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COMPLETENESS OF SOME "DEGENERATING" EXPONENT SYSTEMS IN $\mathbf{L}_p(-\pi,\pi)$, $(1<p<\infty)$

Abstract

In the paper the "double" exponent system with "degenerating" coefficients is considered. At definite conditions on coefficients, the completeness of this system in space $\mathbf{L}_p$, $(1<p<\infty)$ is proved.

By the considering eigen-value problems of some discontinuous differential operators there appears the question on investigation of basic properties of "degenerating" exponent system of the form

$$\left\{A^{+}(t)\cdot\omega^{+}(t)\,e^{int};\quad A^{-}(t)\cdot\omega^{-}(t)\,e^{-i(n+1)t}\right\}_{n\geq 0} \tag{1}$$

in $\mathbf{L}_p(-\pi,\pi)$, $1<p<\infty$, where $A^{\pm}(t)$, $\omega^{\pm}(t)$ are definite functions on the segment $[-\pi,\pi]$ with respect to which we shall assume the fulfillment of the following conditions. Let $\left\{\tau_i^{\pm}\right\}_{i=1}^{e^{\pm}}$ be a set of points of semi-interval $(-\pi,\pi]$ moreover $\left\{\tau_i^{+}\right\}\cap\left\{\tau_i^{-}\right\}$. $\left\{\beta_i^{\pm}\right\}_{i=1}^{e^{\pm}}\subset\mathbb{R}$ is a set of real numbers. The functions $\omega^{\pm}(t)$ have the form

$$\omega^{\pm}(t)\equiv\prod_{i=1}^{e^{\pm}}\left\{\sin\left|\frac{t-\tau_i^{\pm}}{2}\right|\right\}^{\beta_i^{\pm}}. \tag{2}$$

$A^{\pm}(t)\equiv\left|A^{\pm}(t)\right|^{i\alpha^{\pm}(t)}$ are complex-valued functions on the segment $[-\pi,\pi]$ which satisfy the following conditions:

1) $\alpha^{\pm}(t)$ are piecewise Holder functions on the segment $[-\pi,\pi]$, $\{S_i\}_1^r$ is a set of breaking points of the functions $\theta(t)\equiv\alpha^{+}(t)-\alpha^{-}(t)$ on $(-\pi,\pi)$ what is more $\left\{\tau_i^{+};\tau_i^{-}\right\}\cap\{S_i\}_1^r=\{\varnothing\}$.

2) $\left|A^{\pm}(t)\right|$ are measurable functions on $(-\pi,\pi)$ moreover they satisfy the conditions

$$\sup_{(-\pi,\pi)} vrai\ \left\{\left|A^{\pm}(t)\right|\right\}^{\pm 1}\leq M<+\infty.$$

Denote by $\{h_i\}_1^r$ the jumps of the function $\theta(t)$, i.e.,

$$h_i=\theta(S_i+0)-\theta(S_i-0),\quad i=\overline{1,r}.$$

Let $h_0=\theta(-\pi)-\theta(\pi)$.

The following conditions are fulfilled

$$-\frac{2}{p}<\beta_i^{\pm}<\frac{2}{q},\ -\frac{2\pi}{p}<h_k<\frac{2\pi}{q} \tag{3}$$

$$i=\overline{1,e^{\pm}};\quad k=\overline{0,r}.$$

The following is true.

Theorem. *Let the complex-valued functions* $A^{\pm}(t)$ *satisfy conditions 1), 2); the* functions $\omega^{\pm}(t)$ *be determined by formulae (2). If inequalities (3) hold, then system (1) is complete in* $\mathbf{L}_p(-\pi,\pi)$, $(1<p<\infty)$.

The proof of this theorem is based on the investigation of some conjugation problem of theory of analytical functions. At first we shall give necessary notations and definitions.

We shall denote by H_1^+ and H_1^- the ordinary Hardy classes inside and outside of unique circle (functions from H_1^- have finite orders at infinity). Let $\nu(t)$ be some measurable non-negative function on $(-\pi,\pi)$. Denote by $H_{p,v}^{\pm}$ the following classes of analytical functions

$$H_{p,v}^{\pm} \overset{def}{\equiv} \left\{ f\in H_1^{\pm} : \int_{-\pi}^{\pi} \left| f^{\pm}\left(e^{it}\right)\right|^p v(t)\,dt<+\infty \right\},$$

where $f^+(\tau)$ and $f^-(\tau)$ are non-tangent boundary values of the function $f(z)$ inside and outside of unique circle.

