MECHANICS

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INVESTIGATION OF TWO-PHASE FILTERATION PROBLEM OF NONCOMPRESSIBLE VISCOUS FLUIDS IN POROUS MEDIUM

Abstract

In the present paper one-dimensional two-phase filteration problem of non-compressible viscous fluids in porous medium is considered. According to the law of conservation of mass of substance the mathematical model of the investigated process is considered. After definite mathematical transformations the system of linear differential equations of composite type (parabolic and elliptic) is reduced to the system of nonlinear equations of parabolic type.

The formula for determination of the field of pressures depending on saturation and for distribution of saturation having the theoretical and practical important, is obtained.

I.Introduction. Considering the displacement of one fluid by the other one in rectilinear seam, we select the element of seam with the length Δx, height h and width b in the direction perpendicular to the plane (Fig. 1.).

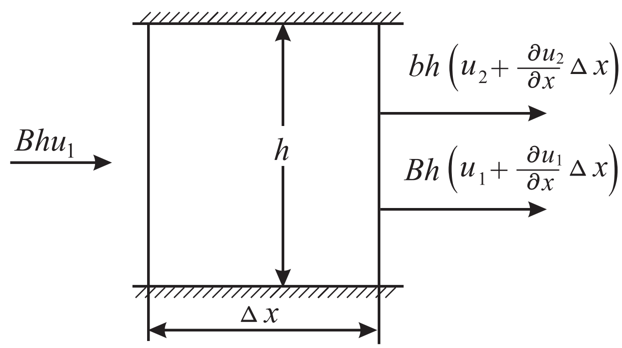

Fig. 1.

In the general case the fluids come in element of seam from the left and flow out from the right. At that the flow of the first fluid is equal to bhu_1 and from the right is $bh\left(u_1+\frac{\partial u_1}{\partial x}\Delta x\right)$. The amount of the first fluid in element of seam composes $bhm\frac{\partial \sigma}{\partial t}\Delta x$($u_1$ is filteration rate of the first fluid; σ is saturation of seam of the first fluid, m is a porosity). According to the law of conservation of mass of substance the difference between rates incoming to the element of seam of the first fluid and flowing from it, is equal to the rate of accumulation of volume of the first fluid in element of seam. Expressing the said one in mathematical form we obtain

$$-bh\left(u_1+\frac{\partial u_1}{\partial x}\Delta x\right)+bhu_1=bhm\frac{\partial \sigma}{\partial t}\Delta x.$$

After cancellation of corresponding terms as $\Delta x \to \infty$ we have

$$\frac{\partial u_1}{\partial x} + m\frac{\partial \sigma}{\partial t} = 0. \tag{1}$$

Since only two fluids (two-phase filtration) are contained in porous medium, then the saturation of porous medium of the second fluid is $\sigma_2 = 1 - \sigma$. Considering analogously to the previous one, the penetration rate of the second fluid in element of seam and its outflow we obtain

$$\frac{\partial u_2}{\partial x} - m\frac{\partial \sigma}{\partial t} = 0. \tag{2}$$

Summing up the equations (1) and (2) we have

$$\frac{\partial}{\partial x}(u_1 + u_2) = 0; \quad u_1 + u_2 = u(t). \tag{3}$$

Thus, total filtration rate of the both fluids do not change on coordinate x which sholud be expected, since the fluids are noncompressible.

Consequently, seam condition is rigid pressure-water. The filtration rates of the both fluids obey the generalized Darcy law such that

$$u_1 = -\frac{k_1(\sigma)}{\mu_1}\frac{\partial P}{\partial x}; \quad u_2 = -\frac{k_2(\sigma)}{\mu_2 \partial x}\frac{\partial P}{}, \tag{4}$$

where $k_1(\sigma), k_2(\sigma), \mu_1, \mu_2$ are respective fluid penetrability and viscosity.

Thus we obtain the following mathematical problem of joint filtration of non-homogeneous noncompressible fluids in porous medium.

