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THE FIRST BOUNDARY VALUE PROBLEM FOR CORDES TYPE LINEAR NON-DIVERGENT PARABOLIC EQUATIONS OF THE SECOND ORDER

Abstract

Dirichlet problem is considered for linear non-divergent parabolic equations of the second order with generally speaking discontinuous coefficients satisfying Cordes condition. The one-valued, strongly (almost everywhere) solvability of this problem is proved in the space $\dot{W}_p^{2,1}$ where p belongs to same segment containing the point 2 .

Introduction. Let $\mathbb{E}_n$ and $\mathbb{R}_{n+1}$ be n-dimensional and $(n+1)$-dimensional Euclidean spaces of the points $x=(x_1,...,x_n)$ and $(x,t)=(x_1,...,x_n,t)$ respectively, $\Omega\in\mathbb{E}_n$ be bounded domain with the boundary $\partial\Omega\in C^2$, $B_R^{x^0}\subset\Omega$ be n-dimensional open sphere of the radius R with the center at the point $x^0=\left(x_1^0,...,x_n^0\right)$, $Q_T=\{(x,t):x\in\Omega,0<t<T<\infty\}$, $S_T=\{(x,t):x\in\partial\Omega,0<t<T\}$, $\Gamma(Q_T)$ be parabolic boundary Q_T, i.e. $\Gamma(Q_T)=S_T\cup\{(x,t):x\in\Omega,t=0\}$, $Q_R^T=B_R^{x^0}\times(0,T)$, $\mathcal{A}\left(Q_R^T\right)$ be the set of all functions $u(x,t)$ from $C^\infty\left(\bar{Q}_R^T\right)$ with support in $B_\rho^{x^0}\times[0,T]$, $\rho<R$ for which $u(x,0)=0$. Consider in the domain Q_T the first boundary value problem for linear parabolic equation of the form

$$\mathcal{L}u=\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2u}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)\,u-\frac{\partial u}{\partial t}=f(x,t), \tag{1}$$

$$u|_{\Gamma(Q_T)}=0, \tag{2}$$

under the assumption that $\|a_{ij}(x,t)\|$ is a real symmetrical matrix, moreover for all $(x,t)\in Q_T$ and $\xi\in\mathbb{E}_n$ the condition

$$\gamma\,|\xi|^2\le\sum_{i,j=1}^n a_{ij}(x,t)\,\xi_i\xi_j\le\gamma^{-1}|\xi|^2,\ \gamma\in(0,1]-const \tag{3}$$

is fulfilled.

Besides we'll suppose that all coefficients of the operator $\mathcal{L}$ are real and measurable in Q_T functions.

The aim of the present paper is finding the conditions on coefficients of equation (1) by fulfilling of which the first boundary value problem (1)-(2) on identically strong (almost everywhere) solvable in the space $\dot{W}_p^{2,1}(Q_T)$ for any $f(x,t)\in\in L_p(Q_T)$, $p\in[p_1,p_2]$, where $p_1\in(1,2)$, $p_2\in(2,\infty)$.

In the case when the coefficients of the equations (1) and its right-hand side satisfy in Q_T Hölder uniform condition, one-valued classical solvability of the problem (1)-(2) is determined in [1]. In [2] it is shown that for correctness of indicated fact we can weaken Hölder uniform condition till the Dini uniform condition. If the leading coefficients of the operator $\mathcal{L}$ are uniformly continuous in Q_T and minor coefficients are elements of corresponding Lebesgue spaces then uniform strong (almost everywhere) solvability of the problem (1)-(2) in the space $\dot{W}_p^{1,2}(Q_T)$, $p\in(1,\infty)$ is proved in [3]-[4]. In [5]-[6] the one-valued strong solvability of the problem (1)-(2) in the absence thereof minor terms in the space $\dot{W}_2^{1,2}(Q_T)$ in case when the matrix of leading coefficients of the operator $\mathcal{L}$ satisfies the Cordes type condition, is determined. The example indicating the exactness of Cordes condition is in [6]. In [7]-[8] the indicated fact is transported on the class of non-linear parabolic equations of the second order. Indicate also papers [9]-[11], in which the strong solvability of Dirichlet problem for elliptic equations of the second order of non-divergent structure with discontinuous coefficients is studied.

1^0. Some auxiliary assertions. Let agree at first in some notation and definition. We'll denote by u_i, u_{ij} and u_t the derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial u}{\partial t}$, respectively; $i,j=1,...,n$. Let $W_p^{1,0}(Q_T)$ and $W_p^{2,1}(Q_T)$ be Banach spaces of the measurable functions $u(x,t)$ given on Q_T with finity norms

$$\|u\|_{W_p^{1,0}(Q_T)} = \left(\int\limits_{Q_T} \left(|u|^p + \sum_{i=1}^n |u_i|^p \right) dxdt \right)^{\frac{1}{p}}$$

and

$$\|u\|_{W_p^{2,1}(Q_T)} = \left(\int\limits_{Q_T} \left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p + |u_t|^p \right) dxdt \right)^{\frac{1}{p}},$$

respectively. Denote by $\dot{W}_p^{2,1}(Q_T)$ the subspace $W_p^{2,1}(Q_T)$, in which dense set is collection of all functions from $C^\infty\left(\bar{Q}_T\right)$ vanishing on $\Gamma(Q_T)$.

The function $u(x,t) \in \dot{W}_p^{2,1}(Q_T)$ is called strong solvability of the first boundary value problem (1)-(2) if it satisfies equation (1) almost everywhere in Q_T.

Further everywhere the note $C(...)$ means that the positive constant C depends only on the contents of parenthesis.

