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INTERPOLATION THEOREM FOR LIZORKIN-TRIEBEL- MORREY TYPE SPACES WITH DOMINANT MIXED DERIVATIVES

Abstract

In the paper differential properties of the functions f, which belong to $\bigcap\limits_{\mu=1}^{N} S^{l^{\mu}}_{p_\mu,\theta_\mu,a,\chi,\tau}F(G)$ are studied, and the fulfillment of Hölder condition of these functions is proved.

Here both differential and difference-differential properties of functions f belonging to the intersection of spaces $S^{l^{\mu}}_{p_\mu,\theta_\mu,a,\chi,\tau}F(G)$, $p_\mu \in (1,\infty)^n$; $\theta_\mu \in (1,\infty)$; $a \in [0,1]^n$; $\tau \in [1,\infty]$; $\chi, l^{\mu} \in (0,\infty)^n$ $(\mu \in 1,2,...,N)$, which will be called Lizorkin- Triebel- Morrey type space with dominant mixed derivatives. Let domain $G \subset R^n$ satisfy the condition of flexible horn introduced by Besov O.V. [1], $e_n = \{1,2,...,n\}$, $e \subseteq e_n, m = (m_1, m_2, ..., m_n)$, m_j are natural numbers, $k = (k_1, k_2, ..., k_n)$, $k_j \geq 0$ are integers, $j \in e_n$; $k^e = (k^e_1, k^e_2, ..., k^e_n)$, $k^e_j = k_j$ for $j \in e$; $k^e_j = 0$ for $j \in e_n \backslash e = e'$; $t_0 \in (0,\infty)^n$ is a fixed vector, $[t]_1 = \min\{1,t\}$ and

$$D^{k^e} f(x) = D_1^{k_1^e} ... D_n^{k_n^e} f(x) ; \quad \int\limits_{a^e}^{b^e} f(x)\, dx^e = \left(\prod_{j\in e} \int\limits_{a_j}^{b_j} dx_j \right) f(x) .$$

Note that in [2] Besov- Morrey type spaces $S^l_{p,\theta,a,\chi,\tau}B(G)$ with dominant mixed derivatives were defined and studied, in [3] interpolation theorems for spaces $S^{l^{\mu}}_{p_\mu,\theta_\mu}F(G)$, which is called Lizorkin- Triebel space with dominant mixed derivatives were proved, and in [4] some properties of spaces $S^l_{p,\theta,a,\chi,\tau}F(G)$ were constructed and studied from the point of view of imbedding theory.

Denote by $S^l_{p,\theta,a,\chi,\tau}F(G)$ a Banach space of locally summable on G functions f with the finite norm $(m_j > l_j > k_j \geq 0,\ j \in e_n;\ p \in (1,\infty)^n, \theta \in (1,\infty)\ )$:

$$\|f\|_{S^l_{p,\theta,a,\chi,\tau}F(G)} = \sum_{e \subseteq e_n} \left\| \left\{ \int\limits_{0^e}^{h^e} \left[\frac{\delta^{m^e - k^e}(h)\, D^{k^e} f}{\prod\limits_{j\in e} h_j^{l_j - k_j}} \right]^{\theta} \prod_{j\in e} \frac{dh_j}{h_j} \right\}^{\frac{1}{\theta}} \right\|_{p,a,\chi,\tau} , \qquad (1)$$

where

$$\delta^{m^e}(h) f(x) = \left(\prod_{j\in e} \delta_j^{m_j}(h_j) \right) f(x) , \ \delta_j^{m_j}(h_j) f(x) = \int\limits_{-1}^{1} \left| \Delta_j^{m_j}(h_j u, G_h) f(x)\, du \right|$$

and

$$\|f\|_{p,a,\chi,\tau,G} = \sup_{x\in G} \left\{ \int\limits_0^{t_{01}} ... \int\limits_0^{t_{0n}} \left[\prod_{j\in e_n} [t_j]^{\frac{\chi_j a}{p_j}} \|f\|_{p,G_{t\chi}(x)} \right]^{\tau} \prod_{j\in e_n} \frac{dt_j}{t_j} \right\}^{\frac{1}{\tau}} .$$

