MATHEMATICS

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ON SPECTRUM OF THE CESARO OPERATOR

Abstract

The fine spectrum of the Cesaro operator in the sequence space l_p $(1 < p < \infty)$ has been examined in the present work. Although this discussion is made for determination of the spectrum in [1]-[5] and others, our consequences are more detailed and include a remark concerning the previous work. At the end, the fine spectrum of the Cesaro operator in the sequence c_0 space has been given.

1. Preliminaries, background and notions. Let E_1 and E_2 be the Banach spaces and $T : E_1 \to E_2$ also be a bounded linear operator. By JmT and $KerT$ we respectively denote the range and the null space of T:

$$JmT = \{y : y = Tx, \quad x \in E_1\},$$

$$KerT = \{x : Tx = 0\}.$$

Let A be any non empty subset of E_1. The annihilator $A^{\perp}$ of A is defined to be set of all bounded linear functionals on E_1 which are zero everywhere on A. Thus $A^{\perp}$ is a closed linear subspace of the dual space E_1^* of E_1.

Now we may give the following two lemmas:

Lemma 1.1 [6]. $KerT^* = \left(\overline{JmT}\right)^{\perp}$, *where* T^* *is a Banach adjoint of* T.

Lemma 1.2 [7]. *The linear subspace* A *is not dense in a Banach space* E_1 *if and only if there exists a bounded linear functional* $f \neq 0$ *such that* $f(x) = 0$ *for any element* $x \in A$.

Let $E \neq \{0\}$ be a complex normed space and $T : D(T) \to E$ also be a linear operator with domain $D(T) \in E$. With T, we associate the operator $T - \lambda I$ where λ is a complex number and I is the identity operator on $D(T)$. If $T - \lambda I$ has an inverse, we denote it by $R_\lambda(T) = (T - \lambda I)^{-1}$ and call it the resolvent operator of T. For our investigation of T, we shall need some concepts which are basic in spectral theory.

Definition 1.3. *A regular value of* T *is a complex number* λ, *such that*

a) $R_\lambda(T)$ *exists,*

b) $R_\lambda(T)$ *bounded,*

c) $R_\lambda(T)$ *is defined on a set which is dense in* E.

The resolvent set $p(T)$ is set of all regular values of T. The spectrum of T is $\sigma(T) = C \backslash p(T)$, where C is a complex plane.

The spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:

1) The point spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exist. $\lambda \in \sigma_p(T)$ is called eigenvalue of T.

2) The continuous spectrum $\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists satisfies c) but not b), that is, $R_\lambda(T)$ is unbounded.

3) The residual spectrum $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists (may be bounded or not) but not satisfy c), that is, the domain of $R_\lambda(T)$ is not dense in E.

The Cesaro operator is represented by the infinity matrix

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

It is known that

1. $C_1 : 1_p \to 1_p$ is a linear operator, where 1_p denotes the space of all p-absolutely convergent series, $1 < p < \infty$;

2. $\|C\| = q$ where $\frac{1}{p} + \frac{1}{q} = 1$;

3. $C_1^* : l_q \to l_q$ and represented by the matrix

$$C_1^* = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} & \dots \\ 0 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

In this paper, our purpose is to investigate the fine spectrum of Cesaro operator in the space l_p $(1 < p < \infty)$.

2. The fine spectrum of the Cesaro operator in the space l_p $(1 < p < \infty)$. In this section the fine spectrum of Cesaro operator in the space l_p $(1 < p < \infty)$ has been examined. Although this discussion was made for determination of the spectrum in [1]-[4] and others, our consequences are more detailed. Furthermore, we give spectrum of the Cesaro operator in the sequence space c_0.

We shall begin with to quote lemmas which are needed in the proof of the theorems.