II. Statement of the problem. Modelling of onedimenisional filtration of two-phase immiscible and noncompressible fluids in homogeneous porous medium is reduced to the following statement of differential problem in which the determination of distribution of saturation and determination of filed of pressures are reduced to the integration of nonlinear partial differential equations. Taking into account that the porous medium is also noncompressible and consequently, $m = const$, then for simplification, we accept $m = 1$

$$\frac{\partial \sigma}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_1(\sigma)}{\mu_1}\frac{\partial P}{\partial x}\right]; \quad \sigma \in [0,1]; \ x \in [0,1], \ t \geq 0, \tag{5}$$

$$\frac{\partial}{\partial x}\left\{\left[\frac{k_1(\sigma)}{\mu_1} + \frac{k_2(\sigma)}{\mu_2}\right]\frac{\partial P}{\partial x}\right\} = 0 \tag{6}$$

with the boundary and initial conditions

$$\sigma(x,t)|_{x=0} = 1; \quad P(\sigma,t)|_{\sigma=1} = P(1,t), \tag{7}$$

$$\sigma(x,t)|_{x=1} = 0; \quad P(\sigma,t)|_{\sigma=0} = P(0,t), \tag{8}$$

$$\sigma(x,t)|_{t=0} = \psi(x); \quad P(\sigma,t)|_{t=0} = P(1,t) - [P(1,t) - P(0,t)]. \tag{9}$$

Denoting

$$f(\sigma,\mu_1,\mu_2)=\frac{k_1(\sigma)}{\mu_1}+\frac{k_2(\sigma)}{\mu_2} \qquad (10)$$

we have from (6)

$$f(\sigma,\mu_1,\mu_2)\frac{\partial P}{\partial x}=u(t). \qquad (11)$$

Taking into account that $k_1(0)=0,\ k_2(0)=1,\ k(1)=1,\ k_2(1)=0$. We have from (10)

$$f(0,\mu_1,\mu_2)=\frac{1}{\mu_2},\quad f(1,\mu_1,\mu_2)=\frac{1}{\mu_1}. \qquad (12)$$

Subject to (12) in (11) we obtain

$$\left.\frac{\partial P}{\partial x}\right|_{\sigma=0}=\mu_2 u(t), \qquad (13)$$

$$\left.\frac{\partial P}{\partial x}\right|_{\sigma=1}=\mu_1 u(t). \qquad (14)$$

After integration of (11) we have

$$P(\sigma,t)=\frac{u(t)\,x}{f(\sigma,\mu_1,\mu_2)}+c(t). \qquad (15)$$

Allowing for (12) in (15) we obtain the following relations

$$P(0,t)=\mu_2 u(t)\,x\ +c(t), \qquad (16)$$

$$P(1,t)=\mu_1 u(t)\,x\ +c(t), \qquad (17)$$

whence we have

$$u(t)\,x=\frac{P(1,t)-P(0,t)}{\mu_2-\mu_1}, \qquad (18)$$

$$c(t)=P(0,t)-\frac{\mu_2}{\mu_1-\mu_2}\left[P(1,t)-P(0,t)\right]=P(1,t)-$$

$$-\frac{\mu_1}{\mu_1-\mu_2}\left[P(1,t)-P(0,t)\right]. \qquad (19)$$

Allowing for (18) and (19) in (15) after simple transformations we obtain the following expression for $P(\sigma,t)$

$$P(\sigma,t)=P(0,t)+\left[\tfrac{1}{f(\sigma,\mu_1,\mu_2)}-\mu_2\right]\tfrac{P(1,t)-P(0,t)}{\mu_1-\mu_2}=$$
$$=P(1,t)+\left[\tfrac{1}{f(\sigma,\mu_1,\mu_2)}-\mu_2\right]\tfrac{P(1,t)-P(0,t)}{\mu_1-\mu_2}. \tag{20}$$

Denoting

$$\varphi_i(\sigma,\mu_0)=\frac{f^{-1}(\sigma,\mu_1,\mu_2)-\mu_i}{\mu_1-\mu_2},\ \ \mu_0=\frac{\mu_1}{\mu_2},\ \ i=1,2. \tag{21}$$