Lemma 1. *If* $u(x,t) \in \mathcal{A}\left(Q_R^T\right)$, *then*

$$\int\limits_{Q_R^T} \left(\sum_{i,j=1}^n u_{ij}^2 + u_t^2 \right) dxdt \le \int\limits_{Q_R^T} (M_0 u)^2 \, dxdt,$$

where $M_0 = \Delta - \frac{\partial}{\partial t}$.

Proof. We have

$$\int\limits_{Q_R^T} (M_0 u)^2 \, dxdt = \int\limits_{Q_R^T} \left((\Delta u)^2 - 2\Delta u \ u_t + u_t^2 \right) dxdt =$$

$$= \int\limits_{Q_R^T} \left(\sum_{i,j=1}^n u_{ii} u_{jj} - 2 \sum_{i=1}^n u_{ii} u_t + u_t^2 \right) dxdt = -\int\limits_{Q_R^T} \sum_{i,j=1}^n u_i u_{jji} dxdt +$$

$$+2\int_{Q_R^T}\sum_{i=1}^{n}u_iu_{it}dxdt+\int_{Q_R^T}u_t^2dxdt=\int_{Q_R^T}\left(\sum_{i,j=1}^{n}u_{ij}^2+u_t^2\right)dxdt+$$

$$+\int_{Q_R^T}\sum_{i=1}^{n}\left(u_i^2\right)_t dxdt=\int_{Q_R^T}\left(\sum_{i,j=1}^{n}u_{ij}^2+u_t^2\right)dxdt+\int_{B_R^{x^0}}\sum_{i=1}^{n}\left(u_i^2\left(x,T\right)-u_i^2\left(x,0\right)\right)dx.$$

Since $u\left(x,0\right)=0$ then hence it follows the required inequality.

Lemma 2. *If* $u\left(x,t\right)\in\mathcal{A}\left(Q_R^T\right)$ and $p\in\left(1,\infty\right)$, then

$$\int_{Q_R^T}\left(\sum_{i,j=1}^{n}\left|u_{ij}\right|^p+\left|u_t\right|^p\right)dxdt\le C_1\left(p,n\right)\int_{Q_R^T}\left|M_0u\right|^p dxdt.$$

Proof. Let

$$F\left(x,t\right)=\Delta u\left(x,t\right)-u_t\left(x,t\right),$$

$$G\left(x,t\right)=\begin{cases} a_0t^{-\frac{n}{2}}\exp\left(-\frac{|x|^2}{4t}\right), & \text{at } t>0,\\ 0, \quad \text{at} \quad t\le 0,\end{cases}$$

where $a_0=2^{-n}\pi^{-\frac{n}{2}}$. Then

$$u\left(x,t\right)=\int_{Q_R^T}G\left(x-y,t-\tau\right)F\left(y,\tau\right)dyd\tau.$$

For $i=1,...,n$ we have

$$u_i\left(x,t\right)=\int_{Q_R^T}G_i\left(x-y,t-\tau\right)F\left(y,\tau\right)dyd\tau=\int_{Q_R^T}G_i\left(y-x,t-\tau\right)F\left(y,\tau\right)dyd\tau=$$

$$=\int_{\mathbb{R}_{n+1}}G_i\left(\vartheta,t-\tau\right)F\left(\vartheta+x,\tau\right)d\vartheta d\tau.$$

Further acting as at differentiation of integrals with weak singularity [12] we obtain

$$u_{ij}\left(x,t\right)=-G_{ij}*F+F\left(x,t\right)\lim_{\rho\to 0}\int_{\partial B_{0,1/\rho}^{(x,t)}}G_i\left(x-y,t-\tau\right)\cos\left(\bar{n},y_j\right)ds_{y,\tau},$$

where

$$G_{ij}*F=\lim_{\rho\to 0}\int_{B_{0,1/\rho}^{(x,t)}}G_{ij}\left(x-y,t-\tau\right)F\left(y,\tau\right)dyd\tau,$$

$$B_{0,1/\rho}^{(x,t)}=\left\{\left(y,\tau\right):0<\frac{G\left(x-y,t-\tau\right)}{t-\tau}<\frac{1}{\rho}\right\}.$$

Let's calculate

$$J_{ij}(\rho)=\int\limits_{\partial B^{(x,t)}_{0,1/\rho}} G_i(x-y,t-\tau)\cos(\bar n,y_j)\,ds_{\tau,y}=\int\limits_{\partial B^{(0,0)}_{0,1/\rho}} G_i(-y,-\tau)\cos(\bar n,y_j)\,ds_{\tau,y}=$$

$$=\tfrac{1}{\rho}\int\limits_{\partial B^{(0,0)}_{0,1/\rho}} \tfrac{y_i}{2}\cos(\bar n,y_j)\,ds_{y,\tau}.$$

If $i\neq j$, then $J_{ij}=0$. Let now $i=j$. Consider for example the case $i=j=n$ since in all remaining cases the proof is analogous. Denote by S_ρ that part $\partial B^{(0,0)}_{0,1/\rho}$ on which $y_n>0$ and by Π_ρ the projection S_ρ on hyperplane $y_n=0$. Then

$$J_{nn}(\rho)=\frac{2}{\rho}\int\limits_{S_\rho}\frac{y_n}{2}\cos(\bar n,y_n)\,ds_{y,\tau}=\frac{2}{\rho}\int\limits_{\Pi_\rho}\frac{y_n}{2}dy_1...dy_{n-1}d\tau=$$

$$=\frac{2}{\rho}\int\limits_{\Pi_\rho}\sqrt{\frac{n+2}{2}(-\tau)\ln\frac{(a_0\rho)^{\frac{2}{n+2}}}{-\tau}-\sum_{i=1}^{n-1}\frac{y_i^2}{4}}\,dy_1...dy_{n-1}d\tau.$$