Consider the following conjugation problem in the classes

$$F^+(\tau)+D(\tau)\cdot F^-(\tau)=g(\tau),\quad |\tau|=1, \tag{4}$$

where $D(\tau)$ is a coefficient, $g(\tau)\in L_p$ is an absolute term. Under the solution of this problem it is understood any pair of the functions $\{F^+(z);\ F^-(z)\}$, $F^{\pm}(z)\in H_{p,v}^{+}$, whose non-tangent boundary values $F^{\pm}(\tau)$ satisfy almost everywhere equation (4). Let us take the following notation

$$v^{\pm}(t)=\left[\omega^{\pm}(t)\right]^{-q}, \tag{5}$$

where here and further q is a conjugate with p number $\frac{1}{p}+\frac{1}{q}=1$.

Using the results of I.I. Privalov monograph [1] and one theorem of V.I. Smirnov [2, p.51] we construct the following lemma which play a vital part at proving the theorem.

Lemma 1. *Let the functions* $A^{\pm}(t)$ *satisfy conditions 1), 2)*; $v^{\pm}(t)$ *be determined from (15). If* $\beta_i^{\pm}>-\frac{1}{p}$, $i=\overline{1,e^{\pm}}$, *then system (1) is complete in* $\mathbf{L}_p(-\pi,\pi)$, $(1<p<\infty)$ *iff homogeneous conjugation problem*

$$\begin{cases} F^+(\tau)-G(\tau)F^-(\tau)=0, & |\tau|=1 \\ F(\infty)=0 \end{cases} \tag{6}$$

has only trivial solution in the classes $H_{q,v^{\pm}}^{\pm}$ *where* $G(\tau)$ *is determined by the formula*

$$G\left(e^{it}\right)=\frac{A^+(t)\cdot\omega^+(t)}{A^-(t)\cdot\omega^-(t)}. \tag{7}$$

So, the question on completeness of the system (1) in L_p is reduced to the investigation of the conjugation problem (6) in the classes $H_{q,v^{\pm}}^{\pm}$. We shall investigate problem (6) by the method worked out in I.I.Danilyuk's monograph [2]. Introduce the notation

$$X_1^{\pm}(z)=\exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{\omega^+(t)}{\omega^-(t)}\cdot\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2^{\pm}(z) = \exp\left\{ \pm \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \left| \frac{A^+(t)}{A^-(t)} \right| \cdot \frac{e^{it}+z}{e^{it}-z} dt \right\},$$

$$X_3^{\pm}(z) = \exp\left\{ \pm \frac{1}{4\pi} \int_{-\pi}^{\pi} \theta(t) \cdot \frac{e^{it}+z}{e^{it}-z} dt \right\}.$$

Now introduce the following functions

$$Z_i(z) = \begin{cases} X_i^+(z), & |z| < 1, \\ \left[X_i^-(z)\right]^{-1}, & |z| > 1. \end{cases}$$

Using Sokhotsky-Plemel formula we obtain the relations

$$\frac{\omega^+(t)}{\omega^-(t)} = Z_1^+\left(e^{it}\right) \cdot \left[Z_1^-\left(e^{it}\right)\right]^{-1},$$

$$\left| \frac{A^+(t)}{A^-(t)} \right| = \frac{Z_2^+\left(e^{it}\right)}{Z_2^-\left(e^{it}\right)},$$

$$\exp i\theta(t) = \frac{Z_3^+\left(e^{it}\right)}{Z_3^-\left(e^{it}\right)}.$$

Thus almost everywhere it holds

$$\frac{A^+(t) \cdot \omega^+(t)}{A^-(t) \cdot \omega^-(t)} = \frac{Z^+\left(e^{it}\right)}{Z^-\left(e^{it}\right)}, \tag{8}$$

where $Z(z) = \prod_i Z_i(z)$. We shall call the function canonical solution of the problem (7).

