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SOLUTION OF STATIONARY AXISYMETRIC GRAVITY

Abstract

The stationary axisymmetric Einstein equations are reduced to the principal chiral field problem with moving poles. Solutions are constructed in terms of Bessel functions. Generalization of solutions by means of applying the discrete symmetry transformations is discussed.

1. The problem of constructing of the solutions of self-dual Yang-Mills (SDYM) model and its dimensional reductions, the principal chiral field problem with moving poles in our case, in the explicit form for semisimple Lie algebra remains important for the present time. The interest arises from the fact that almost all integrable models in one, two and (1+2)-dimensions are symmetry reductions of SDYM or they can be obtained from it by imposing the constraints on Yang-Mills potentials [1-10].

This work is a direct continuation of [13], where the exact solutions of the principal chiral field problem with moving poles have been derived for the case of algebra SL(2,C). The discrete symmetry transformation method [12] allows to generate new solutions from the old ones in much more easier way than applying methods from [11]. In his work the stationary axisymmetric Einstein equations [14-18] are reduced to the principal chiral field problem with moving poles and a new initial , in respect to applying the discrete symmetry transformation, solution is found in terms of Bessel functions.

2. Consider Einstein equations for standard metric of a stationary axisymmetric $4d$space-time

$$ds^2 = w^{-1}[e^{2k}(dx^2 + d\rho^2) + \rho^2 d\varphi^2] - w(dt + W d\varphi)^2, \tag{1}$$

where$(x,\rho)$are canonical Weyl coordinates, $\rho \geq 0,\ x \in R$, and t and φ are time and angular coordinates, respectively. All metric coefficients depend only on (x,ρ) and they can be expressed in terms of complex Ernst potential E[14] as follows:

$$k_\xi = 2i\rho \frac{E_\xi \bar{E}_\xi}{\left(E + \bar{E}\right)^2} \ , \quad W_\xi = 2\rho \frac{\left(E - \bar{E}\right)_\xi}{\left(E + \bar{E}\right)^2} \ , \quad w = ReE, \tag{2}$$

where $\xi = x + i\rho$, $\bar{\xi} = x - i\rho$.

In terms of element of the coset space SL(2,R)/SO(**??**)

$$g = \frac{1}{\left(E + \bar{E}\right)} \begin{pmatrix} 2 & i\left(E - \bar{E}\right) \\ i\left(E - \bar{E}\right) & 2E\bar{E} \end{pmatrix}. \tag{3}$$

Einstein equations then imply the Ernst equation

$$\left(\left(\xi - \bar{\xi}\right) g_{\xi} g^{-1}\right)_{\bar{\xi}} + \left(\left(\xi - \bar{\xi}\right) g_{\bar{\xi}} g^{-1}\right)_{\xi} = 0. \tag{4}$$

In [19] it has been shown how the equation (**??**) can be reduced to the equations of the principal chiral field problem with moving poles

$$(z - \bar{z})\theta_{z,\bar{z}} = [\theta_{\bar{z}}, \theta_z] \tag{5}$$

with additional constraining conditions

$$rank\theta_z = rank\theta_{\bar{z}} = 1, \tag{6}$$

$$sp\theta_z = -1/2 \ , \qquad sp\theta_{\bar{z}} = -1/2 \ , \tag{7}$$

$$\det Re\theta = -(z + \bar{z})/16, \tag{8}$$

where the commutator of two elements of algebra is a standard one: $[A, B] = AB - BA$.

Let's introduce the element$f$:

$$\theta = f - (z + \bar{z})/4.$$

If $f \in SL(2, R)$, i.e. $spf = 0$, we automatically satisfy conditions (**??**), and the condition (**??**) gives the condition on f:

$$\det f_z = \det f_{\bar{z}} = -1/16. \tag{9}$$

Note that the equation for $f$still has the form (**??**).