Let's make change of the variables

$$\vartheta_i=y_i(a_0\rho)^{-\frac{1}{n+2}};\qquad i=1,...,n-1,\quad u=-\tau(a_0\rho)^{-\frac{2}{n+2}}.$$

Let Π^+ be image Π_ρ at such transformation. We have

$$J_{nn}(\rho)=2a_0\int\limits_{\Pi^+}\sqrt{\frac{n+2}{2}u\ln\frac{1}{u}-\sum_{i=1}^{n-1}\frac{\vartheta_i^2}{4}}\,d\vartheta_1...d\vartheta_{n-1}du=$$

$$=\frac{2^{n+1}}{n+2}\int\limits_0^1\sqrt{\ln\frac{1}{r}}dr\int\limits_{\mathbb{E}_{n-1}}\exp\left[-\sum_{i=1}^{n-1}\xi_i^2\right]d\xi_1...d\xi_{n-1},$$

where $\xi_i=\frac{\vartheta_i}{2\sqrt{u}};\ i=1,...,n-1,\ r=\exp\left[\sum\limits_{i=1}^{n-1}\frac{\vartheta_i^2}{4u}-\frac{n+2}{2}\ln\frac{1}{u}\right]$.

It is easy to see that the last integral is equal to $\frac{1}{n+2}$. Subject to these calculations for u_{ij} we have

$$u_{ij}(x,t)=-G_{ij}*F+\frac{\delta_{ij}}{n+2}F(x,t),\quad i,j=1,...,n,\tag{4}$$

where δ_{ij} is Cronecker symbol and $G_{ij}*F$ is a parabolic singular integral with the kernel in G_{ij}. By Jones theorem [13] for $p\in(1,\infty)$, $i,j=1,...,n$

$$\|G_{ij}*F\|_{L_p(Q_R^T)}\le C_{ij}(p,n)\|F\|_{L_p(Q_R^T)}.$$

Subject to this inequality in (4) we'll obtain

$$\sum_{i,j=1}^{n}\|u_{ij}\|_{L_p(Q_R^T)}\le C_1(p,n)\|F\|_{L_p(Q_R^T)}.\tag{5}$$

Now let's show that $\|u_t\|_{L_p(Q_R^T)} \leq C_2(p,n)\|F\|_{L_p(Q_R^T)}$. Really from the relations $u_t = \Delta u - F$ and (5) we have

$$\|u_t\|_{L_p(Q_R^T)} \leq \|\Delta u\|_{L_p(Q_R^T)} + \|F\|_{L_p(Q_R^T)} \leq \sum_{i=1}^{n} \|u_{ii}\|_{L_p(Q_R^T)} +$$

$$+\|F\|_{L_p(Q_R^T)} \leq C_2(p,n)\|F\|_{L_p(Q_R^T)}.$$

Then

$$\left(\int_{Q_R^T} \left(\sum\nolimits_{i,j=1}^{n} |u_{ij}|^p + |u_t|^p\right) dxdt\right)^{\frac{1}{p}} \leq \left(\int_{Q_R^T} \sum\nolimits_{i,j=1}^{n} |u_{ij}|^p\, dxdt\right)^{\frac{1}{p}} +$$

$$+\left(\int_{Q_R^T} |u_t|^p\, dxdt\right)^{\frac{1}{p}} \leq \sum\nolimits_{i,j=1}^{n} \|u_{ij}\|_{L_p(Q_R^T)} + \|u_t\|_{L_p(Q_R^T)} \leq$$

$$\leq C_3(p,n)\left(\int_{Q_R^T} |M_0 u|^p\, dxdt\right)^{\frac{1}{p}}.$$

The lemma is proved.

Denote now by $\mathring{W}_p^{2,1}\left(Q_R^T\right)$ and $\mathring{V}_p^{2,1}\left(Q_R^T\right)$ closures $\mathcal{A}\left(Q_R^T\right)$ by the norms

$$\|u\|_{\mathring{W}_p^{2,1}(Q_R^T)} = \left(\int_{Q_R^T} \left(\sum_{i,j=1}^{n} |u_{ij}|^p + |u_t|^p\right) dxdt\right)^{\frac{1}{p}}$$

and

$$\|u\|_{\mathring{V}_p^{2,1}(Q_R^T)} = \left(\int_{Q_R^T} |M_0 u|^p\, dxdt\right)^{\frac{1}{p}},$$

respectively, $p \in (1,\infty)$. According to the Friedrichs type inequality and lemma 2 functionals determined above are really norms. Denote by $T(p)$ the operator associating to each functions $u(x,t) \in \mathring{V}_p^{2,1}\left(Q_R^T\right)$ itself as element of the space $\mathring{W}_p^{2,1}\left(Q_R^T\right)$. By lemma 2 the operator $T(p)$ is bounded. Denote by $K(p)$ its norm. By lemma 1 $K(2) \leq 1$. Let p_0 be an arbitrary fixed number from the interval $(1,2)$ According to Riez-Thorin theorem on convexity [14] for any $p \in [p_0, 2]$

$$K(p) \leq (K(p_0))^{1-\theta} (K(2))^{\theta} \leq (K(p_0))^{1-\theta},$$

where $\theta = \frac{2(p-p_0)}{p(2-p_0)}$. Thus

$$K(p) \leq K(p_0)^{\frac{p_0(2-p)}{p(2-p_0)}}.$$

Let's fix $p_0 = \frac{5}{3}$ and denote $a = \max\left\{\left(\frac{5}{3}\right)^2, \left(K\left(\frac{5}{3}\right)\right)^2\right\}$. Since for $p \in \left[\frac{5}{3}, 2\right]$ $\frac{p_0(2-p)}{p(2-p_0)} \le \frac{2-p}{2-p_0} = 2(2-p)$, then we finally obtain

$$K(p) \le a^{2-p}.$$

And so we proved the following assertion.