$$G_{t^{\chi}}(x)=G\cap I_{t^{\chi}}(x)\,,\ \ I_{t^{\chi}}(x)=\left\{y:|y_j-x_j|<\frac{1}{2}t_j^{\chi_j},j\in e_n\right\},$$

consequently, $mes\ G_{t^{\chi}}(x)\le mes\ I_{t^{\chi}}(x)=t^{|\chi|},\ |\chi|=\sum\limits_{j\in e_n}\chi_j$. For $\tau=\infty$, space $S^l_{p,a,\chi,\tau}F(G)$ will be called Lizorkin- Triebel- Morrey space with dominant mixed derivatives.

Let

$$\beta_\mu\ge 0,\sum_{\mu=1}^{N}\beta_\mu=1,\ \frac{1}{p}=\sum_{\mu=1}^{N}\frac{\beta_\mu}{p_\mu},\ \frac{1}{\theta}=\sum_{\mu=1}^{N}\frac{\beta_\mu}{\theta_\mu},\ l=\sum_{\mu=1}^{N}l^\mu\beta_\mu,$$

$$\varepsilon_j=\sum_{\mu=1}^{N}l_l^\mu\beta_\mu-\alpha_j-\left(1-\chi_ja_j\right)\left(\frac{1}{p_j}-\frac{1}{q_j}\right);\ \varepsilon_j^0=\sum_{\mu=1}^{N}l_l^\mu\beta_\mu-\alpha_j-\left(1-\chi_ja_j\right)\frac{1}{p_j},$$

and also let $M_e(y,z)\in C^\infty(R^n,R^n)$, $M_e(\cdot,z)$ be infinitely differentiable function with respect to all variables, and uniformly compact function with respect to z from arbitrary compact.

Lemma 1. *Let $1\le p_\mu\le q_\mu\le r_\mu\le\infty, 0<\chi_j\le 1,\ 0<t_j\le T_j\le 1, 0<\rho_j<\infty,\ j\in e_n$; $\delta^{m^e}(\delta u)f\in L_{p,a,\chi,\tau}(G)\,,\ 1\le\tau\le\infty,\ \gamma=(\gamma_1,...,\gamma_n)\,,\ 0<\gamma_j\le T_j,\ \alpha=(\alpha_1,...,\alpha_n)\,,\alpha_j\ge 0$ are integers, $j\in e_n$,*

$$A^e_\gamma(x)=\prod_{j\in e'}T_j^{-1-\alpha_j}\int\limits_{0^e}^{\gamma^e}\prod_{j\in e}t_j^{-2-\alpha_j}I_e\left(x,t^e+T^{e'}\right)dt^e,$$

$$A^e_{\gamma T}(x)=\prod_{j\in e'}T_j^{-1-\alpha_j}\int\limits_{\gamma^e}^{T^e}\prod_{j\in e}t_j^{-2-\alpha_j}I_e\left(x,t^e+T^{e'}\right)dt^e,$$

where

$$I_e\left(x,t^e+T^{e'}\right)=\int\limits_{R^n}M_e^{(\alpha)}\left(\frac{y}{t^e+T^{e'}}\frac{\rho\left(t^e+T^{e'},x\right)}{t^e+T^{e'}}\right)f_e(x+y,t)\,dt,$$

$$|f_e|\le C\int\limits_{-1^e}^{1^e}\delta^{m^e}(\delta u)\,f\left(x+uv^e\right)dv^e.$$

Then the following inequalities hold

$$\sup_{\bar{x}\in U}\left\|A^e_\gamma\right\|_{q,U_{\rho^\chi}(\bar{x})}\le C_1\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t^{-l_j^\mu}\delta^{m^e}(t)\,f\right\|_{p_\mu,a,\chi,\tau}\right\}^{\beta_\mu}\times$$
$$\times\prod_{j\in e_n}\left[\rho_j\right]_1^{\chi_j\frac{a_j}{r_j}}\prod_{j\in e}\gamma_j^{\varepsilon_j},\ \ (\varepsilon_j>0) \tag{2}$$

