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# MULTIPLE EXPANSION ON THE WHOLE AXIS FOR ONE FAMILY OF NOT SELF-ADJOINT SINGULAR DIFFERENTIAL OPERATORS OF COMPLICATED FORM

**Abstract**

In the paper the obtaining of formula is investigated. The formula itself is given for multiple expansion on the whole axis for a family of nonself-adjoint singular differential operators of the high even order with coefficients polynomially depending on complex spectral parameter and not being infinitely small.

The present paper is based on results received in [5] on a spectrum for corresponding family of non-self-adjoint differential singular operators determined on the whole axis.

Consider such family of differential operators:

$$L(\lambda) = L_0 + L_1(\lambda) + \lambda^{2n} \tag{1}$$

where

$$L_0 = \sum_{i=0}^{2n} q_i \frac{d^{2n-i}}{dx^{2n-i}}; \qquad q_0 \equiv 1$$

$$L_1(\lambda) = \sum_{j=2}^{2n} P_j(x,\lambda) \frac{d^{2n-j}}{dx^{2n-j}} \tag{2}$$

$$P_j(x,\lambda) = \lambda^{j-1} p_{j1}(x) + \lambda^{j-2} p_{j2}(x) + ... + p_{jj}(x) \tag{3}$$

The domain of determination of the family of operators is determined analogously [3], p.145.

Here we suppose that $(x+1)\, q_j(x) \in L_1(-\infty,\infty)$.

Denote by:

$$p(\mu) = \mu^{2n} + q_1\mu^{2n-1} + q_2\mu^{2n-2} + ... + q_{2n-1}\mu + q_{2n}. \tag{4}$$

Rewrite the family of the operators (1) in the form:

$$L(\lambda) = \lambda^{2n} + \sum_{i=1}^{2n} A_i \lambda^{2n-i} \tag{5}$$

where

$$A_j = p_{2n,j}(x) + p_{2n-1,j-1}(x)\frac{d}{dx} + ... + p_{2n-j+1,1}\frac{d^{j-1}}{dx^{j-1}}, \quad j = \overline{1, 2n-1}$$

$$A_{2n} = [p_{2n,2n}(x) + q_{2n}] + [p_{2n-1,2n-1}(x) + q_{2n-1}]\frac{d}{dx} + ...+ \tag{5*}$$

$$+ [p_{22}(x) + q_2]\frac{d^{2n-2}}{dx^{2n-2}} + q_1\frac{d^{2n-1}}{dx^{2n-1}} + \frac{d^{2n}}{dx^{2n}}.$$

Using the resolvent determination $R_\lambda$ of the family of the operators $L(\lambda)$, we have

$$L(\lambda) R_\lambda = E \tag{6}$$

or

$$\left(\lambda^{2n} + \sum_{i=1}^{2n} A_i \lambda^{2n-i}\right) R_\lambda = E$$

Hence

$$R_\lambda = \frac{E}{\lambda^{2n}} - \frac{1}{\lambda^{2n}} \left(\sum_{i=1}^{2n} A_i \lambda^{2n-i}\right) R_\lambda$$

Analogously

$$R_\lambda = \sum_{j=1}^{k} (-1)^{j-1} \frac{E}{\lambda^{2nj}} \left(\sum_{i=1}^{2n} A_i \lambda^{2n-i}\right)^{j-1} +$$

$$+ (-1)^k \frac{1}{\lambda^{2nk}} \left(\sum_{i=1}^{2n} A_i \lambda^{2n-i}\right)^k R_\lambda, \quad k = \overline{2, 2n}.$$

Let $f_i$, $i = \overline{0, 2n-1}$ be arbitrary finite functions which are identically equal to zero in some neighborhood of zero and plus infinity, moreover $f_i$ are differentiable $2n-i-2$ times.

Consider an auxiliary Cauchy problem:

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u(x,t) = 0 \tag{7}$$

$$u_t^{(j)}(x,0) = f_j(x), \quad j = \overline{0, 2n-1} \tag{8}$$

where $f_j$, $j = \overline{0, 2n-1}$ are above described functions and

$$L\left(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial x}\right) u(x,t) = u_x^{(2n)}(x,t) + q_1 u_x^{(2n-1)}(x,t) +$$

$$+ \left[q_2 + p_{21}(x) \tfrac{\partial}{\partial t} + p_{22}(x)\right] u_x^{(2n-2)}(x,t) +$$

$$+ \left[q_3 + p_{31}(x) \tfrac{\partial^2}{\partial t^2} + p_{32}(x) \tfrac{\partial}{\partial t} + p_{33}(x)\right] u_x^{(2n-3)}(x,t) +$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$

$$+ \left[q_{2n-1} + p_{2n-1}(x) \tfrac{\partial^{2n-2}}{\partial t^{2n-2}} + p_{2n-1,2}(x) \tfrac{\partial^{2n-3}}{\partial t^{2n-3}} + \ldots + \right.$$

$$\left. + p_{2n-1,2n-2}(x) \tfrac{\partial}{\partial t} + p_{2n-1,2n-1}(x)\right] u_x'(x,t) + \left[q_{2n} + p_{2n,1}(x) \tfrac{\partial^{2n-1}}{\partial t^{2n-1}} + \right.$$

$$\left. + p_{2n,2}(x) \tfrac{\partial^{2n-2}}{\partial t^{2n-2}} + \ldots + p_{2n,,2n-1}(x) \tfrac{\partial}{\partial t} + p_{2n,2n}(x) + \tfrac{\partial^{2n}}{\partial t^{2n}}\right] u(x,t).$$