Lemma 2.1 ([1]-[4]). $\sigma(C_1) = \left\{\lambda : \left|\lambda - \frac{q}{2}\right| \le \frac{q}{2}\right\}$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.2 ([5], pp. 398-399, Weierstrass Criterion). *A series* $\sum\limits_{n=0}^{\infty} a_n$ *of complex terms, for which*

$$\frac{a_{n+1}}{a_n} = 1 - \frac{a}{n} - \frac{b}{n^\gamma}$$

with b_n *bounded, is absolutely convergent if and only if* $\operatorname{Re}(a) > 1$, *where* $\operatorname{Re}(a)$ *denotes the real part of the complex number* $a, \gamma > 1$. *For* $\operatorname{Re}(a) \le 0$ *the series are invariably divergent. If* $0 < \operatorname{Re}(a) \le 1$, *both series*

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n| \ \ and \ \sum_{n=0}^{\infty} (-1)^n a_n$$

are convergent.

Theorem 2.1. $\sigma_p(C_1) = \emptyset$.

Proof. Since $l_p \subset c_0$ and $\sigma_p(C_1) = \emptyset$. in c_0, by ([5], theorem 1, p.265).

Theorem 2.2. $\sigma_c(C_1) = \left\{\lambda : \left|\lambda - \frac{q}{2}\right| = \frac{q}{2}\right\}$, *where* $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. Consider the set

$$\left\{\lambda : \left|\lambda - \frac{q}{2}\right| = \frac{q}{2}\right\}.$$

First of all suppose that $\lambda \neq 0$.

We have in this situation that

$$\left|1 - \frac{q}{2\lambda}\right| = \frac{q}{2|\lambda|}.$$

If $\frac{1}{\lambda} = \alpha + i\beta$, then we have

$$\left|1 - \frac{aq}{2} - i\frac{\beta q}{2}\right| = \frac{q}{2}\sqrt{a^2 + \beta^2}$$

or

$$\mathrm{Re}\left(\frac{1}{\lambda}\right) = a = \frac{1}{q}.$$

Let us take any

$$\lambda \in \left\{\lambda : \mathrm{Re}\left(\frac{1}{\lambda}\right) = \frac{1}{q}\right\}.$$

and solve the equation

$$C_1^* f = \lambda f, \ f = \{f_0, f_1, ...\} \in l_q.$$

Then we have the system of linear equations

$$\begin{gathered} \tfrac{1}{\lambda} f_0 + \tfrac{1}{2\lambda} f_1 + \tfrac{1}{3\lambda} f_2 + ... = f_0 \\ \tfrac{1}{2\lambda} f_1 + \tfrac{1}{3\lambda} f_2 + ... = f_1 \\ \dots\dots\dots\dots\dots \\ \tfrac{1}{(n+1)\lambda} f_n + ... = f_n \\ \dots\dots\dots\dots\dots\dots \end{gathered}$$

whose solution is found as

$$\begin{aligned} f_1 &= \left(1 - \frac{1}{\lambda}\right) f_0, \ f_2 = \left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{1}{2\lambda}\right) f_0, ..., f_n = \\ &= \left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{1}{2\lambda}\right) ... \left(1 - \frac{1}{n\lambda}\right) f_0, ... \end{aligned}$$

and

$$\left(\frac{f_{n+1}}{f_n}\right)^q = \left(1 - \frac{\frac{1}{q} + i\beta}{n+1}\right)^q = 1 - \frac{1 + iq\beta}{n+1} - \frac{A_n}{(n+1)^\gamma},$$

where $\gamma > 1$ and A_n bounded. By Weierstrass Criterion $f \notin l_q$, if $f_0 \neq 0$, since $\mathrm{Re}\,(1 + iq\beta) = 1$. We get that $\lambda \notin \sigma_p\,(C_1^*)$. Hence $\ker\,(C_1^* - \lambda I^*) = \{0\}$ for such λ's which shows

$$\overline{Jm\,(C_1 - \lambda I)} = l_p.$$

Now, suppose that $\lambda = 0$ and consider the equation

$$C_1 x = 0,\ x = \{x_0, x_1 ...\} \in l_p.$$

One can show that $x = 0$. Hence $\ker C_1 = \{0\}$ and C_1 has an inverse.

Let us consider the equation

$$C_1^* f = 0,\ f = \{f_0, f_1 ...\} \in l_q.$$

Its solution is $f = 0$. Hence $\ker C_1^* = \{0\}$ and we therefore see that $\overline{JmC_1} = l_q$. It shows that $\lambda = 0$ and $\left\{\lambda : \mathrm{Re}\left(\frac{1}{\lambda}\right) = \frac{1}{q}\right\}$ belong to $\sigma_c\,(C_1)$. The last assertion completes the proof.

