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ON HÖLDER PROPERTY OF SOLUTION OF SOME RIEMANN PROBLEM WITH DISCONTINUOUS COEFFICIENT

Abstract

By the investigating of basic properties of "double" system of exponents it is important to know the behaviour of solution of corresponding Riemann problem with discontinuous coefficient of the theory of analytical functions. In the given paper it is obtained that at certain conditions on coefficient and absolute term of problem the solution is Hölder on contour.

Theorem. *Let L be simple, smooth, closed contour passing around the direction, $g(t)$ be Hölder, $G(t) \neq 0$ be a piecewise Holder function on L and $G(t)$ have the first kind discontinuity at the point $t_1, t_2, ..., t_m$, moreover $G(t_j \pm 0) \neq 0, \forall\, j = \overline{1,m}$. Then if $0 < \operatorname{Re}\mu_j < 1$ where $\mu_j = \frac{1}{2\pi i} \ln \frac{G(t_j-0)}{G(t_j+0)}$, $j = \overline{1,m}$ then the solution of Riemann boundary value problem*

$$\Phi^+(t) = G(t)\,\Phi^-(t) + g(t)\,, \quad t \in L \backslash \{t_1, t_2, ..., t_m\} \tag{1}$$

satisfies the condition $\Phi^\pm(t_j \pm 0) = \Phi^\pm(t_j \mp 0)\,,\ j = \overline{1,m}$, i.e. $\Phi^\pm(t)$ satisfy Holder condition everywhere on L if we redefine $\Phi^\pm(t_j) = \Phi^\pm(t_j - 0)\,,\quad j = \overline{1,m}$.

Proof. The solution of the Riemann problem (1) are the following Cauchy type integrals [1]:

$$\Phi^+(z) = \prod_{k=1}^m (z - t_k)^{\mu_k} X_1^+(z)\,\Psi^+(z)\,,$$

$$\Phi^-(z) = \prod_{k=1}^m \left(\frac{z - t_k}{z - z_0}\right)^{\mu_k} X_1^-(z)\,\Psi^-(z)\,,$$

where $X_1^\pm(z) = e^{\Gamma^\pm(z)}$

$$\Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\ln\left[\prod_{k=1}^m (\tau - z_0)^{-\mu_k} G(\tau)\right] d\tau}{\tau - z},$$

$$\Psi(z) = \frac{1}{2\pi i} \int_L \frac{\prod_{k=1}^m (\tau - t_k)^{-\mu_k} g(\tau)\, d\tau}{X_1^+(\tau)(\tau - z)}.$$

It is sufficient to carry out the proof for the case $m = 1$. The case $m > 1$ is proved analogously. So, at $m = 1$ we have:

$$\Phi^+(z) = (z - t_1)^{\mu_1} X_1^+(z)\,\Psi^+(z)\,, \tag{2}$$

$$\Phi^{-}(z)=\left(\frac{z-t_1}{z-z_0}\right)^{\mu_1}X_1^{-}(z)\,\Psi^{-}(z)\,. \tag{3}$$

The points of branching of the function $(z-t_1)^{\mu_1}$ and $(z-z_0)^{-\mu_1}$ will be t_1,∞ and z_0,∞ respectively.

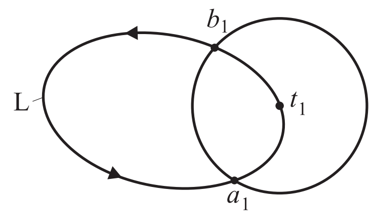

Fig.1.

Let's cut from the point z_0 trough the points t_1 till the infinity on the plane z. On plane being cut by the such way these functions will be single valued, moreover the cut for them will be a discontinuity line.

Let's agree that the section of the discontinuity z_0t_1 lies wholly in integral domain of the contour L.

Take the circle with the center at the point t_1 and let a_1, b_1 be the points of intersection of this circle with the curve L where the point b_1 follows the point a_1. We'll take the radius such small that the circle hasn't with the curve L other points of intersection except a_1 and b_1 (fig.1).

Denote by

$$\varphi^{*}(\tau)=\frac{g(\tau)}{X_1^{+}(\tau)}. \tag{4}$$

It is easy to show that the function $(\tau-z_0)^{-\mu_1}G(\tau)$ is continuous at the point t_1 and so it satisfies Holder condition on the contour L.

Really,

$$\frac{(t_1-0-z_0)^{-\mu_1}G(t_1-0)}{(t_1+0-z_0)^{-\mu_1}G(t_1+0)}=\frac{(t_1+0-z_0)^{-\mu_1}}{(t_1+0-z_0)^{-\mu_1}}e^{-2\pi\mu_1 i}e^{2\pi i\mu_1}=1.$$

Then from the results of F.D.Gakhov [1] it follows that the function $X_1^{\pm}(z)$ satisfies Holder condition on the contour L.