Since at $\sigma=1,\ i=1$ and at $\sigma=0,\ i=2$. Consequently, we obtain

$$P(\sigma,t)=P(1,t)+\varphi_1(\sigma,\mu_0)\left[P(1,t)-P(0,t)\right] \tag{22}$$

$$P(\sigma,t)=P(0,t)+\varphi_2(\sigma,\mu_0)\left[P(1,t)-P(0,t)\right] \tag{23}$$

Subject to (22),(23) and denoting $P(0,t)=P(2,t)$ we have

$$P(\sigma,t)=P(i,t)+\varphi_i(\sigma,\mu_0)\left[P(1,t)-P(0,t)\right];\quad i=1,2. \tag{24}$$

Consequently, we obtain

$$\frac{\partial P}{\partial x}=\frac{\partial P}{\partial \sigma}\frac{\partial \sigma}{\partial x}=-\left[P(1,t)-P(0,t)\right]\varphi_i'(\sigma,\mu_0)\frac{\partial \sigma}{\partial x}. \tag{25}$$

Allowing for (25) in (5) we obtain the following boundary value problem for the determination of $\sigma(x,t)$

$$\frac{\partial \sigma}{\partial t}=-\frac{\left[P(1,t)-P(0,t)\right]}{\mu_1}\frac{\partial}{\partial x}\left[k_1(\sigma)\varphi_i'(\sigma,\mu_0)\frac{\partial \sigma}{\partial x}\right] \tag{26}$$

$$\left.\sigma(x,t)\right|_{x=0}=1;\qquad \left.\sigma(x,t)\right|_{x=1}=0, \tag{27}$$

$$\left.\sigma(x,t)\right|_{t=0}=\psi(x). \tag{28}$$

Determining from the relations (24) $\varphi_i(\sigma,\mu_0)$ we have

$$\varphi_i(\sigma,\mu_0)=\frac{P(\sigma,t)-P(i,t)}{P(1,t)-P(0,t)}, \tag{29}$$

consequently, we have

$$\frac{\partial \varphi_i}{\partial t}=\frac{\partial \varphi_i}{\partial \sigma}\cdot\frac{\partial \sigma}{\partial t}=\frac{1}{P(1,t)-P(0,t)}\frac{\partial P(\sigma,t)}{\partial \sigma}\cdot\frac{\partial \sigma}{\partial t}=$$

$$=\frac{1}{P(1,t)-P(0,t)}\frac{\partial P(\sigma,t)}{\partial t}$$

or

$$\varphi_i(\sigma,\mu_0)\frac{\partial\sigma}{\partial t}=\frac{1}{P(1,t)-P(0,t)}\frac{\partial P(\sigma,t)}{\partial t}. \tag{30}$$

Allowing for (30) in (5) we obtain the following boundary value problem for the determination of the pressure $P(\sigma,t)$

$$\frac{\partial P(x,t)}{\partial t}=\frac{[P(1,t)-P(0,t)]}{\mu_1}\varphi_i'(\sigma,\mu_0)\frac{\partial}{\partial x}\left[k_1(\sigma)\frac{\partial P(x,t)}{\partial x}\right], \tag{31}$$

$$P(x,t)|_{x=0}=P(1,t), \quad P(x,t)|_{x=1}=P(0,t), \tag{32}$$

$$P(x,t)|_{t=0}=P(1,t)-[P(1,t)-P(0,t)]\,x. \tag{33}$$

If we join the problems (26)-(28) and (31)-(31), we obtain the mathematical model of onedimensional two-phase nonstationary filtration of nonhomogeneous incompressible fluid in homogeneous porous medium

$$\frac{\partial\sigma}{\partial t}=-\frac{[P(1,t)-P(0,t)]}{\mu_1}\frac{\partial}{\partial x}\left[k_1(\sigma)\,\varphi_i'(\sigma,\mu_0)\frac{\partial\sigma}{\partial x}\right], \tag{34}$$