Theorem 2.3.

$$\sigma_r\,(C_1) = \left\{\lambda : \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\}.$$

Proof. Consider the next equation $C_1^* f = \lambda f,\ f = \{f_0, f_1 ...\} \in l_q$. As in theorem 2.2 we can show that

$$f_n = \prod_{k=1}^{n} \left(1 - \frac{1}{k\lambda}\right) f_0,\ (n = 1, 2, ...)$$

and

$$\left(\frac{f_{n+1}}{f_n}\right)^q = \left(1 - \frac{\frac{1}{\lambda}}{n+1}\right)^q = 1 - \frac{\frac{q}{\lambda}}{n+1} - \frac{B_n}{(n+1)^\gamma},$$

where $\gamma > 1$ and B_n bounded quantity.

If $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \frac{1}{q}$ and $f_0 \neq 0$, then by Weierstrass Criterion $f = \{f_0, f_1 ...\} \in l_q$.

Hence

$$\lambda \in \sigma_p\,(C_1^*)\,, \tag{2.1}$$

where

$$\mathrm{Re}\left(\frac{1}{\lambda}\right) > \frac{1}{q}.$$

Also we have showed that

$$G_\lambda \equiv \left\{\lambda : \mathrm{Re}\left(\frac{1}{\lambda}\right) > \frac{1}{q}\right\} = \left\{\lambda : \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\}.$$

By (2.1) there exists $f_0 \neq 0$ such that $f \in l_q$ for any $\lambda \in G_\lambda$. Then we have $(C_1^* - \lambda l^*)\,f = 0$. Hence $[(C_1^* - \lambda l^*)\,f]\,(x) = 0$ for any $x \in l_p$ or $f\,[(C_1 - \lambda l)\,x] = 0$. This last equation means by lemma 1.2 that $Jm\,(C_1 - \lambda l)$ is not dense in l_p. Besides, using theorem 2.1 we see that the inverse of $C_1 - \lambda l$ exists for any $\lambda \in C$ and also for any $\lambda \in G_\lambda$. It means that $G_\lambda = \sigma_r\,(C_1)$. The theorem is proved.

Combining lemma 2.1, theorem 2.1-2.3, we have the following main theorem:

Theorem 2.4. *Let C_1 be Cesaro operator and $C_1 : l_p \to l_p$ $(l < p < \infty)$, then the following statements are satisfied:*

1) $\sigma(C_1) = \left\{\lambda : \left|\lambda - \frac{q}{2}\right| \le \frac{q}{2}\right\}$,

2) $\sigma_p(C_1) = \emptyset$,

3) $\sigma_c(C_1) = \left\{\lambda : \left|\lambda - \frac{q}{2}\right| = \frac{q}{2}\right\}$,

4) $\sigma_r(C_1) = \left\{\lambda : \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\}$,

where $\frac{1}{p} + \frac{1}{q} = 1$.

Finally, we give the corresponding theorem to Theorem 2.4 on the fine spectrum of the Cesaro operator in the sequence space c_0 and it may be proved in similarly way to that the l_p above.

Theorem 2.5. *If $C_1 : c_0 \to c_0$, then the following statements are satisfied:*

1) $\sigma(C_1) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}\right\}$,

2) $\sigma_p(C_1) = \emptyset$,

3) $\sigma_c(C_1) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2},\ \lambda \ne 1\right\}$,

4) $\sigma_r(C_1) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2},\right\} U \{1\}$.

Remark. We should remark that the part 1) of theorem 2.4 is established in [1]-[4] and part 2) of this theorem in [3], and the parts 1) and 2) of theorem 2.5 are established in [2], [5] and [8] respectively. In contrast to ([2], p.230, [3], theorem 3, p.206), we have showed that $\sigma_r(C_1)$ does not consist of only the points of

$$F_\lambda = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\}$$

and the point $\lambda = 1$ must be also added to them. Note that in [3] the author asserted that his result ([3], remark 2.7, p.206) is true also, when $q = 1$ and was not proved.

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