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APPROXIMATION BY IBRAGIMOV-GADJIEV OPERATORS IN POLYNOMIAL WEIGHTED SPACE[1]

Abstract

This paper is devoted to study of Ibragimov -Gadjiev operators as an approximation process in the weighted function space of continuous functions on unbounded interval.

1. Introduction

In expensive paper [8] , I. I. Ibragimov, and A. D. Gadjiev, defined a general sequence of positive operators and studied some approximation properties of this operators. Their work was motivated by the development of a general expression that cover other Bernstein type operators. It is shown in [8] that this sequence of operators, in special case, consist of the well known Bernstein, Százs, Bernstein-Chlodovsky and V. A. Baskakov operators(see [2]) .

Also, Ibragimov -Gadjiev operators were studied by some authors. B. Wood and P. Radatz [9] obtained some approximation properties of operators given in [8] to derivative of function. In [3] for these operators were given a modification and the some results in [8] and [9] were also developed. Regarding the same operators, the most recent paper is due to Gadjiev, A. D. and Ispir N. [5] who obtained the theorems on weighted approximation on infinite sets using weighted Korovkin's type theorem given in [6] and [7] . Also Tibor K. Pogány [12] proved general convergence theorem for Ibragimov- Gadjiev operators on the class of finite second-order stochastic processes.

The aim of this paper is to discuss some approximation properties of modified Ibragimov- Gadjiev operators given in [3] in the setting of polynomial weighted function space. The results we shall give different from established in [5] . Our results were inspired by M. Becker paper [11] on Százs and Baskakov operators. Note that this technique will be used for the Balazs- Szabados operators and Százs operators in [1] and [10], respectively.. In the next section we introduce the construction of operators . In section 3 we present some auxiliary result and in section 4 we give pointwise estimate and the rate of convergence with the aid of weighted modulus of continuity.

2. Definitions and construction of operators

Let $C(\mathbb{R}_0)$ be set of all real-valued continuous functions on $\mathbb{R}_0 = [0, \infty)$. By $B(\mathbb{R}_0)$ we denote the subspace of all bounded functions on R_0, $C_m(\mathbb{R}_0)$ stands for consisting of all continuous functions f, such that $\frac{f(x)}{1+x^m} \in B(\mathbb{R}_0)$, with the finite norm

$$\|f\|_{m,\,[0,\,b_\lambda]} := \sup_{0\le x\le b_\lambda} \frac{|f(x)|}{1+x^m},$$

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where $\lim\limits_{\lambda\to\infty} b_\lambda = \infty$.

Considering that, λ positive real number, let $\{\varphi_\lambda(t)\}$ and $\{\psi_\lambda(t)\}$ be a family of functions in $C[0, b_\lambda]$ such that $\varphi_\lambda(0) = 0$, $\psi_\lambda(t) > 0$ for each $t \in [0, b_\lambda]$ and b_λ has infinite limit.

Let $\{\alpha_\lambda\}$ be a family of positive numbers such that

$$\lim_{\lambda\to\infty} \frac{\alpha_\lambda}{\lambda} = 1 \text{ and } \lim_{\lambda\to\infty} \frac{1}{\lambda^2 \psi_\lambda(0)} = 0.$$

Assume that a family of functions of three variables $\{K_\lambda(x, t, u)\}$ $(x, t \in [0, b_\lambda], -\infty < u < \infty, \lambda \geq 0)$ satisfies the following conditions:

1^0. Each function of this family is an entire analytic function with respect to u for fixed x and t on $[0, b_\lambda]$.

2^0. $K_\lambda(x, 0, 0) = 1$ for any $x \in [0, b_\lambda]$ and for any $\lambda \geq 0$.

3^0. $\left\{ (-1)^\upsilon \frac{\partial^\upsilon}{\partial u^\upsilon} K_\lambda(x, t, u)\Big|_{\substack{u=u_1\\ t=0}} \right\} \geq 0 \quad (\forall x \in [0, b_\lambda], \lambda \geq 0, \nu = 0, 1, 2...)$

4^0. $\frac{\partial^\upsilon}{\partial u^\upsilon} K_\lambda(x, t, u)\Big|_{\substack{u=u_1\\ t=0}} = -\lambda x \left[\frac{\partial^{\upsilon-1}}{\partial u^{\upsilon-1}} K_{h(\lambda)}(x, t, u) \right]\Big|_{\substack{u=u_1\\ t=0}}$

$(\forall x \in [0, b_\lambda], \lambda \geq 0, \upsilon = 0, 1, 2...)$, where $h(\lambda)$ is a nonnegative function satisfying the condition $\lim\limits_{\lambda\to\infty} \frac{h(\lambda)}{\lambda} = 1$.