Lemma 3. *If* $u(x,t) \in \overset{\circ}{W}{}_p^{2,1}\left(Q_R^T\right)$, *then for any* $p \in \left[\frac{5}{3}, 2\right]$

$$\|u\|_{\overset{\circ}{W}{}_p^{2,1}\left(Q_R^T\right)} \le a^{2-p}\|u\|_{\overset{\circ}{V}{}_p^{2,1}\left(Q_R^T\right)}.$$

Note that at this the constant $a > 1$ depends only on n. For $p \in \left[\frac{5}{3}, 2\right]$ $\sup\limits_{Q_R^T}\left(\sum\limits_{i,j=1}^{n}|a_{ij}(x,t) - \delta_{ij}|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$ denote by δ_p (for brevity we write sup instead ess sup) and let $\delta_2 = \delta$, $h = \max\left\{\frac{1-\gamma^2}{\gamma}, 1\right\}$.

Lemma 4. *For* $p \in \left[\frac{5}{3}, 2\right]$ *it holds the estimation*

$$\delta_p \le h^{\frac{2-p}{p}}\delta^{\frac{2(p-1)}{p}}.$$

Proof. From the condition (3) it follows that for $i = 1, ..., n$

$$\gamma - 1 \le a_{ii}(x,t) - 1 \le \gamma^{-1} - 1$$

and since $\gamma - 1 \ge 1 - \gamma^{-1}$, then

$$|a_{ii}(x,t) - 1| \le \frac{1-\gamma}{\gamma}. \qquad (6)$$

If $i \ne j$ then

$$2\gamma \le a_{ii}(x,t) + a_{jj}(x,t) + 2a_{ij}(x,t) \le 2\gamma^{-1}.$$

Therefore

$$|a_{ij}(x,t)| \le \frac{1-\gamma^2}{\gamma}. \qquad (7)$$

From (6) and (7) we conclude that for $i, j = 1, ..., n$

$$|a_{ij}(x,t) - \delta_{ij}| \le h. \qquad (8)$$

On the other hand allowing for (8)

$$\delta_p = \sup_{Q_R^T}\left(\sum_{i,j=1}^{n}(a_{ij}(x,t) - \delta_{ij})^2|a_{ij}(x,t) - \delta_{ij}|^{\frac{2-p}{p-1}}\right)^{\frac{p-1}{p}} \le h^{\frac{2-p}{p}}\delta^{\frac{2(p-1)}{p}}$$

and the lemma is proved.

Lemma 5. *Let $\delta < 1$. Then there exists $p_1(\gamma, \delta, n) \in \left[\frac{5}{3}, 2\right]$, such that for all $p \in [p_1, 2]$*

$$a^{2-p}\delta_p \leq \delta^{\frac{1}{3}}.$$

Proof. According to the previous lemma

$$a^{2-p}\delta^p \leq a^{2-p}h^{\frac{2-p}{p}}\delta^{\frac{2(p-1)}{p}}.$$

But $h^{\frac{1}{p}} \leq h^{\frac{2}{3}} = h_1$, $\frac{p-1}{p} \geq \frac{1}{3}$. Therefore

$$a^{2-p}\delta_p \leq (ah_1)^{2-p}\delta^{\frac{2}{3}}. \tag{9}$$

Let now $p_1 = \max\left\{\frac{5}{3}, 2 - \frac{\ln\frac{1}{\delta}}{3\ln(ah_1)}\right\}$. Then at $p \in [p_1, 2]$ $(ah_1)^{2-p} \leq \delta^{-\frac{1}{3}}$ and from (9) it follows the assertion of the lemma.

2^0. Internal priori estimation. Consider the operator

$$\mathcal{L}_0 = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}$$

together with the operator $\mathcal{L}$.

Lemma 6. *If relative to the coefficients of the operator $\mathcal{L}_0$ the condition (3) and $\delta < 1$ is fulfilled, then at all $p \in [p_1, 2]$ for any function $u(x,t) \in \mathring{W}_p^{2,1}\left(Q_R^T\right)$ the estimation*

$$\|u\|_{\mathring{W}_p^{2,1}(Q_R^T)} \leq C_4(\gamma, \delta, n)\|\mathcal{L}_0 u\|_{L_p(Q_R^T)}$$

is true.

Proof. According to lemma 3

$$\|u\|_{\mathring{W}_p^{2,1}(Q_R^T)} \leq a^{2-p}\|M_0 u\|_{L_p(Q_R^T)} \leq a^{2-p}\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} +$$

$$+a^{2-p}\left\|\sum_{i,j=1}^{n}(a_{ij}(x,t) - \delta_{ij})u_{ij}\right\|_{L_p(Q_R^T)} \leq a^{\frac{1}{2}}\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} +$$

$$+ a^{2-p}\left\|\sum_{i,j=1}^{n}(a_{ij}(x,t) - \delta_{ij})u_{ij}\right\|_{L_p(Q_R^T)}. \tag{10}$$

But on the other hand

$$\left\|\sum\nolimits_{i,j=1}^{n}(a_{ij}(x,t) - \delta_{ij})u_{ij}\right\|_{L_p(Q_R^T)} \leq$$

$$\left(\int_{Q_R^T}\sum\nolimits_{i,j=1}^{n}|u_{ij}|^p\left(\sum\nolimits_{i,j=1}^{n}|a_{ij}(x,t) - \delta_{ij}|^{\frac{p}{p-1}}\right)^{p-1}dxdt\right)^{\frac{1}{p}} \leq \delta_p\|u\|_{\mathring{W}_p^{2,1}(Q_R^T)}.$$

Therefore from (10) and lemma 5 we conclude

$$\|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)} \le a^{\frac{1}{2}} \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + a^{2-p}\delta_p \|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)} \le$$

$$\le a^{\frac{1}{2}} \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \delta^{\frac{1}{3}} \|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_T^R)}$$

and the assertion of the lemma is proved.