We'll find the solution in a form:

$$f = \frac{z + \bar{z}}{4} H + \alpha(z, \bar{z}) X^{+}, \tag{10}$$

where H, $X^{\pm}$are generators of $SL(2, R)$algebra having the following commutational relations:

$$[X^{+}, X^{-}] = 0 \ , \ [H, X^{\pm}] = \pm 2X^{\pm}. \tag{11}$$

These generators are (2x2)-matrixes in a spinor representation of $SL(2,R)$algebra:

$$X^{+}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X^{-}=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Substituting (**??**) into equation (**??**) and using (**??**) we have the equation for the function$\alpha(z,\bar{z})$:

$$(z-\bar{z})\alpha_{z,\bar{z}}=\frac{1}{2}(\alpha_z-\alpha_{\bar{z}}). \tag{12}$$

Passing to variables

$$x=z+\bar{z}\ ,\ \ iy=z-\bar{z}\ ,$$

we rewrite the equation (**??**) in a form:

$$\Delta\alpha=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\alpha=\frac{2}{y}\alpha_y.$$

Let's consider this equation in more general form:

$$\Delta\alpha=\frac{\beta}{y}\alpha_y, \tag{13}$$

where $\beta$is some constant.

Let's find the solution of (**??**) in the form:

$$\alpha=\int e^{\lambda x}c(\lambda)d\lambda u(y). \tag{14}$$

Substitution of (**??**) into (**??**) gives the following ordinary differential equation:

$$u_{yy}+\lambda^2 u-\frac{\beta}{y}u_y=0.$$

Making one more substitution $u=y^{\gamma}v$we have

$$v_{yy}+\frac{(2\gamma-\beta)}{y}v_y+(\lambda^2+\frac{\gamma(\gamma-1)-\beta\gamma}{y^2})v=0.$$

Finally, if $\gamma=(\beta+1)/2$we obtain the famous Bessel equation:

$$v_{yy}+\frac{1}{y}v_y+(\lambda^2+\frac{(\beta+1)^2}{4y^2})v=0\ ,$$

having the general solution in terms of two Bessel functions

$$v=c_1J^1_{\frac{\beta+1}{2}}(\lambda y)+c_2J^2_{\frac{\beta+1}{2}}(\lambda y). \tag{15}$$

Thus, making back all necessary substitutions we have the solution of considered axisymmetric stationary gravity equations in terms of Bessel functions.

The next question how to obtain from this solution new solutions using the discrete symmetry transformation:

$$F^{-}=\frac{1}{f^{+}}\ ,$$

$$\frac{\partial F^{0}}{\partial z}=(f^{0}-F^{0}+z)\frac{\partial \ln f^{+}}{\partial z}-\frac{\partial f^{0}}{\partial z}\ ,$$

$$\frac{\partial F^{0}}{\partial \bar z}=(f^{0}-F^{0}+\bar z)\frac{\partial \ln f^{+}}{\partial \bar z}-\frac{\partial f^{0}}{\partial \bar z}\ , \qquad (16)$$

$$\frac{\partial F^{+}}{\partial z}=(f^{0}-F^{0}+z)^{2}\frac{\partial \ln f^{+}}{\partial z}-2f^{+}(f^{0}-F^{0}+z)\frac{\partial f^{0}}{\partial z}-(f^{+})^{2}\frac{\partial f^{-}}{\partial z}\ ,$$

$$\frac{\partial F^{+}}{\partial \bar z}=(f^{0}-F^{0}+\bar z)^{2}\frac{\partial \ln f^{+}}{\partial \bar z}-2f^{+}(f^{0}-F^{0}+\bar z)\frac{\partial f^{0}}{\partial \bar z}-(f^{+})^{2}\frac{\partial f^{-}}{\partial \bar z}\ .$$

Here $f(f^{+},f^{0},f^{-})$is considered to be a known solution of equation (**??**) and $F(F^{+},F^{0},F^{-})$is one to be determined.

