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ON ONE APPROXIMATE METHOD OF SOLUTION OF BUCKLING PROBLEM NONLINEAR ELASTIC RING OF NONUNIFORM BY WIDTH

Abstract

Thin- walled rings having the nonuniformity property in combination with nonlinear elasticity property, are widely used as a bearing member in various constructions. The given paper is devoted to the theoretical investigation of buckling problem of such rings. The posed problem is solved by variational method of mixed type, and this problem is reduced to solving Cauchy problem for the first order nonlinear differential equation. It was brought to light that characterizing parameters of system exert on the value of buckling load.

Suppose that circular ring of radius R and width $2h$ consists of different by width n layers with different coefficients of elasticity E_{k+1} and proportional limits $\sigma^0_{k+1}, [k = 0, 1, 2, ..., (n-1)]$. We will assume that in each layer $E_{k+1} = E_{k+1}(z)$ and $\sigma^0_{k+1} = \sigma^0_{k+1}(z)$, where z is longitudinal coordinate.

Constitutive equations, in general, will be written in the form [1]:

$$\varepsilon^\nu = \frac{\sigma}{E_{k+1}}\left\{1 + \left[\frac{\sigma}{\sigma^0_{k+1}(z)}\right]^m\right\}, \quad a_k \le z \le a_{k+1}. \tag{1}$$

Here $a_k = -h + \sum\limits_{j=0}^{k}\delta_j$, δ_j is width of j-th layer, $\sum\limits_{k=1}^{n}\delta_k = 2h$, moreover, $\delta_0 = 0$ and m is nonlinearity index possessing positive values.

Consider buckling of the chosen ring under the action of compressive load q uniformly distributed on the surface. Introducing polar coordinates and taking into account the nonlinearity just of buckling ω, we will write the functional in the form ([1], (1.9)):

$$J = R\int_{-h}^{h}\int_0^{2\pi}\left\{\dot\sigma\dot\varepsilon + \frac{\sigma}{2R^2}\left[\left(\frac{\partial\dot\omega}{\partial\varphi}\right)^2 + \dot\omega^2\right]\right\}dzd\varphi -$$

$$-\frac{R}{2}\int_0^{2\pi}\sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}\dot\sigma\dot\varepsilon^\nu dzd\varphi + R\int_0^{2\pi}\dot\omega d\varphi. \tag{2}$$

Here we will denote by point the deferentiation with respect to q, i.e. $\dot q = 1$. Taking into account (1) and (2), functional J will be rewritten in the following form

$$J = R\int_{-h}^{h}\int_0^{2\pi}\left\{\dot\sigma\dot\varepsilon + \frac{\sigma}{2R^2}\left[\left(\frac{\partial\dot\omega}{\partial\varphi}\right)^2 + \dot\omega^2\right]\right\}dzd\varphi -$$

$$-\frac{R}{2}\int_0^{2\pi}\sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}\left\{\frac{\dot\sigma^2}{E_{k+1}}\left[1 + (m+1)\left(\frac{\sigma}{\sigma^0_{k+1}}\right)^m\right]\right\}dzd\varphi + R\int_0^{2\pi}\dot\omega d\varphi. \tag{3}$$

By the hypothesis of plane sections we write

$$\varepsilon = \varepsilon_0 - \chi z. \tag{4}$$

Here quantity ε_0 and curvature χ for the case, when the tangent motion can be disregarded, are defined by formulae of the theory of thin shells [2].

$$\varepsilon_0 = \frac{\omega}{R} + \frac{1}{2R^2}\left\{\left(\frac{\partial \omega}{\partial \varphi}\right)^2 + \omega^2\right\},\ \chi = \left(R^2\right)^{-1}\frac{\partial^2 \omega}{\partial \varphi^2}. \tag{5}$$

Putting these expressions into (4) and differentiating, we obtain

$$\dot{\varepsilon} = \frac{\dot{\omega}}{R} + \frac{1}{R^2}\left(\frac{\partial \omega}{\partial \varphi}\frac{\partial \dot{\omega}}{\partial \varphi} + \omega\dot{\omega}\right) - \frac{z}{R^2}\frac{\partial^2 \dot{\omega}}{\partial \varphi^2}. \tag{6}$$