Further everywhere not specifying it we will suppose that the radius R of the sphere $B_R^{x^0}$ ($B_R^{x^0}$ is foundation of the cylinder Q_R^T) doesn't exceed 1.

Lemma 7. *If the conditions of previous lemma are proved then at all* $p \in [p_1, 2]$ *for any funcitons* $u(x,t) \in A\left(Q_R^T\right)$ *the inequality*

$$\|u\|_{W_p^{2,1}(Q_R^T)} \le C_5(\gamma, \delta, n) \|\mathcal{L}_0 u\|_{L_p(Q_R^T)}$$

is true.

It is enough to apply the Friedrichs inequality and lemma 6 for proving.

Now assume the following Cordes condition on leading coefficients of the operator $\mathcal{L}$

$$\sigma = \frac{\sup\limits_{Q_T} \sum\limits_{i,j=1}^{n} a_{ij}^2(x,t)}{\left[\inf\limits_{Q_T} \sum\limits_{i=1}^{n} a_{ii}(x,t)\right]^2} < \frac{1}{n-1}. \tag{11}$$

At this we'll suppose that the condiiton (11) is fulfilled to within non-singular linear transformation, i.e. we can cover the domain Q_T with finite number of the subdomains $Q_1, ..., Q_m$ so in every Q_i there exists non-singular linear transformation at which the image of the operator $\mathcal{L}$ satisfies condition (11) in the image of subdomain Q_i, $i = 1, .., m$.

Lemma 8. *Condition* $\delta < 1$ *to within non-singular linear transformation coincides with the condition (11).*

Proof. Let's make the transformation $\tau = k^2 t$, $y_i = k x_i$; $i = 1, ..., n$, where $k = \left(\frac{\sup\limits_{Q_T} \sum\limits_{i,j=1}^{n} a_{ij}^2(x,t)}{\left[\inf\limits_{Q_T} \sum\limits_{i=1}^{n} a_{ii}(x,t)\right]^2}\right)^{-\frac{1}{2}}$. Then if $\|A_{ij}(y,\tau)\|$ is matrix of leading part of image of the operator $\mathcal{L}$ then $\mathcal{A}_{ij}(y,\tau) = k^2 a_{ij}(x,t)$; $i,j = 1, ..., n$. Condition $\delta < 1$ in new variables will take the form

$$\sup_{\tilde{Q}_T} \sum_{i,j=1}^{n} \mathcal{A}_{ij}^2(y,\tau) - 2\inf_{\tilde{Q}_T} \sum_{i=1}^{n} \mathcal{A}_{ii}(y,\tau) + n < 1, \tag{12}$$

where $\tilde{Q}_T$ is the image of the domain Q_T. It is clear that (12) coincides with the condition

$$\frac{\sup\limits_{Q_T} \sum\limits_{i,j=1}^{n} a_{ij}^2(x,t)}{\left[\inf\limits_{Q_T} \sum\limits_{i=1}^{n} a_{ii}(x,t)\right]^2} < \frac{1}{n-1}.$$

Lemma 9. *Let relative to the coefficients of the operator $\mathcal{L}_0$ the conditions (3) and (11) be fulfilled. Then there exists the constant $C_6(\gamma,\sigma,n)$ such that for any function $u(x,t)\in C^\infty\left(\bar{Q}_R^T\right)$, $u|_{t=0}=0$ at every $p\in[p_1,2]$ and $R_1\in(0,R)$ the estimation*

$$\|u\|_{W_p^{2,1}\left(Q_{R_1}^T\right)}\le C_5\|\mathcal{L}_0u\|_{L_p(Q_R^T)}+\frac{C_6}{(R-R_1)^2}\|u\|_{L_p(Q_R^T)}+$$

$$\frac{C_6}{R-R_1}\|u\|_{W_p^{1,0}(Q_R^T)}$$

is true.

Proof. Let the function $\eta(x)\in C_0^\infty\left(B_R^{x^0}\right)$ be such that $\eta(x)=1$ in $B_{R_1}^{x^0}$, $0\le\eta(x)\le1$, moreover

$$|\eta_i|\le\frac{C_7}{R-R_1},\quad |\eta_{ij}|\le\frac{C_7}{(R-R_1)^2};\quad i,j=1,...,n, \tag{13}$$

where $C_7=C_7(n)$. Applying to the function $u\eta$ lemma 7 we'll obtain

$$\|u\|_{W_p^{2,1}\left(Q_{R_1}^T\right)}\le C_5\|\mathcal{L}_0(u\eta)\|_{L_p(Q_R^T)}. \tag{14}$$

