Melik-Bakhish A. BABAYEV, Nurana A. NOVRUZOVA

ON DE LA VALLE-POUSSIN TYPE THEOREM

Abstract

De La Valle-Poussin type lower estimates of the best approximation by the set of superposiyions of the sums of one-variable functions are established. This set contains some subclasses of Ridge functions set. The approximation is considered in a uniform metrics.

Recently Ridge functions are intensively studied. The function $h : R^n \to R$ of the form

$$h(x_1, ..., x_n) \equiv g(a_1 x_1 + ... + a_n x_n) = g(ax),$$

where $g : R \to R$ and $(a_1, ..., a_n) \in R^n \backslash 0$ is called Ridge functions.

Consider manifold superpositions of sum of functions of one variable

$$g[\varphi(x) + \psi(y)] \tag{1}$$

In the paper de la Valle Poussin type lower bounds of the best approximation of superpositions of monifold of form (1) are constructed.

In particular case $\varphi(x) = ax$, $\psi(y) = by$ the superposition of form (1) terns to Ridge functions of the form

$$g(ax + by).$$

1. Let the boundary function $f = f(x, y)$ be determined on the unique square $T = [0,1]^2$. Denote by $G_{\infty,m,\infty}$ a class of functions of form (1) being polynomial of the order m with respect to x. Let $M = M(T)$ be a set of boundary functions on T with the norm

$$\|f\|_{M(T)} = \sup_{(x,y)\in T} |f(x, y)| .$$

Consider the best approximation of the function $f \in M(T)$ by the class $G_{\infty,m,\infty}$

$$E[f, G_{\infty,m,\infty}, M(T)] = \inf_{g\in G_{\infty,m,\infty}} \|f(x, y) - g[\varphi(x) + \psi(y)]\|_{M(T)} .$$

Assume that there exists the function $g_0 \in G_{\infty,m,\infty}$ for which at the fixed value $y \in [0, 1]$ the difference $f - g_0$ at the points $(x_i, y) \in T$, $i = \overline{1,k}$ satisfying the condition $x_1 < x_2 < ... < x_k$ gets the consequently changing values

$$\lambda_1(y) - \lambda_2(y), ..., (-1)^{k-1} \lambda_k(y);\ \lambda_i(y) > 0. \tag{2}$$

Denote by J the set of these y.

Theorem 1. *At $k \geq m + 2$ the estimation*

$$E[f, G_{\infty,m,\infty}, M(T)] \geq \sup_{y\in J} \min\{\lambda_1(y), ..., \lambda_k(y)\}$$

is true.

We need the auxiliary lemma for proving the theorem

Lemma 1. *For arbitrary $g \in G_{\infty,m,\infty}$, $y \in J$ the estimation*

$$\sup_{x\in[0,1]} |f(x,y) - g[\varphi(x) + \psi(y)]| \geq \min\{\lambda_1(y), ..., \lambda_k(y)\}$$

is true.

Proof of lemma 1. Suppose the contrary. Let there exist the functions $g \in G_{\infty,m,\infty}$ for which the inequality

$$\sup_{x\in[0,1]} |f(x,y) - g[\varphi(x) + \psi(y)]| < \min\{\lambda_1(y), ..., \lambda_k(y)\} \tag{3}$$

is true.

Denote by $\Delta(x,y)$ the difference

$$\Delta(x,y) = g - g_0 = \{f - g_0\} - \{f - g\}.$$

In the considered value y according to the definition $\lambda_1(y)$ we have:

$$\Delta(x_1,y) = [f - g_0](x_1,y) - [f - g](x_1,y) = \lambda_1(y) - [f - g](x_1,y) > 0\ .$$

Analogously by view of the definition $\lambda_2(y)$ we'll obtain:

$$\Delta(x_2,y) = -\lambda_2(y) - [f - g_0](x_2,y) < 0\ .$$

Continuing this process we'll receive that the difference $\Delta(x,y) = g - g_0$ in the point $m+1$ changes the sign consequently. It is impossible since according to the condition the difference $g - g_0$ is polynomial of the power m. Consequently our assumption is wrong, i.e. non function $g \in G_{\infty,m,\infty}$ can satisfy the inequality (3). Consequently for arbitrary function $g \in G_{\infty,m,\infty}$ the relation

$$\sup_{x\in[0,1]} |f(x,y) - g[\varphi(x) + \psi(y)]| \geq \min\{\lambda_1(y), ..., \lambda_k(y)\}$$

is true.

Lemma 1 is proved.