We make the Laplace transformation in equation (7), denoting by

$$y(x,\lambda)=\int\limits_0^{\infty} e^{-\lambda t}u(x,t)\,dt \tag{9}$$

Then integrating by parts we have

$$\int\limits_0^{\infty} e^{-\lambda t}u_t^{(n)}(x,t)\,dt \;=\lambda^n y(x,\lambda)-\sum_{k=0}^{n-1}\lambda^{n-1-k}f_k \tag{10}$$

Taking (9) and (10) into account, equation (7) is transformed into the form:

$$\begin{gathered} y^{(2n)}(x,\lambda)+q_1 y^{(2n-1)}+\left[q_2+P_2(x,\lambda)\right]y^{(2n-2)}(x,\lambda)+\\ +\left[q_3+P_3(x,\lambda)\right]y^{(2n-3)}(x,\lambda)+\cdots+\left[q_{2n-1}+\right.\\ \left.+P_{2n-1}(x,\lambda)\right]y'(x,\lambda)+\left[q_{2n}+P_{2n}(x,\lambda)+\lambda^{2n}\right]y(x,\lambda)=\sum_{i=1}^{2n}\lambda^{2n-i}\varphi_i \end{gathered} \tag{11}$$

where

$$\begin{gathered} \varphi_i=\sum_{j=0}^{i-2}P_{2n-j,i-j-1}(x)\,f_0^{(j)}+\sum_{j=0}^{i-3}P_{2n-j,i-j-2}(x)\,f_1^{(j)}+...+\\ +\sum_{j=0}^{1}P_{2n-j,2-j}f_{i-3}^{(j)}+P_{2n,1}(x)\,f_{i-2}+f_{i-1}=\\ =\sum_{k=0}^{i-2}\sum_{j=0}^{i-k}P_{2n-j,i-j-k-1}f_{k-2}^{(j)}+f_{\;i-1}. \end{gathered} \tag{12}$$

In notations (5*) the expressions (12) will be rewritten in the form:

$$\varphi_i=\sum_{j=1}^{i-1}A_{i-j}f_{j-1}\;+f_{i-1}\;,\qquad i=\overline{1,2n} \tag{13}$$

Then, by the definition of $R_\lambda$, the solution of the equation (11) will be written in the form:

$$y(x,\lambda)=R_\lambda\left(\sum_{i=1}^{2n}\lambda^{2n-i}\varphi_i\right) \tag{14}$$

where $\varphi_i,\ i=\overline{1,2n}$ are determined by the formula (13).

We will use below the known relation:

$$\int\limits_\Gamma \lambda^n d\lambda=\begin{cases}0 & n\neq-1\\ 2\pi & n=-1\end{cases},$$

where $\Gamma$ is a circle of some radius $R$.

Then it is easy to compute that:

$$\left| \int\limits_{\Gamma_{R,\varepsilon}} \lambda^n d\lambda \right| \leq \begin{cases} \varepsilon R^{n+1}, & n \neq -1 \\ 2\pi - \varepsilon, & n = -1 \end{cases} \tag{15}$$

where $\Gamma_{R,\varepsilon}$ is a circle of some radius $R$ with the finite number of rejected arcs of a comman length $\varepsilon$. Let us $D_j$, $j = \overline{1, 2n}$ be the domains consisting of all points $\lambda$ situated on the distance $\varepsilon$ from $p(\mu) = \lambda^{2n}$, where $\varepsilon = \varepsilon(R)$. Then we throw the domains $D_j$, $j = \overline{1, 2n}$ from the circle bounded by $\Gamma_N$. We can take $\varepsilon$ as $\varepsilon(R) = \dfrac{1}{R^{2n+1}}$.

We denote the contour of the remaining part of a circle bounded by a circumference $\Gamma_N$ via $\widetilde{\Gamma}_{N,\varepsilon}$. It is evidently that $\widetilde{\Gamma}_{N,\varepsilon}$ consists of $2n$ domains. We denote the boundaries of the domains $D_j$, $j = \overline{1, 2n}$ by $M_j$ and $M_{j+1}$ and the arcs of the contour $\Gamma_N$ between $D_j$ and $D_{j+1}$ by $S_{j,N\varepsilon}, \overline{1, 2n}$ . Let

$$\sum_{j=1}^{2n} S_{j,N\varepsilon} = \Gamma_{N,\varepsilon}$$

where $\Gamma_{N,\varepsilon}$ is a bound passing in a positive direction. We suppose $\varepsilon$ so small and $R$ so big that $R\sigma(L) \subset \bigcup\limits_{j=1}^{n} D_j$ (see [5], p.118).

