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MATHEMATICAL MODELS OF QUEUEING SYSTEMS WITH CYCLIC SERVICES

Abstract

In the given paper the mathematical model of queueing systems is constructed and operating by cyclic moment expected waiting time is minimized.

One of the main characteristics of queueing systems is the expected waiting time before service. Because of nonregularity of times when service starts, short intervals between times when service starts can appear. Therefore one of the control possibilities, improving service, is delay of service start. In [1-3] control problems in a class of delays are considered for some class of queueing systems.

Let us consider a queueing system, where $t_1, t_2, ..., t_n, ...$ are times when service starts. Denote $\eta_1 = t_1,\ \eta_i = t_i - t_{i-1},\ i = 2, 3, ...$ Let $\eta_1, \eta_2, ..., \eta_n, ...$ be independent and differently distributed random variables. The aim of the present paper is controlling the times when service starts, to minimize the expected waiting time before service.

The flow of service requests is stationary with intensity μ and does not depend on $t_1, t_2, ..., t_n$. It is assumed that service time is time after the service request to nearest time when service starts. At the instant t_i all service requests, which arrived during the time interval $[t_{i-1}, t_i)$ are served immediately.

Denote by W, σ^2 be the expectations and variance of the waiting times until service.

Let s_i be the epoch of the arrival of the customer i, $\psi(s_i)$ be the number of customers, who arrived at the epoch s_i, $\phi_i = \{s : \psi(s) > 0,\ s \in [t_{i-1}, t_i)\}$, V_i be the waiting time of all customers who arrived during the time interval η_i, $R(t)$ be the number of customers, who arrived during the time interval $[0, t)$.

Accoring to Campbell's formula [4]

$$E(V_i/\eta_i) = E\left\{\sum_{s\in\phi_i}(t_i - s)\,\psi(s)/\eta_i\right\} = \int\limits_{t_{i-1}}^{t_i}(t_i - s)\,\mu ds = \mu\frac{\eta_i^2}{2}.$$

Denote by W_N the conditional expectation of the waiting time of all customers for given $\eta_1, \eta_2, ..., \eta_N$. Then

$$W_N = E\{E(W_n/\eta_1, \eta_2, ..., \eta_N)\} = \sum_{i=1}^{N} E\{E(V_i/\eta_1, \eta_2, ..., \eta_N)\} =$$

$$= \sum_{i=1}^{N} E(EV_i/\eta_i) = \sum_{i=1}^{N} E\left(\frac{\mu\eta_i^2}{2}\right) = \mu\frac{\sum\limits_{i=1}^{N} E\eta_i^2}{2},$$

$$W=\lim_{N\to\infty}\frac{W_N}{R\left(t\right)}=\lim_{N\to\infty}\frac{\sum\limits_{i=1}^{N}E\eta_i^2}{2\sum\limits_{i=1}^{N}E\eta_i}. \tag{1}$$

Similarly, for the variance we have

$$E\left(V_i/\eta_i\right)=E\left\{\sum_{s\in\phi_i}\left(t_i-s\right)^2\psi\left(s\right)/\eta_i\right\}=\int\limits_{t_{i-1}}^{t_i}\left(t_i-s\right)^2\mu ds=\frac{\mu\eta_i^3}{3},$$

$$W_N=\mu\frac{\sum\limits_{i=1}^{N}E\eta_i^3}{3},$$

$$W=\lim_{N\to\infty}\frac{\sum\limits_{i=1}^{N}E\eta_i^3}{3\sum\limits_{i=1}^{N}E\eta_i},$$

$$\sigma^2=\lim_{N\to\infty}\frac{\sum\limits_{i=1}^{N}E\eta_i^3}{3\sum\limits_{i=1}^{N}E\eta_i}-\left(\lim_{N\to\infty}\frac{\sum\limits_{i=1}^{N}E\eta_i^2}{2\sum\limits_{i=1}^{N}E\eta_i}\right)^2. \tag{2}$$

The analogous expressions for the case of identically distributed random variables were obtained in [5].