So, the function $\varphi^{*}(\tau)$ satisfies on $L\backslash\{t_1\}$ Holder condition as ratio of two functions satisfying Holder condition. The function $\Psi(z)$ can be represented in the form

$$\Psi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi^{*}(\tau)\,d\tau}{(\tau-t_1)^{\mu_1}(\tau-z)}. \tag{5}$$

Denote by $\left[(z-t_1)^{-\mu_1}\right]^{*}$ the branch of the analytical functions $(z-t_1)^{-\mu_1}$ on the plane cut along the line t_1b_1 (fig.2) which takes the value $(t-t_1)^{-\mu_1}$ on the left hand side t_1b_1.Then

$$\left[(t-t_1)^{-\mu_1}\right]^{*}\equiv(t-t_1)^{-\mu_1}\,,\quad t\in a_1t_1$$

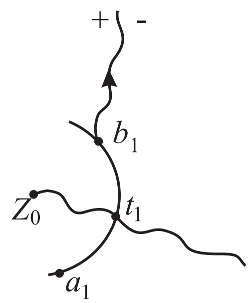

Fig. 2.

Using the results of book [1, p.74] we can represent the function $\Psi(t)$ near the point $z = t_1$ in the form

$$\Psi(t) = \left[\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0) - \frac{ctg\mu_1\pi}{2i}\varphi^*(t_1-0)\right]\frac{1}{[(t-t_1)^{\mu_1}]^*} +$$

$$+\Phi_1(t), \quad t \in a_1t_1, \tag{6}$$

$$\Psi(t) = \left[\frac{ctg\mu_1\pi}{2i}\varphi^*(t_1+0) - \frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right]\frac{1}{[(t-t_1)^{\mu_1}]^*} +$$

$$+\Phi_1(t), \quad t \in t_1b_1, (7) \tag{7}$$

where $\left[(t-t_1)^{-\mu_1}\right]^*$ is a limit of the function $\left[(z-t_1)^{-\mu_1}\right]^*$ at z tending to t from the left of the curve a_1t_1 and

$$|\Phi_1(t)| \leq \frac{c}{|t-t_1|^{\rho_1}}, \quad (\rho_1 < \operatorname{Re}\mu_1). \tag{8}$$

Applying Sokhotsky-Plemel formulae, from the representation we'll get

$$\Phi^+(t) = (t-t_1)^{\mu_1} X_1^+(t)\Psi^+(t) = (t-t_1)^{\mu_1} X_1^+(t) \times$$

$$\times\left[\tfrac{1}{2}\varphi^*(t)(t-t_1)^{-\mu_1} + \tfrac{1}{2\pi i}\int_L \tfrac{\varphi^*(\tau)d\tau}{(\tau-t_1)^{\mu_1}(\tau-t)}\right] = (t-t_1)^{\mu_1} X_1^+(t) \times$$

$$\times\left[\tfrac{1}{2}\varphi^*(t)(t-t_1)^{-\mu_1} + \left(\tfrac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0) - \tfrac{ctg\mu_1\pi}{2i}\varphi^*(t_1-0)\right)\times\right.$$

$$\left.\times\left[(t-t_1)^{-\mu_1}\right]^* + \Phi_1(t)\right] = X_1^+(t)\left[\tfrac{1}{2}\varphi^*(t) +\right.$$

$$\left.+\tfrac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0) - \tfrac{ctg\mu_1\pi}{2i}\varphi^*(t_1-0) + (t-t_1)^{\mu_1}\Phi_1(t)\right], \quad t \in a_1t_1 .$$

Passing to the limit at $t \to t_1 - 0$ and allowing for the relation (8) we have:

$$\Phi^+(t_1-0) = X_1^+(t_1)\left[\frac{1}{2}\varphi^*(t_1-0) + \frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0) - \frac{ctg\mu_1\pi}{2i}\varphi^*(t_1-0)\right] =$$

$$= X_1^+(t_1)\left[\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0) - \frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right].$$

If $t \in t_1 b_1$ then

$$\Phi^+(t) = (t-t_1)^{\mu_1} X_1^+(t)\left[\frac{1}{2}\varphi^*(t)(t-t_1)^{-\mu_1} + \right.$$

$$\left. + \left(\frac{ctg\mu_1\pi}{2i}\varphi^*(t_1+0) - \frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right)\left[(t-t_1)^{-\mu_1}\right]^* + \Phi_1(t)\right] =$$

$$= X_1^+(t)\left[\frac{1}{2}\varphi^*(t) + \frac{ctg\mu_1\pi}{2i}\varphi^*(t_1+0) - \frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0) + \Phi_1(t)(t-t_1)^{\mu_1}\right].$$

Passing again to the limit at $t \to t_1 + 0$ and allowing for (8)we have:

$$\Phi^+(t_1+0) = X_1^+(t_1)\left[\frac{1}{2}\varphi^*(t_1+0) + \frac{ctg\mu_1\pi}{2i}\varphi^*(t_1+0) - \frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right] =$$

$$= X_1^+(t_1)\left[\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0) - \frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right].$$