$$\frac{\partial P(x,t)}{\partial t}=\frac{[P(1,t)-P(0,t)]}{\mu_1}\varphi_i'(\sigma,\mu_0)\frac{\partial}{\partial x}\left[k_1(\sigma)\frac{\partial P(x,t)}{\partial x}\right], \tag{35}$$

$$\sigma(x,t)|_{x=0}=1; \quad P(x,t)|_{x=0}=P(1,t), \tag{36}$$

$$\sigma(x,t)|_{x=1}=0; \quad P(x,t)|_{x=1}=P(0,t), \tag{37}$$

$$\sigma(x,t)|_{t=0}=\psi(x); \quad P(x,t)|_{t=0}=P(1,t)-[P(1,t)-P(0,t)]\,x.$$

Thus we have proved the following property of solution of the equation (35) and we have constructed the mathematical model of the stated problem.

Property. *If the function $P(x,t)$ is determined and continuous in the domain $x\in[0,1]$, $t\in(0,\infty)$ and satisfies the equation (5),(6) at the points of the domain $x\in[0,1]$, $t>0$, then it is a composite function on x, since $P(x,t)=P(\sigma(x,t),t)$.*

III. Investigation of solution of problem. As it is obvious the boundary value problem (5)-(9) modelling the displacement process of one fluid by the other one in porous medium is a nonlinear problem.

The one of effective and universal numerical methods for solution of such problems is the method of finite differences.

However the question of applicability of net method for the solution of system of equations of composite type is not enough investigated. Starting from these we reduced the boundary value problem (5) to the boundary value problem (34)-(37) by above mentioned mathematical transformations.

For these nonlinear problems the proof of stability and consequently, of convergence of difference schemes is associated with great mathematical difficulties. Reason of this are principal difficulties both of theoretical and practical character in this direction.

We investigate one partial solution of the problem (34)-(37) by simplifying it.

Such simplification must admit us to obtain the analytical expression for the corresponding problem which admits by construction and numerical solution to compare the difference scheme of obtained numerical results with already known exact solutions of initial problems with identical initial-boundary conditions.

At the first approximation we assume that the dependence of pressure on saturation and distribution of saturation of the investigated sample of the first fluid along the sample is a linear function of saturation and coordinate X respectively (Fig.2).

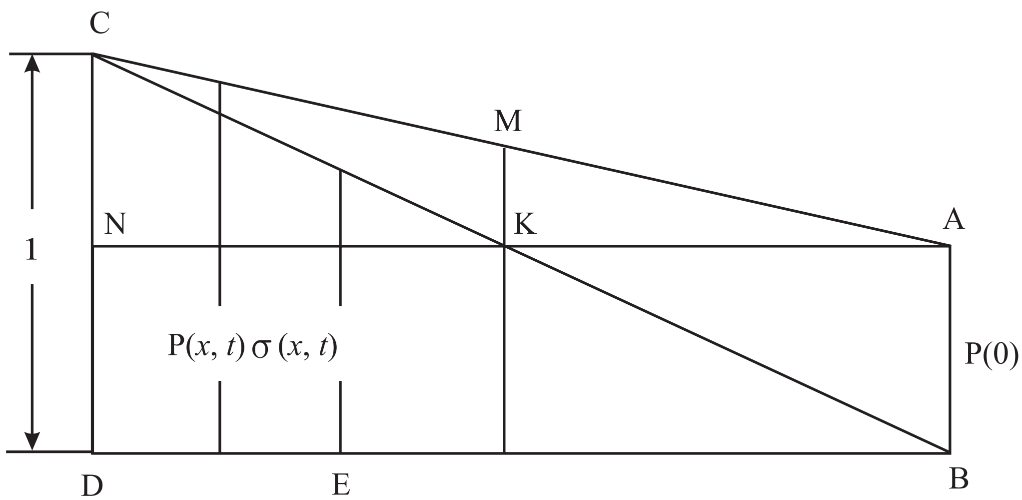

Fig. 2.