By virtue of the thin- wall property we will accept the distribution law of tangential stress on width

$$\sigma = -\frac{qR}{2h} + \frac{3}{2h^3}Mz,\ \dot{\sigma} = -\frac{R}{2h} + \frac{3}{2h^3}\dot{M}z. \tag{7}$$

In order to find the stationary value of functional (3), we apply Rietz method. Approximation functions will be given in the form [3]

$$\omega = \omega_0(q) + \omega_1(q)\cos 2\varphi\ ;\ M = m(q)\cos 2\varphi. \tag{8}$$

Differentiating, we will write

$$\dot{\omega} = \dot{\omega}_0(q) + \dot{\omega}_1(q)\cos 2\varphi\ ;\ \dot{M} = \dot{m}(q)\cos 2\varphi. \tag{9}$$

We substitute relations (6)- (9) in the expression for the functional (3), and after integration we find it as function $\omega_0,\ \omega_1,\ m$ and derivatives of these quantities with respect to q. Further, equating

$$\frac{\partial J}{\partial \dot{\omega}_0} = \frac{\partial J}{\partial \dot{\omega}_1} = \frac{\partial J}{\partial \dot{m}} = 0,$$

we will obtain a system of two ordinary differential equations. Combining them, finally we have the following result

$$\pi\left(\frac{4}{R} - \frac{2,8R^2q}{h^6}\Phi_2\right)\dot{\omega}_1 - \pi\frac{2,8R^2}{h^6}\Phi_2\omega_1 +$$
$$+\frac{q^mR^{m+2}}{h^2 2^{m+2}}\sum_{p=0}^{m}\left(\Phi^v_{p+1}K_{p+1} - \frac{3,7}{h^2}\Phi^v_{p+2}K_{p+2}\omega_1 -\right.$$
$$\left.-\frac{3,7q}{h^2}\Phi^v_{p+2}K_{p+2}\dot{\omega}_1\right)\frac{3^{p+1}\cdot 5^p\cdot \mathbf{C}^p_m(m+1)(-1)^{m-p}}{4^p\cdot h^{m+2p+2}}\omega_1^p = 0. \tag{10}$$

Here

$$\Phi_2 = \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}\frac{z^2dz}{E_{k+1}},\ \Phi^v_{p+j} = \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}\frac{z^{p+j}dz}{E_{k+1}\left(\sigma^0_{k+1}\right)^m},\ j = 1, 2,$$

$$K_p = 2\sqrt{\pi}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)},$$

and Γ- is Eulier gamma function.

It is necessary to complete equation (10) by the initial conditions, which consist in the absence of moment for $q = 0$ and the presence of initial imperfection, i.e.

$$m(0) = 0;\ \omega(0) = \omega_1^0 \cos 2\varphi, \tag{11}$$

where ω_1^0 is a given amplitude of initial imperfection.

For example, consider the buckling of three-layered ring ($n = 3$). Suppose that $E_1 = E_3$, $\delta_1 = \delta_3$ and $\sigma_{0,1} = \sigma_{0,3}$. We introduce the following dimensionless quantities $\alpha = E_1E_2^{-1}, \beta = \delta_2\delta_1^{-1}, \xi = hR^{-1}, \lambda = E_1\sigma_{0,1}^{-1}, \gamma = \sigma_{0,1}\sigma_{0,2}^{-1}, a = \omega_1 h^{-1}$, $\tau = qE_1^{-1}, \varphi_2 = \dfrac{E_1}{h^3}\Phi_2, \varphi_2^v = \dfrac{E_1^3}{h^3}\Phi_2^v,\ \varphi_4^v = \dfrac{E_1^3}{h^5}\Phi_4^v$.

We will take the nonlinearity index as $m = 2$. Using the first condition (11), substituting the dimensionless quantities into system (10), after some transformations we will reduce (10) to the differential equation

$$\frac{d\tau}{da} = \frac{1}{a}\frac{4 - 0,94\xi^{-3}\varphi_2\tau - 0,7\xi^{-5}\lambda^2\varphi_2^v\tau^3 - 4,45\xi^{-5}\lambda^2\varphi_4^v a^2\tau^3}{0,94\xi^{-3}\varphi_2 + 2,11\xi^{-5}\lambda^2\varphi_2^v\tau^2 + 4,45\xi^{-5}\lambda^2\varphi_4^v a^2\tau^2}, \tag{12}$$