Consider the family of linear operators;

$$L_\lambda(f; x) = \sum_{\upsilon=0}^{\infty} f\left(\frac{\upsilon}{\lambda^2 \psi_\lambda(0)}\right) \left\{ \frac{\partial^\upsilon}{\partial u^\upsilon} K_\lambda(x, t, u) \Big|_{\substack{u=u_1 u=\alpha_\lambda \psi_\lambda(t)\\ t=0}} \right\} \frac{[-\alpha_\lambda \psi_\lambda(0)]^\upsilon}{\upsilon!} \qquad (1)$$

Note that for $\lambda = n$, $h(\lambda) = m + n$ $(m + n = 0, 1, 2....)$ and $\lim\limits_{\lambda\to\infty} b_\lambda = A$ the operators defined by (1) reduced to the operators defined in [8].

3. Preliminary Result

By elementary calculation from (1) we obtain

$$\begin{aligned} &L_\lambda(1; x) = 1 \\ &L_\lambda(s - x; x) = \left(\tfrac{\alpha_\lambda}{\lambda} - 1\right) x \\ &L_\lambda\left((s-x)^2; x\right) = \left(\left(\tfrac{\alpha_\lambda}{\lambda}\right)^2 \tfrac{h(\lambda)}{\lambda} - 2\tfrac{\alpha_\lambda}{\lambda} + 1\right) x^2 + \tfrac{\alpha_\lambda}{\lambda} \tfrac{1}{\lambda^2 \psi_\lambda(0)} x \end{aligned} \qquad (2)$$

Now, we calculate m-th moments for L_λ using similar technique in [3]

Lemma 1. *For any natural* m

$$L_\lambda(s^m; x) = A_m(\lambda) + \sum_{k=1}^{m-1} B_{k,m}(\lambda) x^k, \qquad (3)$$

where $\lim\limits_{\lambda\to\infty} A_m(\lambda) = 1$ *and* $\lim\limits_{\lambda\to\infty} B_{k,m}(\lambda) = 1$ *for* $k = 1, 2, ..., m-1$.

Proof. For the proof we use induction method. We can choose suitable constant $a_1, a_2, ...a_{m-1}$ such that

$$\left(\frac{\upsilon}{\lambda^2 \psi_\lambda(0)}\right)^m = \frac{\upsilon(\upsilon-1)...(\upsilon-m+1)}{\left(\lambda^2 \psi_\lambda(0)\right)^m} + \sum_{j=1}^{m-1} a_j \left(\frac{\upsilon}{\lambda^2 \psi_\lambda(0)}\right)^j \frac{1}{\left(\lambda^2 \psi_\lambda(0)\right)^{m-j}}.$$

Thus

$$L_\lambda(s^m;x)=\frac{1}{\left(\lambda^2\psi_\lambda(0)\right)^m}\sum_{\upsilon=m}^{\infty}\left\{\frac{\partial^\upsilon}{\partial u^\upsilon}K_\lambda(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\right\}\frac{[-\alpha_\lambda\psi_\lambda(0)]^\upsilon}{(\upsilon-m)!}+$$

$$+\sum_{j=1}^{m-1}a_j\frac{1}{\left(\lambda^2\psi_\lambda(0)\right)^{m-j}}L_\lambda\left(s^j;x\right)=$$

$$=\frac{1}{\left(\lambda^2\psi_\lambda(0)\right)^m}\sum_{\upsilon=0}^{\infty}\left\{\frac{\partial^{\upsilon+m}}{\partial u^{\upsilon+m}}K_\lambda(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\right\}\frac{[-\alpha_\lambda\psi_\lambda(0)]^{\upsilon+m}}{(\upsilon)!}+$$

$$+\sum_{j=1}^{m}a_j\frac{1}{\left(\lambda^2\psi_\lambda(0)\right)^{m+1-j}}L_\lambda\left(s^j;x\right)$$