For systems with recurrent service one of the possible methods of service improving is introduction of delays at which intervals between successive services can increase. Therefore from the sequence $t_1,t_2,...,t_n,...$ we pass to new sequence $t_1^*,t_2^*,...,t_n^*,...$, for which $\eta_1^*=t_1^*-t_{i-1}^*=\eta_i+g_i\left(\eta_i\right),\ \ \eta_1=t_1^*,\ g_i\in G$ where G is a class of measurable nonnegative functions. Let $W^*,\ \sigma^{*2}$ be the expectations and variance of the delayed waiting times until service for the sequence $\eta_1^*,\eta_2^*,...,\eta_n^*,...$. By analogy with (1) and (2) we have

$$W^*=\lim_{N\to\infty}\frac{\sum\limits_{i=1}^{N}E\eta_i^{*2}}{2\sum\limits_{i=1}^{N}E\eta_i^*}, \tag{1'}$$

$$\sigma^{*2}=\lim_{N\to\infty}\frac{\sum\limits_{i=1}^{N}E\eta_i^{*3}}{3\sum\limits_{i=1}^{N}E\eta_i^*}-\left(\lim_{N\to\infty}\frac{\sum\limits_{i=1}^{N}E\eta_i^{*2}}{2\sum\limits_{i=1}^{N}E\eta_i^*}\right). \tag{2'}$$

Denote the difference $W^* - W$ by $M_{F_i}(g_i)$

$$M_{F_i}(g_i) = \lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N} E\eta_i^{*2}}{2\sum_{i=1}^{N} E\eta_i^{*}} - c_N\right) =$$

$$= \lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N} E\left(\eta_i^2+2\eta_i g_i(\eta_i)+g_i^2(\eta_i)\right)}{2\sum_{i=1}^{N} E(\eta_i+g_i(\eta_i))} - c_N\right) =$$

$$= \lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N}\int_0^\infty \left(x^2/2+xg_i(x)+g_i^2(x)/2\right)dF_i(x) - c_N\sum_{i=1}^{N}\int_0^\infty x dF_i(x) - c_N\sum_{i=1}^{N}\int_0^\infty g_i(x)dF_i(x)}{\sum_{i=1}^{N}E\eta_i + \sum_{i=1}^{N}\int_0^\infty g_i(x)dF_i(x)}\right) =$$

$$= \lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N}\int_0^\infty g_i(x)(g_i(x)/2+x-c_N)dF_i(x) + \sum_{i=1}^{N}\int_0^\infty \frac{x^2}{2}dF_i(x) - c_N\sum_{i=1}^{N}\int_0^\infty x dF_i(x)}{\sum_{i=1}^{N}E\eta_i + \sum_{i=1}^{N}\int_0^\infty g_i(x)dF_i(x)}\right) =$$

$$= \lim_{N\to\infty}\frac{\sum_{i=1}^{N}\int_0^\infty g_i(x)\left(g_i(x)/2+x-c_N\right)dF_i(x)}{\sum_{i=1}^{N}\left(E\eta_i + \int_0^\infty g_i(x)\,dF_i(x)\right)}. \tag{3}$$

$$c_N = \frac{\sum_{i=1}^{N} E\eta_i^2}{2\sum_{i=1}^{N} E\eta_i}. \tag{4}$$

In order to gain in expected waiting time before service is derived the existence for $F_i(x)$ of a function $g_i(x) \in G$ such that $M_{F_i}(g_i) < 0$, i.e.

$$\sum_{i=1}^{N}\int_0^\infty g_i(x)\left(g_i(x)/2+x-c_N\right)dF_i(x) < 0.$$

is necessary.

Definition 1. *We call queue discipline (q.d.) with distribution function (d.f.) $F_i(x)$ improved if there exists $g_i(x) \in G$, such that $M_{F_i}(g_i) < 0$.*

Theorem 1. *In order q.d. with d.f.* $F(x) = (F_1(x), F_2(x), ..., F_N(x))$ *be improved it is necessary and sufficient that there exist such point* $x < c_N$ *and such* i *that* $F_i(x) > 0$*, i.e.* $F_i(x)$ *have growing point situated at the left of point* c_N.