But on the other hand

$$|\mathcal{L}_0(u\eta)|\le|\mathcal{L}_0u|+|u|\left|\sum_{i=1}^n a_{ij}(x,t)\eta_{ij}\right|+2\left|\sum_{i,j=1}^n a_{ij}(x,t)u_i\eta_i\right|, \tag{15}$$

and further allowing for (13)

$$\left|\sum_{i,j=1}^n a_{ij}(x,t)\eta_{ij}\right|\le\frac{C_8(\gamma,n)}{(R-R_1)^2},$$

$$2\left|\sum_{i,j=1}^n a_{ij}(x,t)u_i\eta_j\right|\le2\left(\sum_{i,j=1}^n a_{ij}(x,t)u_iu_j\right)^{\frac12}\left(\sum_{i,j=1}^n a_{ij}(x,t)\eta_i\eta_j\right)^{\frac12}\le$$

$$\le2\gamma^{-1}\left(\sum_{i=1}^n u_i^2\right)^{\frac12}\left(\sum_{i=1}^n \eta_i^2\right)^{\frac12}\le2\gamma^{-1}\sum_{i=1}^n|u_i|\sum_{i=1}^n|\eta_i|\le\frac{2n\gamma^{-1}C_7}{R-R_1}\sum_{i=1}^n|u_i|.$$

Thus from (15) we conclude

$$\|\mathcal{L}_0(u\eta)\|_{L_p(Q_R^T)}\le\|\mathcal{L}_0u\|_{L_p(Q_R^T)}+\frac{C_8}{(R-R_1)^2}\|u\|_{L_p(Q_R^T)}+$$

$$+\frac{C_9(\gamma,n)}{R-R_1}\sum_{i=1}^n\|u_i\|_{L_p(Q_R^T)}\le\|\mathcal{L}_0u\|_{L_p(Q_R^T)}+\frac{C_8}{(R-R_1)^2}\|u\|_{L_p(Q_R^T)}+$$

$$+\frac{C_9}{R-R_1}\|u\|_{W_p^{1,0}(Q_R^T)}. \tag{16}$$

Subject to (16) in (14) and denoting by C_{10} the $\max\{C_5C_8, C_5C_9\}$ we arrive at the required estimation (13).

Lemma 10. *Let relative to the coefficients $\mathcal{L}_0$ the conditions of the previous lemma be fulfilled. Then there exists the constant $C_{11}(\gamma,\sigma,n)$ such that for any function $u(x,t)\in C^\infty\left(\bar{Q}_R^T\right)$, $u|_{t=0}=0$ at any $\varepsilon>0$ and $p\in[p_1,2]$ the estimation*

$$\|u\|_{W_p^{2,1}\left(Q^T_{\frac{R}{2}}\right)} \le C_5\|\mathcal{L}_0u\|_{L_p(Q_R^T)} + \varepsilon\|u\|_{W_p^{2,1}(Q_R^T)} + \frac{C_{11}}{\varepsilon R^2}\|u\|_{L_p(Q_R^T)}$$

is true.

Proof. We'll use the following interpolation inequality ([3]): let $p\in(1,\infty)$ then for any function $u(x,t)\in W_p^{2,1}\left(Q_R^T\right)$ at any $\varepsilon>0$ and $p\in[p_1,2]$ the estimation

$$\|u\|_{W_p^{1,0}(Q_R^T)} \le \varepsilon\|u\|_{W_p^{2,1}(Q_R^T)} + \frac{C_{12}(p,n)}{\varepsilon}\|u\|_{L_p(Q_R^T)}. \tag{17}$$

is true.

Let's fix an arbitrary $\varepsilon>0$ and let $\varepsilon_1>0$ be a number which will be choosen later. According to lemma 9 and the inequality (17)

$$\|u\|_{W_p^{2,1}\left(Q^T_{\frac{R}{2}}\right)} \le C_5\|\mathcal{L}_0u\|_{L_p(Q_R^T)} + \frac{4C_6}{R^2}\|u\|_{L_p(Q_R^T)} + \frac{2C_6}{R}\|u\|_{W_p^{1,0}(Q_R^T)} \le$$

$$\le C_5\|L_0u\|_{L_p(Q_R^T)} + \frac{4C_6}{R^2}\|u\|_{L_p(Q_R^T)} + \frac{2C_6\varepsilon_1}{R}\|u\|_{W_p^{2,1}(Q_R^T)} + \frac{2C_6C_{13}}{R\varepsilon_1}\|u\|_{L_p(Q_R^T)},$$

where $C_{13}=\sup\limits_{p\in[p_1,2]}C_{12}(p,n)$. Now it is enough to choose $\varepsilon_1=\dfrac{\varepsilon R}{2C_6}$, the lemma is proved.

Remark. If the minor coefficients of the operator $\mathcal{L}$ are bounded, then there exists such $R_0(\gamma,\sigma,n,\mathbb{B},c)$ that at $R\le R_0$ the assertion of lemma 10 is also true for the operator $\mathcal{L}$. Here $\mathbb{B}=(b_1(x,t),...,b_n(x,t))$.

For $\rho>0$ the set $\{x: x\in\Omega, dist(x,\partial\Omega)>\rho\}$ denote by Ω_ρ.

Lemma 11. *Let relative to the coefficients of the operator $\mathcal{L}_0$ the conditions (3) and (11) be fulfilled. Then for any function $u(x,t)\in C^\infty\left(\bar{Q}_R^T\right)$, $u|_{t=0}=0$ at any $\varepsilon>0$, $\rho>0$ and $p\in[p_1,2]$ the estimation*

$$\|u\|_{W_p^{2,1}(\Omega_\rho\times(0,T))} \le C_{14}(\gamma,\sigma,n,\rho,\Omega)\|\mathcal{L}_0u\|_{L_p(Q_T)} +$$

$$+\varepsilon\|u\|_{W_p^{2,1}(Q_T)} + \frac{C_{15}(\gamma,\sigma,n,\rho,\Omega)}{\varepsilon}\|u\|_{L_p(Q_T)}.$$

is true.