On the other hand, by using 4^0 we get

$$\left[\frac{\partial^{\upsilon+m}}{\partial u^{\upsilon+m}}K_\lambda(x,t,u)\right]\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}=$$

$$=(-1)^m x^m\lambda h(\lambda)h_2(\lambda)\dots h_{(m-1)}(\lambda)\left[\frac{\partial^\upsilon}{\partial u^\upsilon}K_{h_{(m)}(\lambda)}(x,t,u)\right]\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}},$$

where

$$h_{(m)}(\lambda)=\underbrace{h(h(...h}_{\text{m-times}}(\lambda)))).$$

Using this equality and making simplification we can write

$$L_\lambda(s^m;x)=\left(\frac{\alpha_\lambda}{\lambda}\right)^m\left(\frac{h(\lambda)h_2(\lambda)\dots h_{(m-1)}(\lambda)}{\lambda^m}\right)x^m+$$

$$+\left(\frac{\alpha_\lambda}{\lambda}\right)^{m-1}\frac{\left(h(\lambda)h_2(\lambda)\dots h_{(m-2)}(\lambda)\right)}{\lambda^{m-2}}\frac{a_{m-1}}{\left(\lambda^2\psi_\lambda(0)\right)}x^{m-1}+$$

$$+\left(\frac{\alpha_\lambda}{\lambda}\right)^{m-2}\frac{\left(h(\lambda)h_2(\lambda)\dots h_{(m-3)}(\lambda)\right)}{\lambda^{m-3}}\frac{\{a_{m-2}+a_{m-2}c_{1,m-2}\}}{\left(\lambda^2\psi_\lambda(0)\right)^2}x^{m-2}$$

$$\vdots$$

$$+\left(\frac{\alpha_\lambda}{\lambda}\right)\frac{\{a_1+a_2c_{1,2}+\dots+a_{m-1}c_{m-1,m}\}}{\left(\lambda^2\psi_\lambda(0)\right)^m}x,$$

where

$$c_{1,m+1}=a_mc_{2,m+1}=a_{m-1}+a_{m-1}c_{1,m-1}$$

$$\vdots$$

$$c_{m,m+1}=a_1+a_2c_{1,2}+\dots+a_mc_{m-1,m}$$

and

$$A_m(\lambda)=\left(\frac{\alpha_\lambda}{\lambda}\right)^m\left(\frac{h(\lambda)\,h_2(\lambda)\dots h_{(m-1)}(\lambda)}{\lambda^{m-1}}\right),\ B_{k,m}(\lambda)=A_k(\lambda)\frac{C}{\left(\lambda^2\psi_\lambda(0)\right)^{m-k}}.$$

Lemma 2. *For each* $m\in\mathbb{N}_0=\{0\}\cup\mathbb{N}$ *and all* f *belongs to* $C_m(\mathbb{R}_0)$ *there exists a constant* λ_0 *such that for* $\lambda>\lambda_0$

$$\|L_\lambda(1+s^m;x)\|_m\le 2 \tag{4}$$

and the inequality

$$\|L_\lambda(f;x)\|_m\le 2\|f\|_m \tag{5}$$

holds.

The equality (5) shows that L_λ is a positive linear operators from the space $C_m(\mathbb{R}_0)$ to $C_m(\mathbb{R}_0)$, (see [7]).

Proof. For $x\ge 0$ and $\upsilon=1,2,3,...,m$ since $\frac{x^\upsilon}{1+x^m}\le 1$ by Lemma 1 we have

$$\frac{L_\lambda(s^m;x)}{1+x^m}\le A_m(\lambda)+\sum_{k=1}^{m-1}B_{k,m}(\lambda),$$

where $A_m(\lambda)$ and $B_{k,m}(\lambda)$ $(k=1,2,..,m-1)$ defined as in (3) . Since $\lim\limits_{\lambda\to\infty}A_m(\lambda)=1$, and $\lim\limits_{\lambda\to\infty}B_{k,m}(\lambda)=0$ the right hand side of the above inequality tends to 1 as $\lambda\to\infty$. Consequently, there exist a constant λ_0 such that for $\lambda>\lambda_0$

$$\frac{L_\lambda(1+s^m;x)}{1+x^m}\le 1.$$

This implies that

$$\sup_{0\le x\le b_\lambda}\frac{(1+L_\lambda(s^m;x))}{1+x^m}\quad\le\quad 1+\|L_\lambda(s^m;x)\|_{m,\,[0,\,b_\lambda]}\le 2$$