Lemma 2. *Let conditions 1), 2) hold and inequalities (3) be fulfilled. Then the general solution of the conjugation problem*

$$F^+(\tau) - G(\tau) F^-(\tau) = 0, \quad |\tau| = 1 \tag{9}$$

in the classes $H^{\pm}_{q,v^{\pm}}$ which has the order $\leq m$ on the infinity is of the form

$$F(z) = Z(z) \cdot P_m(z),$$

where $Z(z)$ is a canonical solution of the homogeneous problem (8), $P_m(z)$ is an arbitrary polynomial of order $\leq m$.

Proof. Taking into account that $G(\tau)$ is determined by formula (7) and subject to expression (8) in (9) we have

$$\frac{F^+(\tau)}{Z^+(\tau)} = \frac{F^-(\tau)}{Z^-(\tau)}, \quad |\tau| = 1. \tag{10}$$

Denote $\Phi(z) = \frac{F(z)}{Z(z)}$. Since $Z(z)$ has no zeros and poles for $|z| \neq 1$ then $\Phi(z)$ and $F(z)$ have the same orders at infinity. Show that at fulfilment of conditions of lemma 2 $\Phi(z) \in H_1^{\pm}$. It is evident that $F(z) \in H_1^{\pm}$ and what is more as it follows from the results of the paper [2, p.74] $Z(z) \in H_\delta^{\pm}$ at sufficiently small δ.

If at that $\frac{F^{\pm}(\tau)}{Z^{\pm}(\tau)} \in L_1$, then by Smirnov's known theorem $\Phi(z) \in H_1^{\pm}$.

By requirement of the problem $\frac{F^{-}(e^{it})}{\omega^{-}(\tau)} \in L_q$. Therefore in order $\frac{F^{-}(\tau)}{Z^{-}(\tau)} \in L_1$ that $\frac{\omega^{-}(t)}{Z^{-}(e^{it})} \in L_p$ is sufficient. So, we have $Z^{-}(e^{it})$. By Sokhotsky-Plemel formulae we have

$$X_1^{-}(e^{it}) = \exp\frac{1}{2}\ln\frac{\omega^{+}(t)}{\omega^{-}(t)}\exp\left\{-\frac{i}{4\pi}\int_{-\pi}^{\pi}\ln\frac{\omega^{+}(\tau)}{\omega^{-}(\tau)}\cdot ctg\frac{t-\tau}{2}d\tau\right\},$$

$$X_2^{-}(z) = \exp\frac{1}{2}\ln\left|\frac{A^{+}(t)}{A^{-}(t)}\right|\exp\left\{-\frac{i}{4\pi}\int_{-\pi}^{\pi}\ln\left|\frac{A^{+}(\tau)}{A^{-}(\tau)}\right|\cdot ctg\frac{t-\tau}{2}d\tau\right\},$$

$$X_3^{-}(z) = \exp\frac{i}{2}\theta(t)\cdot\exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta(\tau)\cdot ctg\frac{t-\tau}{2}d\tau\right\}.$$

From these representations it follows immediately

$$\left|Z_1^{-}(e^{it})\right| = \left[\frac{\omega^{-}(t)}{\omega^{+}(t)}\right]^{\frac{1}{2}}$$

$$\sup_{(-\pi,\pi)} vrai\left\{\left|Z_2^{-}(e^{it})\right|^{\pm 1}\right\} \le C < +\infty.$$

Further represent the function $\theta(t)$ in the form $\theta(t) = \theta_0(t) + \theta_1(t)$, where $\theta_0(t)$ is a continuous part, $\theta_1(t)$ is a jump function which is determined by the formula

$$\theta_1(-\pi) = 0,\ \theta_1(t) = \sum_{-\pi<s_k<t} h_{k+}\left[\theta(t) - \theta(t-0)\right]$$

at $-\pi < t < \pi$.

