu_0^+ - \lambda_k\right) \times$$

$$\times \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\left(\mu_k^- - \lambda_k\right)\left(\mu_k^+ - \lambda_k\right)}{k^2} \geq 0$$

and $sign\, u_1(\lambda_k) = (-1)^k$. Therefore the inequalities $u_1(\lambda_{2m-1}) \leq -2,\ u_1(\lambda_{2m}) \geq 2$ $(m = 0, \pm 1, \pm 2, ...)$ are valid. Then there exists such h_k that

$$u_1(\lambda_k) = 2(-1)^k ch\ h_k$$

Consider the function $u_2(z)$ such that

$$|u_2(\lambda_k)| = \sqrt{u_1^2(\lambda_k) - 4} = 2|sh\ h_k|.$$

Therefore if we requaire that $u_2(\lambda_k) = 2sh\ h_k$, we determine not only $|u_2(\lambda_k)|$ but also $sign\ u_2(\lambda_k) = sign\ h_k$. Now we construct the function $u_2(z)$ in the following form

$$u_2(z) = s_1(z) \sum_{k=-\infty}^{\infty} \frac{2sh\ h_k}{(z - \lambda_k)\, s_1'(\lambda_k)}. \tag{11}$$

As in [3, p.239-240] we may prove that $\sum\limits_{k=-\infty}^{\infty} h_k^2 < \infty$. Then by virtue of theorem 28 of [7], the function $u_2(z)$ given by the formula (11) has the form

$$u_2(z) = \int_{-\pi}^{\pi} \widetilde{\psi}(t) e^{itz} dt,\ \widetilde{\psi}(t) \in L_2[-\pi,\pi].$$

Using this formula and (8) we find

$$s_2(z) = \frac{1}{2}\left[u_1(z) - u_2(z)\right] = \cos\pi z + \int_{-\pi}^{\pi} g(t) e^{itz} dt,\quad g(t) \in L_2[-\pi,\pi].$$

Zeros of this function satisfy such asymptotic equalities (see [5])

$$\nu_k = k - \frac{1}{2} + \xi_k,\ \sum_{k=-\infty}^{\infty} \xi_k^2 < \infty. \tag{12}$$

It is clear that $s_2(\lambda_k) = (-1)^k ch\ h_k - sh\ h_k = (-1)^k ch\ h_k \left[1 - (-1)^k th\ h_k\right]$. Hence it follows in view of the inequality $th\ h_k < 1$ that $sign\ s_2(\lambda_k) = (-1)^k$. This equality shows that in each interval $(\lambda_k, \lambda_{k+1})$ there is at least one zero ν_{k+1} of the function $s_2(z)$ and by virtue of (12) of the other zeros it cannot have.

Thus zeros of the functions $s_1(z)$ and $s_2(z)$ alternate and satisfy the asymptotic formulae (10), (12). Then by virtue of theorem 2 of [5] there exists the matrix-function $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$ (whose elements belong to $L_2[0,\pi]$ such that the considered sequences of zeros $\{\lambda_k\}$, $\{\nu_k\}$ are spectra of boundary value problems generated by the same Dirac equation with the coefficient of $Q(x)$ and boundary conditions $y_1(0) = y_2(\pi) = 0$ and $y_1(0) = y_2(\pi) = 0$ and consequently $s_1(z) = s_1(z,\pi)$, $s_2(z) = s_2(z,\pi)$ where $\begin{pmatrix} s_1(z,x) \\ s_2(z,x) \end{pmatrix}$ is a solution of this equation under the initial data $\begin{pmatrix} s_1(z,0) \\ s_2(z,0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

As in [5] it is easily established that the characteristic functions of periodic and antiperiodic boundary value problems generated by constructed Dirac equation are equal to $u_1(z) \mp 2$. The theorem is proved.

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