Proof. Let's fix an arbitrary $\varepsilon>0$, $\rho>0$ and let $\varepsilon_2>0$ be a number which will be choosen later. Let cover $\bar{\Omega}_\rho$ by the system of spheres $\left\{B^{x^i}_{\frac{\rho}{2}}\right\}$ and choose from this cover the finite subcovering $B^1,...,B^N$. It is evident that the number N depends only on ρ, n and $diam\Omega$. Applying for every $i=1,...,N$ lemma 10 we obtain

$$\|u\|^p_{W_p^{2,1}(B^i\times(0,T))} \le 3^{p-1}\left(C_5^p\|\mathcal{L}_0u\|^p_{L_p(Q_T)} + \varepsilon_2^p\|u\|^p_{W_p^{2,1}(Q_T)} + \frac{C_{11}^p}{\varepsilon_2^p\rho^{2p}}\|u\|^p_{L_p(Q_T)}\right).$$

Summarizing this inequality by i from 1 to N we conclude

$$\|u\|_{W_p^{2,1}(\Omega_\rho\times(0,T))}\le 3^{p-1}N\left(C_5^p\|\mathcal{L}_0u\|^p_{L_p(Q_T)}+\varepsilon_2^p\|u\|^p_{W_p^{2,1}(Q_T)}+\frac{C_{11}^p}{\varepsilon_2^p\rho^{2p}}\|u\|^p_{L_p(Q_T)}\right).$$

Now it is sufficient to choose $\varepsilon_2=\dfrac{\varepsilon}{3N}$ and the lemma is proved.

3⁰. Basic coercive estimation. The assertion of lemma 11 is true without any demands relative to the domain $\partial\Omega$. All next assertions of the present paper hold under the condition $\partial\Omega\in C^2$ which we'll always suppose as fulfilled one.

Lemma 12. *Let relative to the coefficients of the operator $\mathcal{L}_0$ the conditions (3), (11) be fulfilled. Then there exist positive constants ρ_1, C_{16} and C_{17} depending on γ,σ,n and the domain Ω such that for any function on $u(x,t)\in\dot{W}_p^{2,1}(Q_T)$ at every $\varepsilon>0$ and $p\in[p_1,2]$ the estimation*

$$\|u\|_{W_p^{2,1}\left((\Omega\backslash\Omega_{\rho_1})\times(0,T)\right)}\le C_{16}\|\mathcal{L}_0u\|_{L_p(Q_T)}+\varepsilon\|u\|_{W_p^{2,1}(Q_T)}+\frac{C_{17}}{\varepsilon}\|u\|_{L_p(Q_T)}.$$

is true.

Proof. It is sufficient to prove the lemma for the functions $u(x,t)\in C^\infty\left(\bar{Q}_T\right)$, $u|_{\Gamma(Q_T)}=0$. Besides not losing generality we'll suppose that the coefficients of the operator $\mathcal{L}_0$ are infinite differentiable $\bar{Q}_T$. Let's fix an arbitrary $\varepsilon>0$ and the point $x^0\in\partial\Omega$. Make orthogonal transformation of the coordinate $x\to y$ such that the tangent hyperplane to $\partial\tilde{\Omega}$ at the point y^0 will be perpendicular to the axis Oy_n. Here $\tilde{\Omega}$ and y^0 are images of the domain Ω and the point x^0 respectively at such transformation. Denote by $\tilde{u}(y,t)$ the image of the function $u(x,t)$. We'll suppose for simplicity that the domain $\partial\tilde{\Omega}$ at intersection $\partial\tilde{\Omega}$ with some neighbourhood O_h of the point y^0 is given by the equation $y_n=\varphi(y_1,...,y_{n-1})$ with twice continuously differentiable function φ and the part $\tilde{\Omega}$ adjacent to $\partial\tilde{\Omega}\cap O_h$ belongs to the set $\{y:y_n>\varphi(y_1...y_{n-1})\}$. Let $\mathcal{A}(x,t)=\|a_{ij}(x,t)\|$ be a matrix of leading coefficients of the operator $\mathcal{L}_0$, $\widetilde{\mathcal{A}}(y,t)=\|\tilde{a}_{ij}(y,t)\|$, where $\tilde{a}_{ij}(y,t)$ are leading coefficients of the image $\tilde{\mathcal{L}}_0$ of the operator $\mathcal{L}_0$ at our transformation; $i,j=1,...,n$. Show now that the eigen numbers of the matrices $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ coincide. Really, fix an arbitrary point $(x,t)\in Q_T$ and let λ be an arbitrary eigen number of the matrix $\mathcal{A}$, and x^λ be corresponding to it eigen vector. By virtue of orthogonality of our transformation there exists a non-degenerated matrix T such that $\widetilde{\mathcal{A}}=T^{-1}\mathcal{A}T$. Denote by $T^{-1}x^\lambda$ the y^λ. We have

$$\widetilde{\mathcal{A}}y^\lambda=T^{-1}\mathcal{A}x^\lambda=\lambda T^{-1}x^\lambda=\lambda y^\lambda.$$