By direct check one can be convinced it that the transformation (**??**) preserves a determinant, i.e.

$$\det F_z=\begin{vmatrix} F_z^{0} & F_z^{+} \\ F_z^{-} & -F_z^{0}\end{vmatrix}=\begin{vmatrix} f_z^{0} & f_z^{+} \\ f_z^{-} & -f_z^{0}\end{vmatrix}=\det f_{\bar z}$$

That means that the transformation (**??**) automatically satisfies the property (**??**) of the solution.

New solutions generated by the discrete symmetry transformations (**??**) applied to the initial solution (**??**) will be the subject of future investigations.

References

[1]. Ward R.S. *Intergrable and Solvable Systems and Relations Among Them*, Phil. Trans. R. Soc. Lond.A, 1985, v. 315, pp. 451-457.

[2]. Mason L.J. and J. Sparlin G.A. *Nonlinear Shrodinger and Korteweg de Vries are Reductions of Yang-Mills*, Phys. Lett.A, 1989, v. 137, pp. 29-33.

[3]. Bakas and Depireux D.A. *A fractional KdV hierarchy*, Mod. Phys. Lett. A, 1991, v. 6, pp. 1561-1574.

[4]. Ablowitz M.J., Chakravarty S. and Takhtajan L.A. *A Self-Dual Yang-Mills Hierarchy and Its Relations to Integrable Systems*, Comm. Math. Phys., 1993, v. 158, pp. 1289-1297.

[5]. Ivanova T.A. and Popov A.D. *Solution Equations and Self-Dual Gauge Fields*, Phys. Lett. A, 1992, v. 170, pp. 293-298.

[6]. Ivanova T.A. and Popov A.D. *Some New Integrable Equations From the Self-Dual Yang-Mills Equations*, Phys. Lett. A, 1995, v. 250, pp. 158-166.

[7]. Ivanova T.A. and Popov A.D. *Self-Dual Yang-Mills Fields in d=7,8 Octonions and Ward Equations*, Lett. Math. Phys. , 1992, v. 24, pp. 85-92.

[8]. Legare M. and Popov A.D. *Lax pair reduction for Yang-Mills self-duality*, JETP Lett., 1994, v. 59, pp. 845-850.

[9]. Belavin A.A. and Zakharov V.E. *Yang-Mills equations as inverse scattering problems*, Phys. Lett. B, 1978, v.73, pp. 53-57.

[10] Leznov A.N. and Mukhtarov M.A. *Deformation of algebras and solution of self-duality equations*, J. Math. Phys., 1987, 28 (14), pp. 2574-2579.

[11]. Leznov A.N. *Backlund transformation for integrable system*, IHEP preprint-92/87, 1990, 10 p.

[12]. Leznov A.N., Mukhtarov M.A. and Zakrzewski W.J. *Discrete Symmetry and Solution of Principle Chiral Field Problem*, Tr. J. of Physics 1995, v. 19, pp. 416-420.

[13]. Mukhtarov M.A. *New Solutions of the Principle Chiral Field Problem with Moving Poles*, Proc.Ins.Math.Mech., 10(18), 2000, pp. 123-125.

[14]. Ernst F. *New Formulation of the Axially Symmetric Gravitational Field Problem*, Phys. Rev., 1968, v.167, pp. 1175-1191.

[15]. Belinskii V.A. and Zakharov V.A. *Stationary gravitational solitons with axial symmetry*, Sov. Phys. JETP 1979, v.77, pp. 3-19.

[15]. Korotkin D. A. and Matveev V. B. *Algebro-geometrical solutions of gravitational equations*, Leningrad Math. J. ,1990, v. 1., pp. 379-408.

[16]. Korotkin D.A. *Elliptic solutions of stationary axisymmetric Einstein equations*, Class. Quantum Gravity, 1993, v. 10, pp. 2587-2613.

[17]. Korotkin D.A. and Nicolai H. *Izomonodromic quantization of dimensionally reduced gravity*, Nucl. Phys. B, 1996, v. 475, pp. 397-439.

[18]. Mukhtarov M.A. *Stationary axisymmetric gravity as a principal chiral field model*, Fizika, 2003, 9 (2), pp. 59-60.

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