Also, from (4), for every $f\in C_m(\mathbb{R}_0)$ we obtain

$$\|L_\lambda(f;x)\|_{m,\,[0,\,b_\lambda]}\le\left\|L_\lambda\left((1+s^m)\,|f(s)|\,\frac{1}{(1+s^m)};x\right)\right\|_{m,\,[0,\,b_\lambda]}\le$$

$$\le\|L_\lambda((1+s^m);x)\|_{m,\,[0,\,b_\lambda]}\,\|f\|_{m,\,[0,\,b_\lambda]}\,leq2\,\|f\|_{m,\,[0,\,b_\lambda]}\,.$$

Lemma 3. *For each* $m\in\mathbb{N}_0=\{0\}\cup\mathbb{N}$, $x\in[0,b_\lambda]$ *and* $\lambda>0$ *we have*

$$\frac{L_\lambda\left((1+s^m)(s-x)^2;x\right)}{(1+x^m)}\le H_m(\lambda)\left(x^2+x+2\right),\quad x\in\mathbb{R}_0 \tag{6}$$

holds, where

$$H_m(\lambda)=\max\{|A_{m+2}(\lambda)-2A_{m+1}(\lambda)+A_m(\lambda)|,$$

$$|B_{m+1,m+2}(\lambda)-2B_{m,m+1}(\lambda)+B_{m-1,m}(\lambda)|,$$

$$\sum_{k=1}^{m}\left|B_k(\lambda)-2B_{k-1}(\lambda)+B_{k-2}(\lambda)\right|,\ \left|\left(\frac{\alpha_\lambda}{\lambda}\right)^2\frac{h(\lambda)}{\lambda}-2\frac{\alpha_\lambda}{\lambda}+1\right|+\frac{\alpha_\lambda}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)}\Bigg\}.$$

Proof. From lemma 1 we get

$$L_\lambda\left((s-x)^2 s^m;x\right)=L_\lambda\left(s^{m+2};x\right)-2xL_\lambda\left(s^{m+1};x\right)+x^2L_\lambda\left(s^m;x\right)$$

$$\le\left|A_{m+2}(\lambda)-2A_{m+1}(\lambda)+A_m(\lambda)\right|x^{m+2} \tag{7}$$

$$+\sum_{k=1}^{m+1}\left|B_{k,m+2}(\lambda)-2B_{k-1,m+1}(\lambda)+B_{k-2,m}(\lambda)\right|x^k,$$

where we choose for the convenience $B_{-1,m+1}(\lambda)=B_{0,m}(\lambda)=0$. Using (7) we can write

$$\frac{L_\lambda\left((1+s^m)(s-x)^2;x\right)}{(1+x^m)}\le\left|\left(\frac{\alpha_\lambda}{\lambda}\right)^2\frac{h(\lambda)}{\lambda}-2\frac{\alpha_\lambda}{\lambda}+1\right|+\frac{\alpha_\lambda}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)}+$$

$$+\left|A_{m+2}(\lambda)-2A_{m+1}(\lambda)+A_m(\lambda)\right|x^2+$$

$$+\left|B_{m+1,m+2}(\lambda)-2B_{m,m+1}(\lambda)+B_{m-1,m}(\lambda)\right|x+$$

$$+\sum_{k=1}^{m}\left|B_{k,m+2}(\lambda)-2B_{k-1,m+1}(\lambda)+B_{k-2,m}(\lambda)\right|.$$

This is claimed result.

4. Main Results

In this section we give pointwise estimates in the polynomial weighted function space denote by

$$C_m^2(\mathbb{R}_0):=\left\{f:\ f,f',f''\in C_m(\mathbb{R}_0)\right\}.$$

Also, we define weighted modulus of continuity similar given in [5] and give theorem on the order of approximation of function by the operators (1) on infinite interval $\mathbb{R}_0$.