Assume that the line AC corresponds to pressure change and line CB corresponds to distribution of saturation. As it is obvious from Fig.2

$$\frac{MK}{CN} = \frac{AK}{AN} = \frac{BE}{BD} = \frac{KE}{CD}$$

or

$$\frac{MK}{CN} = \frac{KE}{CD}$$

consequently, we have

$$\frac{P(\sigma,t) - P(0)}{1 - P(0)} = \sigma. \tag{38}$$

Note that in the considered case it is accepted that

$$P(\sigma,t) = \frac{P^*(\sigma,t)}{P(1)}; \quad P(0,t) = \frac{P^*(0)}{P(1)}$$

then we obtain from (38) that

$$P(\sigma,t) = p(0) + [1 - P(0)]\,\sigma. \tag{39}$$

It is known that in the absence thereof mutual breaking of fluid we may accept [3] that

$$K_1(\sigma) = \sigma. \tag{40}$$

Comparing (39) with (20) or (24) we obtain

$$\varphi_1(\sigma,\mu_0) = \sigma. \tag{41}$$

Assume that we may represent the distribution function of saturation along the investigated sample in the following form:

$$\sigma(x,t) = a(t)\,x + b(t) \tag{42}$$

where $a(t)$ and $b(t)$ are indeterminate functions.

Then it is obvious that

$$\frac{\partial\sigma}{\partial t} = a'(t)\,x + b'(t)\,. \tag{43}$$

Allowing for (42) and (43) in (34) we obtain

$$a'(t)\,x + b'(t) = \frac{1-P(0)}{\mu_1}\frac{\partial}{\partial x}\left[a^2(t)\,x + a(t)\,b(t)\right]$$

or

$$a'(t)\,x + b'(t) = \frac{1-P(0)}{\mu_1}a^2(t)\,. \tag{44}$$

Comparing the both sides of the equation (44) we obtain the following system:

$$\left.\begin{array}{r} a'(t) = 0 \\ b'(t) = \frac{1-P(0)}{\mu_1}a^2(t) \end{array}\right\} \tag{45}$$

If we solve (45) with respect to $a(t)\,,\ b(t)$, then we have

$$\left.\begin{array}{r} a(t) = const = a \\ b(t) = \frac{1-P(0)}{\mu_1}a^2(t) + c \end{array}\right\}. \tag{46}$$

Allowing for (46) in (42) we have

$$\sigma(x,t) = ax + \frac{1-P(0)}{\mu_1}a^2(t) + c \tag{47}$$

For determination of the coefficients a and c we can use (9), i.e. the initial condition for $\sigma(x,t)$ in the following form

$$\left.\sigma(x,t)\right|_{t=0} = 1 - x. \tag{48}$$

Consequently, we have

$$\sigma(x,t)|_{t=0} = ax + c = 1 - x,$$

whence we have $a = -1,\ c = 1$. Consequently, we obtain that

$$\sigma(x,t) = 1 - x + \frac{1 - P(0)}{\mu_1} t. \tag{49}$$

Thus we obtain that the solution of the initial problem

$$\frac{\partial \sigma}{\partial t} = \frac{1-P(0)}{\mu_1} \frac{\partial}{\partial x} \left[\sigma \frac{\partial \sigma}{\partial x}\right]$$

$$\frac{\partial P}{\partial t} = \frac{1-P(0)}{\mu_1} \frac{\partial}{\partial x} \left[\sigma \frac{\partial p}{\partial x}\right]$$

$$\sigma(x,t)|_{t=0} = 1 - x;\ \ P(x,t)|_{t=0} = 1 - [1 - P(0)]\, x.$$

may be represented in the following form

$$\left.\begin{aligned} P(x,t) &= 1 - [1 - P(0)]\, x + \frac{[1-P(0)]^2}{\mu_1} t \\ \sigma(x,t) &= 1 - x + \frac{1-P(0)}{\mu_1} t \end{aligned}\right\}, \tag{50}$$

where

$$x, \sigma \in [0,1];\ \ t \geq 0.$$

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