Proof of theorem 1. According to lemma 1 we have

$$\sup_{\substack{x\in[0,1]\\ y\in[0,1]}} |f(x,y) - g[\varphi(x) + \psi(y)]| \geq \min\{\lambda_1(y), ..., \lambda_k(y)\}$$

The left hand side of the last inequality doesn't depend on y. It allows to write

$$\|f(x,y) - g[\varphi(x) + \psi(y)]\|_{M(T)} \geq \sup_{y\in J} \min\{\lambda_1(y), ..., \lambda_k(y)\}$$

Finally allowing for that the right hand side of obtained inequality doesn't depend on $g \in G_{\infty,m,\infty}$ we receive

$$E[f, G_{\infty,m,\infty}, M(T)] = \inf_{g\in G_{\infty,m,\infty}} \|f(x,y) - g[\varphi(x) + \psi(y)]\|_{M(T)} \geq$$
$$\geq \sup_{y\in J} \min\{\lambda_1(y), ..., \lambda_k(y)\}\ .$$

Theorem 1 is proved.

2. Denote by $G_{\infty,\infty,n}$ the set of functions of the form $g[\varphi(x)+\psi(y)]$ being polynomial of the order n relative to the variable y and consider the approximation $E[f, G_{\infty,\infty,n}, M(T)]$. Analogously to lemma 1 and theorem 1 we can prove the following.

Lemma 2. *Let for some function $g_0 \in G_{\infty,\infty,n}$ the difference $f-g_0$ for the fixed value $x\in[0,1]$ in $N\geq n+2$ points $(x,y_i)\in T$ satisfying the conditions $y_1<y_2<...<y_N$ consequently changing the sign take the value*

$$\lambda_1^{(1)}(x), -\lambda_2^{(1)}(x), ..., (-1)^{N-1}\lambda_N^{(1)}(x), \quad \lambda_i(x)>0 \ . \qquad (4)$$

Denote by I the set of the values $x\in[0,1]$ for which the relations (4) are fulfilled. Then for the arbitrary function $g\in G_{\infty,\infty,n}$ the relation $(x\in I)$ is true

$$\sup_{y\in[0,1]} |f(x,y)-g[\varphi(x)+\psi(y)]| \geq \min\left\{\lambda_1^{(1)}(x), ..., \lambda_N^{(1)}(x)\right\}.$$

Theorem 2. *At $N\geq n+2$ the lower bound is true*

$$E[f, G_{\infty,\infty,n}, M(T)] \geq \sup_{x\in I}\min\left\{\lambda_1^{(1)}(x), ..., \lambda_N^{(1)}(x)\right\}.$$

Denote by $G_{\infty,m,n}$ the set of functions of the form (1) being polynomial of the m-th order by x, and n-th order by y.

Corollary 1. *At the conditions of the theorems 1 and 2 the estimation*

$$E[f, G_{\infty,m,n}, M(T)] \geq$$
$$\geq \max\left\{\sup_{x\in J}\min\left[\lambda_1^{(1)}(x), ..., \lambda_N^{(1)}(x)\right], \sup_{x\in I}\min\left[\lambda_1(y), ..., \lambda_k(y)\right]\right\}.$$

is true.

The functions which satisfy the relation (2) may be more than one. Denote by H the set of such functions of the form (1). Thus for every function $h\in H$ in K_h points $(x_{h,i}, y_h)\in T$ satisfying the condition

$$x_{h,1}<x_{h,2}<...<x_{h,k}$$

the difference $f-h$ changing the sign consequently gets the values

$$\lambda_{h,1}(y_h), -\lambda_{h,2}(y_h), ..., (-1)^{k_n-1}\lambda_{h,k_n}(y_h), \lambda_{h,i}(y_h)>0, \quad i=\overline{1,k_h} \ .$$

Denote by J_h the set of the points y_h for which above mentioned conditions for the function $h\in H$ are fulfilled.

Then from the theorem 1 we obtain

Corollary 2. *At $\inf_{h\in H} k_h \geq m+2$ the estimation*

$$E[f, G_{\infty,m,n}, M(T)] \geq \sup_{h\in H}\sup_{y\in J_h}\min\left[\lambda_{h,1}(y_h), ..., \lambda_{h,k_h}(y_h)\right]$$

is true.

Analogously to theorem 2 and corollary 1 we can reduce the best approximation of the estimations

$$E[f, G_{\infty,\infty,n}, M(T)] \ .$$

References

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Melik-Bakhish A. Babayev
Institute of Mathematics and Mechanics of NAS Azerbaijan.
9, F.Agayev str.,Az1141, Baku, Azerbaijan.
Tel.: 38-62-17 (off.)

Nurana A. Novruzova
Lenkaran State University.
50, General Hazi Aslanov str.,Az3730,Lenkaran, Azerbaijan.

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