Theorem 4. *Let $f\in C_m^2(\mathbb{R}_0)$. For the linear positive operator $L_\lambda(f;x)$ the inequality*

$$\left\|\frac{L_\lambda(f;x)-f(x)}{2+x+x^2}\right\|_{m,[0,b_\lambda]}\le\left\|f'\right\|_m\left|\frac{\alpha_\lambda}{\lambda}-1\right|+$$

$$+\left\|f''\right\|_m\left\{\left|\left(\frac{\alpha_\lambda}{\lambda}\right)^2\frac{h(\lambda)}{\lambda}-2\frac{\alpha_\lambda}{\lambda}+1\right|+\frac{\alpha_\lambda}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)}+H_m(\lambda)\right\}$$

holds, where $H_m(\lambda)$ given in Lemma 3.

Proof Let $x\in\mathbb{R}_0$ be a fixed point. For $f\in C_m^2(\mathbb{R}_0)$ and $s\in\mathbb{R}_0$ we can write

$$f(s)=f(x)+f'(x)(s-x)+\int_x^s(s-y)f''(y)\,dy.$$

Since L_λ is a linear positive operator and $L_\lambda(1;x)=1$ we get

$$L_\lambda(f(s);x)=f(x)+f'(x)L_\lambda((s-x);x)+L_\lambda\left(\int_x^s (s-y)f''(y)\,dy;x\right). \quad (8)$$

The estimation

$$\left|\int_x^s (s-y)f''(y)\,dy\right|\le \left\|f''\right\|_{m,[0,b_\lambda]}(2+s^m+x^m)(s-x)^2$$

yields

$$\frac{1}{1+x^m}L_\lambda\left(\left|\int_x^s (s-y)f''(y)\,dy\right|;x\right)\le \left\|f''\right\|_{m,[0,b_\lambda]}\left\{L_\lambda\left((s-x)^2;x\right)+\right.$$

$$\left.+\frac{1}{1+x^m}L_\lambda\left((1+x^m)(s-x)^2;x\right)\right\}$$

From (2), Lemma 3 and (8) it follows that

$$\frac{|L_\lambda(f;x)-f(x)|}{1+x^m}\le \left\|f'\right\|_{m,[0,b_\lambda]}|L_\lambda((s-x);x)|+$$

$$+\left\|f''\right\|_{m,[0,b_\lambda]}\left\{\frac{1}{1+x^m}L_\lambda\left((s-x)^2(1+s^m);x\right)+L_\lambda\left((s-x)^2;x\right)\right\}\le$$

$$\le \left\|f'\right\|_{m,[0,b_\lambda]}\left|\frac{\alpha_\lambda}{\lambda}-1\right|x+\left\|f''\right\|_{m,[0,b_\lambda]}\left\{H_m(\lambda)\left(x^2+x+1\right)+\right.$$

$$\left.+\left|\left(\frac{\alpha_\lambda}{\lambda}\right)^2\frac{h(\lambda)}{\lambda}-2\frac{\alpha_\lambda}{\lambda}+1\right|x^2+\frac{\alpha_\lambda}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)}x\right\}.$$

Thus we have

$$\left\|\frac{L_\lambda(f;x)-f(x)}{2+x+x^2}\right\|_{m,[0,b_\lambda]}\le \left\|f'\right\|_{m,[0,b_\lambda]}\left|\frac{\alpha_\lambda}{\lambda}-1\right|+\left\|f''\right\|_{m,[0,b_\lambda]}\left\{H_m(\lambda)+\right.$$

$$\left.+\left|\left(\frac{\alpha_\lambda}{\lambda}\right)^2\frac{h(\lambda)}{\lambda}-2\frac{\alpha_\lambda}{\lambda}+1\right|+\frac{\alpha_\lambda}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)}\right\}.$$

Now, we give an estimation for the rate of convergence in weighted space. In process we use following weighted space $C_K(\mathbb{R}_0)$ defined

$$C_K(\mathbb{R}_0)=\left\{f\in C_m(\mathbb{R}_0):\lim_{x\to\infty}\frac{f(x)}{1+x^m}=K_f\right\}$$

where K_f is a constant depending on f.

It is known that usual first modulus of continuity $\omega(f;\delta)$ does not tends to zero as $\delta\to 0$ on the infinite interval. Now we give weighted modulus of continuity $\Omega_m(f;\delta)$ and $\Omega_m^2(f;\delta)$ which tens to zero as $\delta\to 0$ on the infinite interval.