Taking (13), (14) and (15) into account we have for $\lambda \epsilon \rho(L)$, where $\rho(L)$ is a resolvent set of the family $L(\lambda)$ and for $\lambda$ sufficiently big:

$$\frac{1}{2\pi i} \sum_{j=1}^{2n} \int\limits_{S_{j,N\varepsilon}} y(x,\lambda)\, d\lambda = \frac{1}{2\pi i} \int\limits_{\Gamma_{N,\varepsilon}} y(x,\lambda)\, d\lambda = f_0 + o(\varepsilon) + O\left(\frac{1}{|\lambda|^2}\right) \tag{16}$$

Here we took into account the boundedness on the kernel $K(x,\xi,\lambda)$ of the resolvent $R_\lambda$ into account which we introduce:

$$|K(x,\xi,\lambda)| \leq \frac{c}{|\lambda|^{2n-1}} \tag{17}$$

for $\lambda \epsilon \rho(L)$ and $\lambda$ sufficiently big, $c = const$, uniformly relative to $x, \xi$ in every finite square $0 \leq x,\ \xi \leq a,\ a > 0$. Let the coefficients $p_{ki}(x)$ satisfy the condition:

$$|p_{ki}(x)| \leq ce^{-\varepsilon|x|},\ c = const,\ i \leq k. \tag{18}$$

Evidently the right hand side of (16) is $f_0$ as $|\lambda| \to \infty$.

Analogously we can obtain:

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{N,\varepsilon}} \lambda y(x,\lambda)\, d\lambda = f_1 + o(\varepsilon) + O\left(\frac{1}{|\lambda|^2}\right) or$$

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{N,\varepsilon}} \lambda^k y(x,\lambda)\, d\lambda = f_k + o(\varepsilon) + O\left(\frac{1}{|\lambda|^2}\right) \tag{19}$$

and

$$\frac{1}{2\pi i}\int\limits_{\Gamma_N}\lambda^k y(x,\lambda)\,d\lambda = f_k + O\left(\frac{1}{|\lambda|^2}\right) \tag{20}$$

For receiving a multiple expansion of arbitrary finite sufficiently smooth functions we shall suppose that

$$W_j(\lambda)\neq 0 \quad and \quad W_j^*(\lambda)\neq 0 \tag{21}$$

where $W_j(\lambda)$ and $W_j^*(\lambda)$ are determined in [5], p.118.

Then using the generalized Cauchy theorem we have

$$\frac{1}{2\pi i}\lim_{N\to\infty}\int\limits_{\Gamma_{N,\varepsilon}} R_\lambda\left(\sum_{i=1}^{2n}\lambda^{2n-i}\varphi_i\right)d\lambda =$$

$$=\frac{1}{2\pi i}\lim_{N\to\infty}\sum_{j=1}^{2n}\int\limits_{M_j}\left(R_{\lambda-i_0}-R_{\lambda+i_0}\right)\left(\sum_{i=1}^{2n}\lambda^{2n-i}\varphi\right)d\lambda =$$

$$=\frac{1}{2\pi i}\lim_{N\to\infty}\sum_{j=1}^{2n}\int\limits_{M_j}F(\lambda)\,M(x,\lambda)\,d\lambda = \frac{1}{2\pi i}\sum_{j=1}^{2n}\int\limits_{M_j}B(x,\lambda)\,d\lambda$$

where

$$B(x,\lambda)=F(\lambda)\,M(x,\lambda),$$

$$F(\lambda)=\int\limits_{-\infty}^{+\infty}\chi(\xi,\lambda)\,d\xi$$

and the functions $\chi(\xi,\lambda)$ are determined by some combinations of the functions (13) and solutions of the equation $p(\mu)=\lambda^{2n}$ by $\lambda$ and $M(x,\lambda)$ are determined by some solutions of the equation $L(\lambda)\,y=0$ moreover under the conditions (17) and (21) the limits of integrals exist.

Thus we have

$$f_0=\frac{1}{2\pi i}\sum_{j=1}^{2n}\int\limits_{M_j}B(x,\lambda)\,d\lambda \tag{22}$$

Analogously we obtain

$$f_j=\frac{1}{2\pi i}\sum_{j=1}^{2n}\int\limits_{M_j}\lambda^j B(x,\lambda)\,d\lambda \tag{23}$$

Thus we prove the following

**Theorem.** *Let the conditions (18) and (21) be fulfilled. Then for arbitrary finite sufficiently smooth functions* $f_k,\ k=\overline{0,2n-1}$ *$2n$-multiple expansion of the form (23) holds where the integrals converge uniformly and absolutely for all* $x\in(-\infty,\infty)$.

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