Proof. Let there exist $F_i(x)$ and $g_i(x) \in G$ so that

$$\int\limits_0^\infty g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) < 0. \tag{5}$$

If $F_i(x) = 0$ for any $x < c_N$, then

$$\int\limits_0^\infty g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) = \int\limits_0^{c_N} g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) +$$

$$+\int\limits_{C_N}^\infty g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) = \int\limits_{c_N}^\infty g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) \geq 0,$$

which contradicts to (5).

Let $F_i(x)$ have growing point at the left of c_N. Then

$$\int\limits_0^\infty g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) =$$

$$= \int\limits_0^{C_N} g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) + \int\limits_{c_N}^\infty g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x)$$

We choose $0 < g_i(x) < \max(2(c_N - x), 0)$. Then

$$\int\limits_0^\infty g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) = \int\limits_0^{c_N} g_i(x)\left(g_i(x)/2 + x - c_N\right) dF_i(x) < 0,$$

i.e. q.d. with d.f. $F(x)$ is improved. Thereby the theorem is proved.

Definition 2. *We call function* $\widetilde{g}_i(x) \in G$ *an optimal function for queue discipline with d.f.* $F_i(x)$ *if*

$$\min_{g_i \in G} M_{F_i}(g_i) = M_{F_i}(\widetilde{g}_i).$$

Theorem 2. *Under the conditions of theorem 1, optimal function is*

$$\widetilde{g}_i(x) = (c_N - x)^+ = \max(0, (c_N - x))$$

where c_N *is a unique solution of the equation*

$$c_N E\eta_i + \int\limits_0^{c_N} (c_N - x) F_i(x)\, dx = \frac{E\eta_i^2}{2}. \tag{6}$$

Proof. Let $\eta_i^* = \eta_i + g_i(\eta_i) = \varphi_i(\eta_i)$ where $\varphi_i(x) \geq x$ because $g_i(x) \geq 0$. Let c_N satisfy the equation

$$c_N^2 = \int_{c_N}^{\infty} (x - c_N)^2 \, dF_i(x). \tag{7}$$

Consider the difference

$$W^* - c_N = \lim_{N\to\infty} \frac{\sum_{i=1}^{N} E\eta_i^{*2}}{2\sum_{i=1}^{N} E\eta_i^*} - c_N =$$

$$= \lim_{N\to\infty} \frac{\sum_{i=1}^{N} \int_0^{\infty} (\varphi_i(x) - c_N)^2 \, dF_i(x) - \sum_{i=1}^{N} \int_{c_N}^{\infty} (x - c_N)^2 \, dF_i(x)}{2\sum_{i=1}^{N} \int_0^{\infty} \varphi_i(x) \, dF_i(x)}. \tag{8}$$

The numerator of the function in (8) is nonnegative because $\varphi_i(x) \geq x$ and it equals to zero if

$$\varphi_i(x) = x + (c_N - x)^+ .$$

Consequently, $\tilde{g}_i(x) = (c_N - x)^+$.

Equation (7) is nothing but other notation of equation (6). Indeed,

$$c_N^2 = \int_{c_N}^{\infty} (x - c_N)^2 \, dF_i(x),$$

$$c_N^2 = \int_0^{\infty} (x - c_N)^2 \, dF_i(x) - \int_0^{c_N} (x - c_N)^2 \, dF_i(x),$$

$$c_N^2 = \int_0^{\infty} x^2 dF_i(x) - 2\int_0^{\infty} c_N x dF_i(x) + \int_0^{\infty} c_N^2 dF_i(x) - \int_0^{c_N} (x - c_N)^2 \, dF_i(x),$$

$$2c_N E\eta_i + \int_0^{c_N} (x - c_N)^2 \, dF_i(x) = E\eta_i^2,$$

$$c_N E\eta_i + \frac{1}{2}\int_0^{c_N} (x - c_N)^2 \, dF_i(x) = \frac{E\eta_i^2}{2},$$

$$c_N E\eta_i + \int_0^{c_N} (c_N - x) \, F_i(x) \, dx = \frac{E\eta_i^2}{2}.$$

As the left part of the equation is a strictly monotone function of c_N and the right part is constant solution of equation (6) is unique. Hence, theorem 2 is proved.

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