Let

$$\Omega_m(f;\,\delta)=\sup_{|h|\le\delta,\,x\in[0,\,b_\lambda],}\frac{|f(x+h)-f(x)|}{(1+x^m)(1+h^m)},\tag{9}$$

$$\Omega_m^2(f;\,\delta)=\sup_{|h|\le\delta,\,x\in[0,\,b_\lambda]}\frac{|f(x+2h)-2f(x+h)+f(x)|}{(1+x^m)(1+h^m)},\tag{10}$$

where $f\in C_K(\mathbb{R}_0)$.

For a lfrge x we have the estimate

$$\frac{|f(x+h)-f(x)|}{(1+x^m)(1+h^m)}\le 2^m\left|\frac{f(x+h)}{1+(x+h)^m}-K_f\right|+\left|\frac{f(x)}{1+x^m}-K_f\right|+\frac{2^m}{x}K_f.$$

Therefor, by the definition of $C_K(\mathbb{R}_0)$ for given $\varepsilon>0$ we can find $x_0=x_0(\varepsilon)$ such that the inequalities

$$\left|\frac{f(x+h)}{1+(x+h)^m}-K_f\right|<\frac{\varepsilon}{2^m\cdot 3},\ \left|\frac{f(x)}{(1+x^m)}-K_f\right|<\frac{\varepsilon}{3},\ \frac{1}{x}<\frac{\varepsilon}{2^m\cdot K_f\cdot 3}$$

holds for $x>x_0$. Therefore

$$\Omega_m(f;\,\delta)\le\sup_{0\le x\le x_0,\,|h|\le\delta}|f(x+h)-f(x)|+\sup_{x_0\le x\le b_\lambda,\,|h|\le\delta}\left|\frac{f(x+h)}{(1+(x+h)^m)}-K_f\right|\cdot 2^m+$$

$$+\sup_{x_0\le x\le b_\lambda}\left|\frac{f(x)}{(1+x^m)}-K_f\right|+\sup_{x_0\le x\le b_\lambda,\,|h|\le\delta}\frac{2^m}{x}K_f\le\omega(f;\,\delta)+\varepsilon.$$

Since $\overline{\lim_{\delta\to 0}}\,\Omega_m(f;\,\delta)<\varepsilon$ for any $\varepsilon>0$ we have $\lim\limits_{\delta\to 0}\Omega_m(f;\,\delta)=0$ for $f\in C_K(\mathbb{R}_0)$.

Theorem 5. *Let $f\in C_K(\mathbb{R}_0)$. Then*

$$\left\|\frac{L_\lambda(f;\,x)-f(x)}{2+x+x^2}\right\|_{m,\,[0,\,b_\lambda]}\le C_m\left(1+[H_m(\lambda)]^m\right)\times$$

$$\times\left(\Omega_m(f;\,K_m(\lambda))+\Omega_m^2(f;\,K_m(\lambda))\right)+\left|\left(\frac{\alpha_\lambda}{\lambda}\right)^2\frac{h(\lambda)}{\lambda}-2\frac{\alpha_\lambda}{\lambda}+1\right|+\frac{\alpha_\lambda}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)},$$

where C_m is a constant.

Proof. Using the Stieklov means for $f\in C_K(\mathbb{R}_0)$ given by

$$f_h(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\left[2f(x+s+t)-f(x+2(s+t))\right]dsdt,$$

we get

$$\frac{|f_h(x)-f(x)|}{1+x^m}\le\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\frac{|2f(x+s+t)-f(x+2(s+t))-f(x)|}{(1+x^m)}dsdt\le$$

$$\le\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\frac{|f(x+s+t)-f(x+2(s+t))|\,(1+(s+t)^m)}{(1+x^m)(1+(s+t)^m)}dsdt+$$

$$+\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\frac{|f(x+s+t)-f(x)|\,(1+(s+t)^m)}{(1+x^m)\,(1+(s+t)^m)}dsdt\leq$$

$$\leq\{\Omega_m(f;\,h)+\Omega_m(f;\,h)\}\,(1+h^m)=2\Omega_m(f;\,h)\,(1+h^m).$$

Therefore

$$\|f_h-f\|_{m,\,[0,\,b_\lambda]}\leq 2\Omega_m(f;\,h)\,(1+h^m). \tag{11}$$

Also, we have similarly in (11)

$$f_h'(x)=\frac{1}{h^2}\int_0^{\frac{h}{2}}8\left\{f\left(s+x+\frac{h}{2}\right)-f(s+x)\right\}-2\left\{f(2s+x+h)-f(2s+h)\right\}ds.$$