$$\sup_{\bar{x}\in U}\left\|A^e_\gamma\right\|_{q,U_{\rho^\chi}(\bar{x})}\le C_1\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t^{-l_j^\mu}\delta^{m^e}(t)\,f\right\|_{p_\mu,a,\chi,\tau}\right\}^{\beta_\mu}\times$$

$$\times \prod_{j\in e_n} \left[\rho_j\right]_1^{\chi_j \frac{a_j}{r_j}} \left\{ \begin{array}{l} \prod\limits_{j\in e} T_j^{\varepsilon_j},\ \varepsilon_j > 0 \\ \prod\limits_{j\in e} \ln \frac{T_j}{\gamma_j}, \varepsilon_j = 0 \\ \prod\limits_{j\in e} \gamma_j^{\varepsilon_j},\ \varepsilon_j < 0 \end{array} \right. \tag{3}$$

Lemma 2. *Let* $1 \le p_\mu \le q_\mu < \infty,\ 0 < \chi_j \le 1,\ 0 < t_j \le T_j \le 1,\ \alpha = (\alpha_1, ..., \alpha_n),\ \alpha_j \ge 0$ *be integers,* $j \in e_n$,$1 \le \tau_1 < \tau_2 \le \infty$ *then the following inequalities hold*

$$\|A_T^e\|_{q,b,\chi,\tau_2;U} \le C \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j\in e} t^{-l_j^\mu} \delta^{m^e}(t) f \right\|_{p_\mu,a,\chi,\tau_1} \right\}^{\beta_\mu} \tag{4}$$

where A_T^e *was defined in lemma 1.*

Theorem. *Let open set* $G \subset R^n$ *satisfy the condition of flexible horn,* $1 \le p_\mu \le q_\mu \le \infty,\quad 1 < \theta \le \theta_1 < \infty,\ \alpha = (\alpha_1, ..., \alpha_n),\ \alpha_j \ge 0$ *be integers,* $j \in e_n$, $1 \le \tau_1 < \tau_2 \le \infty$ *and let* $\varepsilon_j > 0$ for $j \in e_n$, $f \in \bigcap\limits_{\mu=1}^{N} S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1} F(G)$, *then the following inequalities hold:*

$$\|D^\alpha f\|_{q,G} \le C_1 A_1(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1} F(G)} \right\}^{\beta_\mu}, \tag{5}$$

$$\|D^\alpha f\|_{q,b,\chi,\tau_2;G} \le C_2 \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1} F(G)} \right\}^{\beta_\mu}, p_\mu \le q_\mu < \infty, \tag{6}$$

and if $\varepsilon_j - l_j^1 > 0$ *for* $j \in e_n$, *then*

$$\|D^\alpha f\|_{S^{l^1}_{q,\theta_1} F(G)} \le C_3 A_2(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1} F(G)} \right\}^{\beta_\mu}, p_\mu \le q_\mu < \infty, \tag{7}$$

$$\|D^\alpha f\|_{S^{l^1}_{q,\theta_1,b,\chi,\tau_2} F(G)} \le C_4 \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1} F(G)} \right\}^{\beta_\mu}, p_\mu \le q_\mu < \infty, \tag{8}$$

$$\|D^\alpha f\|_{S^{l^1}_{q,\theta_1} B(G)} \le C_5 A_2(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1} F(G)} \right\}^{\beta_\mu}, \tag{9}$$

$$\|D^\alpha f\|_{S^{l^1}_{q,\theta_1,b,\chi,\tau_2} B(G)} \le C_6 \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1} F(G)} \right\}^{\beta_\mu}, p_\mu \le q_\mu < \infty, \tag{10}$$

where

$$A_1(T) = \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}}, A_2(T) = \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}-l_j^1},\ s_{e,j} =$$

$$= \left\{ \begin{array}{l} \varepsilon_j,\ j \in e \\ -\alpha_j - \left(1 - \chi_j a_j\right) \left(\frac{1}{p_j} - \frac{1}{q_j}\right),\ j \in e' \end{array} \right.$$