It is easy to note that $\Phi^+(t_1+0) = \Phi^+(t_1-0)$. And now let's investigate the boundary property of the function $\Phi^-(z)$. If apply Sokhotsky-Plemel formula analogously we'll obtain:

$$\Phi^-(t) = \left(\frac{t-t_1}{t-z_0}\right)^{\mu_1} X_1^-(t)\left[-\frac{1}{2}\varphi^*(t)(t-t_1)^{-\mu_1} + \frac{1}{2\pi i}\int_L \frac{\varphi^*(\tau)\,d\tau}{(\tau-t_1)^{\mu_1}(\tau-t)}\right].$$

Further subject to the relations (6), (7) and (8) in this expression

$$\Phi^-(t) = \left(\frac{t-t_1}{t-z_0}\right)^{\mu_1} X_1^-(t)\left[-\frac{1}{2}\varphi^*(t)(t-t_1)^{-\mu_1} + \right.$$

$$\left. + \left(\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0) - \frac{ctg\mu_1\pi}{2i}\varphi^*(t_1-0)\right)\left[(t-t_1)^{-\mu_1}\right]^* + \Phi_1(t)\right] =$$

$$= (t-z_0)^{-\mu_1} X_1^-(t) \times$$

$$\times\left[-\frac{1}{2}\varphi^*(t)+\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0)-\frac{ctg\mu_1\pi}{2i}\varphi^*(t_1-0)+\Phi_1(t)(t-t_1)^{\mu_1}\right]\ ,\ t\in a_1t_1\ .$$

Passing to the limit at $t\to t_1-0$:

$$\Phi^-(t_1-0)=(t_1-0-z_0)^{-\mu_1}X_1^-(t_1)\left[-\frac{1}{2}\varphi^*(t_1-0)+\right.$$

$$\left.+\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0)-\frac{ctg\mu_1\pi}{2i}\varphi^*(t_1-0)\right]=(t_1-0-z_0)^{-\mu_1}\times$$

$$\times X_1^-(t_1)\left[\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0)-\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right]=$$

$$=\frac{e^{i\mu_1\pi}}{2i\sin\mu_1\pi}(t_1-0-z_0)^{-\mu_1}X_1^-(t_1)\left[\varphi^*(t_1+0)-\varphi^*(t_1-0)\right]\ .$$

From $t\in t_1b_1$ it follows that

$$\Phi^-(t)=\left(\frac{t-t_1}{t-z_0}\right)^{\mu_1}X_1^-(t)\left[-\frac{1}{2}\varphi^*(t)(t-t_1)^{\mu_1}+\right.$$

$$\left.+\left(\frac{ctg\mu_1\pi}{2i}\varphi^*(t_1+0)-\frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right)\left[(t-t_1)^{\mu_1}\right]^*+\Phi_1(t)\right]=$$

$$=(t-z_0)^{-\mu_1}X_1^-(t)\times$$

$$\times\left[-\frac{1}{2}\varphi^*(t)+\frac{ctg\mu_1\pi}{2i}\varphi^*(t_1+0)-\frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)+\Phi_1(t)(t-t_1)^{\mu_1}\right].$$

Analogously we have:

$$\Phi^-(t_1+0)=(t_1+0-z_0)^{-\mu_1}X_1^-(t_1)\left[-\frac{1}{2}\varphi^*(t_1+0)+\right.$$

$$\left.+\frac{ctg\mu_1\pi}{2i}\varphi^*(t_1+0)-\frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right]=$$

$$=(t_1+0-z_0)^{-\mu_1}X_1^-(t_1)\left[\frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1+0)-\right.$$

$$\left.-\frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}\varphi^*(t_1-0)\right]=\frac{e^{-i\mu_1\pi}}{2i\sin\mu_1\pi}(t_1+0-z_0)^{-\mu_1}X_1^-(t_1)\times$$

$$\times\left[\varphi^*(t_1+0)-\varphi^*(t_1-0)\right]\ .$$

It is evident that

$$(t_1 - 0 - z_0)^{-\mu_1} = (t_1 + 0 - z_0)^{-\mu_1} e^{-2\pi i \mu_1}$$

or

$$(t_1 + 0 - z_0)^{-\mu_1} = e^{2\pi i \mu_1} (t_1 - 0 - z_0)^{-\mu_1} \quad .$$

Then

$$\Phi^{-}(t_1 + 0) = \frac{e^{i\mu_1 \pi}}{2i \sin \mu_1 \pi} (t_1 - 0 - z_0)^{-\mu_1} X_1^{-}(t_1) \left[\varphi^{*}(t_1 + 0) - \varphi^{*}(t_1 - 0)\right].$$

Hence, it follows that $\Phi^{-}(t_1 - 0) = \Phi^{-}(t_1 + 0)$.
The theorem is proved.

References

[1]. Gakhov F.D. *Boundary value problems.* Moscow, 1977. (Russian)

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