$$\frac{f_h'(x)}{(1+x^m)}=\frac{1}{h^2}\int_0^{\frac{h}{2}}\frac{8\left\{f\left(s+x+\frac{h}{2}\right)-f(s+x)\right\}-2\left\{f(2s+x+h)-f(2s+t)\right\}}{(1+x^m)}ds=$$

$$=\frac{1}{h^2}\int_0^{\frac{h}{2}}\frac{8\left\{f\left(s+x+\frac{h}{2}\right)-f(s+x)\right\}(1+(x+s)^m)\left(1+\left(\frac{h}{2}\right)^m\right)}{(1+x^m)\,(1+(x+s)^m)\left(1+\left(\frac{h}{2}\right)^m\right)}ds+$$

$$+\frac{1}{h^2}\int_0^{\frac{h}{2}}(1+(x+2s)^m)\,(1+h^m)\,\frac{2\left\{f(2s+x+h)-f(2s+t)\right\}}{(1+x^m)\,(1+(x+2s)^m)\,(1+h^m)}ds.$$

Using following inequalities for $x\geq 0$ and $s\geq 0$

$$\frac{(1+(x+s)^m)}{(1+x^m)}\leq 2^{m-1}\,(1+s^m),$$

we can write

$$\frac{f_h'(x)}{(1+x^m)}\leq\frac{1}{h^2}\int_0^{h/2}2^{m-1}\,(1+s^m)\left(1+\left(\frac{h}{2}\right)^m\right)\frac{8\left\{f\left(s+x+\frac{h}{2}\right)-f(s+x)\right\}}{(1+(x+s)^m)\left(1+\left(\frac{h}{2}\right)^m\right)}ds+$$

$$+\frac{1}{h^2}\int_0^{h/2}2^{m-1}\,(1+s^m)\,(1+h^m)\,\frac{2\left\{f(2s+x+h)-f(2s+t)\right\}}{(1+(x+2s)^m)\,(1+h^m)}ds\leq$$

$$\leq 8\frac{2^{m-1}}{h^2}\left(1+\left(\frac{h}{2}\right)^m\right)\int_0^{h/2}(1+s^m)\,\Omega_m\left(f;\,\frac{h}{2}\right)ds+$$

$$+2\frac{2^m}{h^2}\,(1+h^m)\int_0^{h/2}(1+s^m)\,\Omega_m(f;\,h)\,ds\leq 10\frac{2^m}{h}\,(1+h^m)\,\Omega_m(f;\,h).$$

Consequently
$$\left\| f_h^{'}(x) \right\|_{m,\,[0,\,b_\lambda]} \le \frac{2^m}{h}(1+h^m)\,\Omega_m(f;\,h)\,. \tag{12}$$

Moreover we can write
$$\frac{\left|f_h^{''}(x)\right|}{(1+x^m)} = h^{-2}\frac{8\left\{f(x+h)-2f\left(\frac{h}{2}+x\right)-f(x)\right\}\left(1+\left(\frac{h}{2}\right)^m\right)}{(1+x^m)\left(1+\left(\frac{h}{2}\right)^m\right)}+$$
$$+h^{-2}\frac{\{f(x+2h)-2f(x+h)+f(x)\}(1+h^m)}{(1+x^m)(1+h^m)} \le h^{-2}9(1+h^m)\,\Omega_m^2(f;\,h)\,.$$

Thus
$$\left\| f_h^{''}(x) \right\|_{m,\,[0,\,b_\lambda]} \le h^{-2}9(1+h^m)\,\Omega_m^2(f;\,h)\,. \tag{13}$$

From (1) we can write
$$\begin{aligned} \left\| \tfrac{L_\lambda(f;\,x)-f(x)}{2+x+x^2} \right\|_{m,\,[0,\,b_\lambda]} &\le \left\| \tfrac{L_\lambda(f-f_h;\,x)}{2+x+x^2} \right\|_{m,\,[0,\,b_\lambda]} \\ &+ \left\| \tfrac{L_\lambda(f_h;\,x)-f_h(x)}{2+x+x^2} \right\|_{m,\,[0,\,b_\lambda]} + \left\| \tfrac{f_h(x)-f(x)}{2+x+x^2} \right\|_{m,\,[0,\,b_\lambda]}\,. \end{aligned} \tag{14}$$