In particular, if $\varepsilon_j^0 > 0$ *for* $j \in e_n$, *then* $D^\alpha f$ *is continuos on* G *and*

$$\sup_{x\in G}|D^{\alpha}f|\le C_1A_1^0(T)\prod_{\mu=1}^{N}\left\{\|f\|_{S^{l^{\mu}}_{p_{\mu},\theta_{\mu},a,\chi,\tau_1}F(G)}\right\}^{\beta_{\mu}} \tag{11}$$

where $A_1^0(T)=\sum\limits_{e\subseteq e_n}\prod\limits_{j\in e_n}T_j^{s^0_{e,j}},\ s^0_{e,j}=\begin{cases}\varepsilon_j^0,\ j\in e\\ -\alpha_j-\left(1-\chi_ja_j\right)\frac{1}{p_j},\ j\in e'\end{cases}.$

Proof. Let $f\in S^{l^{\mu}}_{p_{\mu},\theta_{\mu},a,\chi,\tau_1}F(G)\to f\in S^{l^{\mu}}_{p_{\mu},\theta_{\mu},a,\chi}F(G)\to S^{l^{\mu}}_{p_{\mu},\theta_{\mu}}F(G)$. If $\varepsilon_j>0, l_j-\alpha_j>0,\ 0\le a_j\le 1,\ j\in e_n,\ p_{\mu}\le q_{\mu}\ (\mu=1,2,...,N)$ then by theorem 1 from [4] $D^{\alpha}f$ exists and $D^{\alpha}f\in L_q(G)$. Then for almast every point $x\in G$ the following integral representation is valid:

$$D^{\alpha}f=\sum_{e\subseteq e_n}(-1)^{|\omega^e|+|\alpha|}\,2^{|\omega^e|}\prod_{j\in e'}T_j^{-1-\alpha_j}\int\limits_{0^e}^{T^e}\frac{dt^e}{\prod\limits_{j\in e}t_j^{2+\alpha_j}}\times$$
$$\times\int M_e^{(\alpha)}\left(\frac{y}{t^e+T^{e'}},\frac{\rho\left(t^e+T^{e'},x\right)}{t^e+T^{e'}}\right)f_e\left(x+y,t^e\right)dy \tag{12}$$

where $\omega=(1,1,...,1)$ and supports of function $M_e^{(\alpha)}$ are contained in I_1, therefore supports of representation (12) are contained in flexible horn which is defined in [2].

By means of Minkowsky inequality we obtain that

$$\|D^{\alpha}f\|_{q,G}\le C\sum_{e\subseteq e_n}\|A_T^e\|_{q,G}\,. \tag{13}$$

By means of inequality (2) from inequality (3) taking into account that $\max p_{\mu j}\le\theta_{\mu},\ \ j\in e_n\ (\mu=1,2,...,N)$ we obtain inequality (5). In order to prove the rest of inequalities we will estimate the norm $\left\|\Delta^{M^e}(h,G_h)\,D^{\alpha}f\right\|_{q,G}$. After some transformations, from representation (12) we will obtain the following inequality

$$\left|\Delta^{M^e}(h,G_h)\,D^{\alpha}f\right|\le C_1\sum_{e\subseteq e_n}\prod_{j\in e'}T_j^{-1-\alpha_j}\int\limits_{0^e}^{H^e}\frac{dt^e}{\prod\limits_{j\in e}t_j^{2+\alpha_j}}\times$$
$$\times\int\left|M_e^{(\alpha)}\right|\left|\Delta^{M^e}(h)\,f_e\left(x+y,t^e\right)\right|dy+C_2\prod_{j\in e}h_j^{M_j}\sum_{e\subseteq e_n}\prod_{j\in e'}T_j^{-1-\alpha_j}\int\limits_{H^e}^{T^e}\frac{dt^e}{\prod\limits_{j\in e}t_j^{2+\alpha_j}}\times$$
$$\times\int\left|M_e^{(\alpha+M^e)}\right|\int\limits_{0^e}^{1^e}\left|f_e\left(x+y+M_1^eh_1^eu_1^e+...,M_n^eh_n^eu_n^e,\ t^e\right)\right|du^edy=$$
$$=C_1\sum_{e\subseteq e_n}Z_e^1(x)+C_2\sum_{e\subseteq e_n}Z_e^2(x)\,,$$

$$\left\|\Delta^{M^e}(h,G_h)\,D^{\alpha}f\right\|_{q,G_{\rho\chi}(x)}\le$$
$$\le C_1\sum_{e\subseteq e_n}\left\|Z_e^1(x)\right\|_{q,G_{\rho\chi}(x)}+C_2\sum_{e\subseteq e_n}\left\|Z_e^2(x)\right\|_{q,G_{\rho\chi}(x)} \tag{14}$$