Following the paper [2] we shall determine the values $h_0^{(1)}, h_0^{(0)}$: $h_0^{(1)} = \theta_1(-\pi) - \theta_1(\pi)$, $h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi)$. It is easy to note that

$$h_0 = h_0^{(1)} - h_0^{(0)} = \theta(-\pi+0) - \theta(\pi-0).$$

Denote

$$U^{\pm}(t) = \prod_k\left\{\sin\left|\frac{t-s_k^{\pm}}{2}\right|\right\}^{\frac{h_k^{\pm}}{2\pi}}.$$

Then by the results of the paper [2] for $Z_3^{-}(e^{it})$ we obtain the representation

$$\left|Z_3^{-}(e^{it})\right| = U_0(t)\cdot\left[U^{+}(t)\right]^{-1}\cdot U^{-}(t)\cdot\left\{\sin\left|\frac{\sigma-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}},$$

where

$$U_0(t) \equiv \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}}\cdot\exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta(\tau)\cdot ctg\frac{t-\tau}{2}d\tau\right\}.$$

As it is shown in the paper [2, p.79]

$$|U_0(t)|^{\pm 1} \le const \ \ on\,[-\pi,\pi]\,.$$

It is easy to note that

$$\left|\frac{\omega^-(t)}{Z_1^-(e^{it})}\right| = \left[\omega^-(t)\cdot\omega^+(t)\right]^{\frac{1}{2}}.$$

Consequently, as a result we obtain

$$\left|\frac{\omega^-(t)}{z^-(e^{it})}\right| = [\omega^-(t)\cdot\omega^+(t)]^{\frac{1}{2}}\cdot\left|Z_2^-(e^{it})\right|^{-1}\cdot\left|Z_3^-(e^{it})\right| = |U_0(t)|^{-1}\cdot\left|Z_2(e^{it})\right|^{-1}\cdot$$

$$\cdot\left|U^+(t)\right|\cdot\left|U^-(t)\right|\cdot\left\{\sin\left|\tfrac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}}\cdot[\omega^-(t)\cdot\omega^+(t)]^{\frac{1}{2}}.$$

From this representation it follows that at fulfilment the inequalities $\beta_i^{\pm} > -\frac{2}{p}$; $i=\overline{1,e^{\pm}}$; $\frac{h_k^-}{2\pi} < \frac{1}{p}$; $\frac{h_0}{2\pi} > -\frac{1}{p}$ the function $\left|\frac{\omega^-(t)}{z^-(e^{it})}\right|$ belongs to the space $L_p(-\pi,\pi)$. As a result, by Smirnov theorem the function $\Phi(z)$ belongs to the class $H_i^{\pm}$. Then from (10) it follows that $\Phi(z)$ is a polynomial of order $\le m$, i.e., $\Phi(z) = P_m(z)$ and as a result

$$F(z) = Z(z)\cdot P_m(z)\,. \tag{11}$$

It remains to prove that $F(z)$ is a solution of the problem (9), i.e., $F(z)\in H^{\pm}_{q,v^{\pm}}$. We have

$$\left|\frac{z^-(e^{it})}{\omega^-(t)}\right| = |U_0(t)|^{-1}\cdot\left|Z_2(e^{it})\right|\cdot\left|U^+(t)\right|^{-1}\cdot\left|U^-(t)\right|\cdot\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{\frac{h_0}{2\pi}}\cdot$$

$$\cdot\left[\omega^-(t)\,\omega^+(t)\right]^{-\frac{1}{2}}.$$

It is clear that for $\frac{h_k^+}{2\pi} < \frac{1}{q}$, $\frac{h_0}{2\pi} < \frac{1}{q}, \beta_i^{\pm} < \frac{2}{q}$, $i=\overline{1,e^{\pm}}$; the relation $\left|\frac{z^-}{\omega^-}\right|$ belongs to the space $L_q(-\pi,\pi)$. Then, $F(z)\in H_1^{\pm}$ is evident.

Thus the constructed function $F(z)$ is a general solution of problem (9) in the classes $H^{\pm}_{q,v^{\pm}}$.

The lemma is proved.

Now it is easy to note that the proof of the theorem immediately follows from lemmas 1,2. In fact if conditions 1), 2) are fulfilled, then the completeness of the system (1) in L_p is equivalent to the trivial solvability of the homogeneous problem (6) in the classes $H^{\pm}_{q,v^{\pm}}$. If it holds inequality (3), then as it follows from lemma 2, the solution of problem (6) can be represented in the form (11). Then from $F(\infty)=0$ it follows that $P_m(z)\equiv 0$, so $F(z)\equiv 0$. Consequently, by lemma 1 system (1) is complete in L_p.

The theorem is proved.

The author expresses his gratitude to prof. S.S.Mirzoyev for his attention to the work.

References

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Received October 1, 2002; Revised May 28, 2003.
Translated by Mamedova V.I.