By (11) we get
$$\left\| \frac{L_\lambda(f-f_h;\,x)}{2+x+x^2} \right\|_{m,\,[0,\,b_\lambda]} \le \|f_h-f\|_{m,\,[0,\,b_\lambda]} \le 2\Omega_m(f;\,h)(1+h^m)\,. \tag{15}$$

Also using Theorem 1, (13), (14) and (15) we get
$$\left\| \frac{L_\lambda(f_h;\,x)-f_h(x)}{2+x+x^2} \right\|_{m,\,[0,\,b_\lambda]} \le \left\| f_h^{'} \right\|_{m,\,[0,\,b_\lambda]} \left| \frac{\alpha_\lambda}{\lambda}-1 \right| + \left\| f_h^{''} \right\|_{m,\,[0,\,b_\lambda]} \{H_m(\lambda)+$$
$$+\left| \left(\frac{\alpha_\lambda}{\lambda}\right)^2 \frac{h(\lambda)}{\lambda} - 2\frac{\alpha_\lambda}{\lambda}+1 \right| + \frac{\alpha_\lambda}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)} \Big\} \le$$
$$\le \frac{2^m}{h}(1+h^m)\,\Omega_m(f;\,h)\left|\frac{\alpha_\lambda}{\lambda}-1\right| x + h^{-2}9(1+h^m)\,\Omega_m^2(f;\,h)\{H_m(\lambda)+$$
$$+\left| \left(\frac{\alpha_\lambda}{\lambda}\right)^2 \frac{h(\lambda)}{\lambda} - 2\frac{\alpha_\lambda}{\lambda}+1 \right| + \frac{\alpha_\lambda}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)} \Big\}\,.$$

If we choose $h = K_m(\lambda) = \max\left\{ \sqrt{H_m(\lambda)},\ \sqrt{\left|\frac{\alpha_\lambda}{\lambda}-1\right|} \right\}$ we obtain claimed result.

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References

[1]. Agratini O. *On approximation properties of Balázs-Szabados operators and their Kantorovich extension.* Korean J. Comput. and Appl. Math., v.9, (2002), No 2, pp. 361-372

[2] Altomare F., Campiti M. *Korovkin-Type Approximation Theory and Its Applications.* 1994, New York.

[3]. Dogru O. *On a certain family of linear positive operators.* Tr. J. of Math., v. 21, (1997), pp.337-389.

[4]. Gadjiev A. D., Atakut C. *On approximation of unbounded functions by the generalized Baskakov operators.* Trans. Acad. Sci. Azerb., XXII, No:1, (2003), pp.33-42.

[5]. Gadjiev A. D., Ispir N. *On a sequence of linear positive operators in weighted space.* Proc. of Imm of Azerb. As., v. XI (XIX), (1999), pp. 45-56.

[6]. Gadziev A. D. *The convergence problem for a sequence of positive linear operators on bounded sets and theorems analogous to that of P. P. Korovkin.* Soviet Math. Dokl., v.15, No:5, (1974)

[7]. Gadziev A. D. *On P. P. Korovkin type theorems.* Math. Zamet., 20 (1976) pp. 781-786. (in Russian)

[8] Ibragimov I.I., Gadziev A. D. *On a sequence of linear positive operators, Soviet Math.* Dokl., v.11, (1970) No:4, pp. 1092-1095.

[9]. Radatz P. and Wood B. *Approximation derivatives of functions unbounded on the positive axis with linear operators.* Rev. Roum. Math. Pures et Appl. Bucarest, v. XXIII, No. 5 (1078), pp. 771-781.

[10]. Lesniewicz M. and Rempulska L. *Approximation by some operators of the Százs- Mırakyan type in exponential weighted space.* Glasnik Math. v. 32 (52) (1997), pp. 57-69.

[11]. Becker M. *Global approximation theorems for Szá zs-Mirakyan and Baskakov operators in polynomial weighted space.* Indiana Univ. Math. Math. J. v. 27 (1) (1978), pp. 127-142.

[12]. Pogány Tibor K. *Some Korovkin-type theorems for stochastic processes.* Teor. Imovi. Mat. Stat. No. 61 (1999), 145–151; translation in Theory Probab. Math. Statist. No. 61 (2000), pp. 153–159 (2001)

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