From inequality (2) for $H\equiv T,\ \rho\to\infty$ we will obtain

$$\left\|Z_e^1(x)\right\|_{q,G}\le C_3\prod_{j\in e_n}T_j^{s_{e,j}}\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e}t_j^{-l_j^{\mu}}\Delta^{M^e}(h)\,\delta^{m^e}(t)\,f\right\|_{p_{\mu},a,\chi,\tau}\right\}^{\beta_{\mu}}\le$$

$$\leq C_4 \prod_{j\in e_n} T_j^{s_{e,j}} \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j\in e} t_j^{-l_j^{\mu}} \delta^{m^e}(t) f \right\|_{p_\mu,a,\chi,\tau} \right\}^{\beta_\mu}. \tag{15}$$

From inequality (3) as $\rho \to \infty$ $\left(l_j^1 \leq M_j,\ j \in e \right)$

$$\left\| Z_e^2(x) \right\|_{q,G} \leq C_5 \prod_{j\in e} h_j^{M_j} \prod_{j\in e_n} T_j^{s_{e,j}-M_j} \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j\in e} t_j^{-l_j^{\mu}} \delta^{m^e}(t) f \right\|_{p_\mu,a,\chi,\tau} \right\}^{\beta_\mu} \leq$$

$$\leq C_5 \prod_{j\in e} h_j^{l_j^1} \prod_{j\in e_n} T_j^{s_{e,j}-l_j^1} \left\{ \left\| \prod_{j\in e} t_j^{-l_j^{\mu}} \delta^{m^e}(t) f \right\|_{p_\mu,a,\chi,\tau} \right\}^{\beta_\mu}. \tag{16}$$

By means of inequalities (14)-(16) under the condition that if $\max p_{\mu j} \leq \theta_\mu$, $j \in e_n$, we obtain

$$\left\| D^\alpha f \right\|_{S^{l^1}_{q,\theta_1}B(G)} \leq C_6 A_2(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1}F(G)} \right\}^{\beta_\mu},$$

and if $\theta_1 \leq \min q_j, j \in e_n$ then we get

$$\left\| D^\alpha f \right\|_{S^{l^1}_{q,\theta_1}F(G)} \leq C_6 A_2(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1}F(G)} \right\}^{\beta_\mu}.$$

Estimates (6), (8) and (10) can be similarly established on the basis of inequality (4).

Let now $\varepsilon_j^0 > 0,\ j \in e_n$. Let us show that in this case $D^\alpha f$ is continuous on G. On the basis of identity (10) and inequality (6) for $q \equiv \infty,\ \varepsilon_j = \varepsilon_j^0 > 0$ we have

$$\left\| D^\alpha f - f_T^{(\alpha)} \right\|_{\infty,G} \leq C \sum_{\varnothing \neq e \subseteq e_n} \prod_{j\in e_n} T_j^{s^0_{e,j}} \prod_{\mu=1}^{N} \left\{ \|f\|_{S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1}F(G)} \right\}^{\beta_\mu}.$$

$\lim\limits_{T\to 0} \left\| D^\alpha f - f_T^{(\alpha)} \right\|_{\infty,G} = 0$. Since $f_T^{(\alpha)}$ is continuous on G, convergence in $L_\infty(G)$ in this case coincides with uniformity, and consequently $D^\alpha f$ is continuous on G.

It was also proved that for f which belong to the intersection $S^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1}F(G)$ $(\mu = 1,2,...,N)$, generalized derivative $D^\alpha f$ on G satisfies the Hölder condition in metrics L_q.

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