here

$$\varphi_2 = LD^{-3}, \varphi_2^v = MD^{-3},\ \varphi_4^v = ND^{-5} \text{ and } L = 2 + 3\beta + 1,5\beta^2 + 0,25\alpha\beta^3,$$

$$M = 2 + 3\beta + 1,5\beta^2 + 0,25\alpha\beta^3\gamma^2,$$

$$N = 2 + 5\beta + 5\beta^2 + 2,5\beta^3 + 0,625\beta^4 + 0,0625\alpha\beta^5\gamma^2,$$

$$D = 1 + 0,5\beta.$$

Assuming $\xi = 10^{-1}$, $\lambda = 3 \cdot 10^2$, be Runge- Kutta method we solve Cauchy problem for equation (12) at the initial condition $a(0) = 10^{-1}$. Relations between the buckling load τ_{bl} and quantities α, β and γ are performed in the tables.

$\gamma = 0,25;\ \beta = 4;\ 0,4$ (respectively)

α	0,25	0,5	0,75	1	1,25	1,5	1,75	2
τ_{bl}	51849	51154	50470	49798	49136	48485	47845	47216
τ_{bl}	46219	46211	46202	46194	46186	46177	46169	46161

(Here and later on we will assume that all the values τ_{bl} are multiplied on 10^{-8}.)

$\alpha = \gamma = 0,25$

β	0,5	1	1,5	2	2,5	3	3,5	4
τ_{bl}	46269	46709	47385	48196	49079	49996	50924	51489

$\alpha = 0,25; \beta = \ 4;\ 0,4$

γ	0,25	0,5	0,75	1	1,25	1,5	1,75	2
τ_{bl}	51849	51610	51221	450699	50059	49323	47511	47641
τ_{bl}	46219	46217	46217	46209	46202	46195	46186	46175

We will take the nonlinearity index $m = 4$. Then equation (10) will be

$$\frac{d\tau}{da} = \frac{1}{a}\left(4 - 0,94\xi^{-3}\varphi_2\tau - 0,29\xi^{-7}\lambda^4\varphi_2^v\tau^5 - \right.$$

$$\left. -11,12\xi^{-7}\lambda^4 a^2\varphi_4^v\tau^5 - 15,52\xi^{-7}\lambda^4 a^4\varphi_6^v\tau^5\right) /$$

$$/\left(0,94\xi^{-3}\varphi_2 + 1,47\xi^{-7}\lambda^4\varphi_2^v\tau^4 + 18,54\xi^{-7}\lambda^4 a^2\varphi_4^v\tau^4 + 15,52\xi^{-7}\lambda^4 a^4\varphi_6^v\tau^4\right). \quad (13)$$

Here

$$\varphi_2 = LD^{-3}, \varphi_2^v = M_1 D^{-3},\ \varphi_4^v = N_1 D^{-5},$$

$$\varphi_6^v = QD^{-7},\ \ L = 2 + 3\beta + 1,5\beta^2 + 0,25\alpha\beta^3,$$

$$D = 1 + 0,5\beta,\ M_1 = 2 + 3\beta + 1,5\beta^2 + 0,25\alpha\beta^3\gamma^4,$$

$$N_1 = 2 + 5\beta + 5\beta^2 + 2,5\beta^3 + 0,625\beta^4 + 0,0625\alpha\beta^5\gamma^4,$$

$$Q = 2 + 7\beta + 10,5\beta^2 + 8,75\beta^3 + 4,375\beta^4 + 1,3125\beta^5 + 0,21875\beta^6 + 0,015625\alpha\beta^7\gamma^4.$$

Relations between the buckling load τ_{bl} and quantities α, β and γ are performed in the tables.

$$\gamma = 0,25;\ \beta = 4;\ 0,4$$

α	0,25	0,5	0,75	1	1,25	1,5	1,75	2
τ_{bl}	45260	44859	44057	43057	43657	43258	42859	42461
τ_{bl}	42271	42266	42260	42255	42250	42244	42239	42233

$$\alpha = \gamma = 0,25$$

β	0,5	1	1,5	2	2,5	3	3,5	4
τ_{bl}	42297	42526	42883	43313	43784	44273	44768	45260

$$\alpha = 0,25;\ \beta = 4;\ 0,4$$

γ	0,25	0,5	0,75	1	1,25	1,5	1,75	2
τ_{bl}	45260	45227	45083	44710	43987	42844	41312	39504
τ_{bl}	42271	42271	42270	42267	42260	42249	42229	42200

We will take the nonlinearity index $m = 6$. Then, analogously, equation (10) will be

$$\frac{d\tau}{da} = \frac{1}{a}\left(4 - 0,94\xi^{-3}\varphi_2\tau - 0,1\xi^{-9}\lambda^6\varphi_2^v\tau^7 - 9,73\xi^{-9}\lambda^6 a^2\varphi_4^v\tau^7 - \right.$$

$$\left. -81,47\xi^{-9}\lambda^6 a^4\varphi_6^v\tau^7 - 51,98\xi^{-9}\lambda^6 a^6\varphi_8^v\tau^7\right)\cdot\left(0,94\xi^{-3}\varphi_2 + 0,72\xi^{-9}\lambda^6\varphi_2^v\tau^6 + \right.$$

$$\left. +22,71\xi^{-9}\lambda^6 a^2\varphi_4^v\tau^6 + 114,06\xi^{-9}\lambda^6 a^4\varphi_6^v\tau^6 + 51,98\xi^{-9}\lambda^6 a^6\varphi_8^v\tau^6\right)^{-1}, \quad (14)$$

here

$$\varphi_2 = LD^{-3}, \varphi_2^v = M_2 D^{-3},\ \varphi_4^v = N_2 D^{-5},$$

$$\varphi_6^v = Q_1 D^{-7},\ \ \varphi_8^v = RD^{-9}, L = 2 + 3\beta + 1,5\beta^2 +$$

$+0,25\alpha\beta^3, D = 1{+}0,5\beta,\ M_2 = 2{+}3\beta{+}1,5\beta^2{+}0,25\alpha\beta^3\gamma^6,\ N_2 = 2{+}5\beta{+}5\beta^2{+}2,5\beta^3 +$
$+0,625\beta^4 + 0,0625\alpha\beta^5\gamma^6, Q_1 = 2 + 7\beta + 10,5\beta^2 + 8,75\beta^3 + 4,375\beta^4 + 1,3125\beta^5 +$
$+0,21875\beta^6{+}0,015625\alpha\beta^7\gamma^6,\ R = 2{+}9\beta{+}13,5\beta^2{+}21\beta^3{+}15,75\beta^4{+}7,875\beta^5{+}2,625\beta^6 +$

$$+0,421875\beta^7 + +0,0703\beta^8 + 0,0039\alpha\beta^9\gamma^6.$$

Relations between the buckling load τ_{bl} and quantities $\alpha,\ \beta$ and γ are performed in the tables.

$\gamma = 0,25;\ \beta = 4;\ 0,4$

α	0,25	0,5	0,75	1	1,25	1,5	1,75	2
τ_{bl}	43844	43493	43140	42786	42430	42074	41717	41360
τ_{bl}	41756	41751	41746	41741	41736	41731	41726	41721

$\alpha = \gamma = 0,25$

β	0,5	1	1,5	2	2,5	3	3,5	4
τ_{bl}	41774	41934	42182	42484	42813	43155	43501	43844

$\alpha = 0,25;\ \beta = 4;\ 0,4$

γ	0,25	0,5	0,75	1	1,25	1,5	1,75	2
τ_{bl}	43844	43839	43787	43530	42741	41099	38671	35896
τ_{bl}	41756	41756	41756	41754	41748	41731	41694	41619

Thus, for the chosen values of parameters the following conclusions may be made:

1) As it should be expected, taking into account of physical nonlinearity decreases the values of buckling load;

2) Even partition of layers significantly increases the values of buckling roads in comparison with odd partition and uniformity;

3) Changing of values of α and γ does not change the qualitative picture of critical state;

4) For the odd number of layers in terms of stability, the case $n = 3$ is more preferable;

5) In homogeneous case ($\alpha = \beta = \gamma = 1$) for $m = 2; 4; 6$, respectively, we obtain $\tau_{bl} = 46152 \cdot 10^{-8};\ 42237 \cdot 10^{-8};\ 41732 \cdot 10^{-8}$.

Thus, it is brought to light that it is possible to increase (decrease) the buckling load by constructing the nonuniformity, and by that, in a certain sence, to optimize the construction.

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