On the other hand we can write condition (11) in the following form

$$\sigma=\sup_{Q_T}\frac{\sum\limits_{i=1}^n\lambda_i^2(x,t)}{\left[\sum\limits_{i=1}^n\lambda_i(x,t)\right]^2}<\frac{1}{n-1},$$

where $\lambda_i(x,t)$ are eigen numbers of the matrix $\mathcal{A}(x,t)$; $i=1,...,n$. Thus the condition (11) is fulfilled also for the operator $\tilde{\mathcal{L}}_0$, moreover with the same constant σ. Analogously it is shown that for the operator $\tilde{\mathcal{L}}_0$ the conditions (3) are fulfilled (with the same constant γ). Let's make one more transformation $z_i=y_i$; $i=$

$1,...,n-1,\ z_n=y_n-\varphi(y_1,...,y_{n-1})$. Let $\mathcal{L}'_0,\ \Omega'$ and z^0 be images of the operator $\tilde{\mathcal{L}}_0$, of the domain $\tilde{\Omega}$ and the point y^0 respectively at our transformation, and $a'_{ij}(z,t)$ be leading coefficients of the operator $\mathcal{L}'_0;\ i,j=1,...,n$. It is easy to see that

$$a'_{ij}(z,t)=\sum_{k,l=1}^{n}\tilde{a}_{kl}(y,t)\frac{\partial z_i}{\partial y_k}\frac{\partial z_j}{\partial y_l};\quad i,j=1,...,n.$$

Therefore

$$a'_{ij}(z,t)=\tilde{a}_{ij}(y,t)\quad \text{if}\quad 1\le i,j\le n-1,$$

$$a'_{nj}(z,t)=-\sum_{k=1}^{n-1}\tilde{a}_{kj}(y,t)\frac{\partial\varphi}{\partial y_k}+\tilde{a}_{nj}(y,t)\quad \text{if}\quad 1\le j\le n-1,$$

$$a'_{nn}(z,t)=\sum_{k,l=1}^{n}\tilde{a}_{kl}(y,t)\frac{\partial\varphi}{\partial y_k}\frac{\partial\varphi}{\partial y_l}-2\sum_{k=1}^{n-1}\tilde{a}_{nk}(y,t)\frac{\partial\varphi}{\partial y_k}+\tilde{a}_{nn}(y,t).$$

Since $\frac{\partial\varphi}{\partial y_i}(y^0)=0$ for $i=1,...,n-1$ then there exists $h_1(y^0,\varphi)$ such that at $h\le h_1$ at intersection $\Omega'\cap\left(B_h^{z^0}\times(0,T)\right)$ the condition (11) (with the constant $\sigma'=\dfrac{\sigma+\frac{1}{n-1}}{2}$) is fulfilled. Besides for the operator $\mathcal{L}'_0$ in indicated intersection the conditions (3) are fulfilled (with the constant $\dfrac{\gamma}{2}$). Assume that $r=r(z^0)=h_1(y_0,\varphi)$ and let $u'(z,t)$ be image of the function $\tilde{u}(y,t)$ at our transformation. It is clear that in variables z the intersection $\Omega'\cap B_r^{z^0}$ represent hemisphere $B_r^+=\left\{z:\left|z-z^0\right|<r,\ z_n>0\right\}$. Continue the function $u'(z,t)$ by the odd form and coefficients of the operator $\mathcal{L}'_0$ by the even form by the hyperplane $z_n=0$ in $B_r^{z^0}\backslash B_r^+$ and denote by $u'(z,t)$ and $\mathcal{L}'_0$ the obtained in this way function and the operator respectively. Since $u'(z,t)\in W_p^{2,1}\left(B_r^{z^0}\times(0,T)\right)$ then according to lemma 10

$$\left\|u'\right\|_{W_p^{2,1}\left(B_{\frac{r}{2}}^{z^0}\times(0,T)\right)}\le C_5\left\|\mathcal{L}'_0u'\right\|_{L_p\left(B_r^{z^0}\times(0,T)\right)}+\varepsilon_3\left\|u'\right\|_{W_p^{2,1}\left(B_r^{z^0}\times(0,T)\right)}+$$

$$+\frac{C_{11}}{\varepsilon_3 r^2}\left\|u'\right\|_{L_p\left(B_r^{z^0}\times(0,T)\right)},\tag{18}$$

where $\varepsilon_3>0$ will be choosen later. But on the other hand each of norms at the right-hand side (18) represent the corresponding norm taken by semi-cylinder $Q_r^+=B_r^+\times(0,T)$ and multiplied by $2^{\frac{1}{p}}$. Therefore from (18) we conclude

$$\left\|u'\right\|_{W_p^{2,1}\left(Q_{\frac{r}{2}}^+\right)}\le C_5\left\|\mathcal{L}'_0u'\right\|_{L_p(Q_r^+)}+\varepsilon_3\left\|u'\right\|_{W_p^{2,1}(Q_r^+)}+\frac{C_{11}}{\varepsilon_3 r^2}\left\|u'\right\|_{L_p(Q_r^+)}.\tag{19}$$

Cover $\partial\Omega'$ by the system of spheres $\left\{B_{\frac{r}{2}}^{z^i}\right\}$ and choose from this cover finite subcoverining $B^1,...,B^M$. At this the number M is determined only by the quantities $\gamma,\ \sigma,\ h$ and the domain Ω. Writing out the inequality of the form (19) for every semi-cylinder $B_r^+(z^i)\times(0,T)$; $i=1,...,M$ raising both sides of obtained inequalitites to power p and summarizing by i from 1 to M we obtain

$$\|u'\|^p_{W_p^{2,1}(\mathcal{B}\times(0,T))}\le 3^{p-1}M\left(C_5\|\mathcal{L}_0'u'\|^p_{L_p(\Omega'\times(0,T))}+\varepsilon_3^p\|u'\|_{W_p^{2,1}(\Omega'\times(0,T))}+\right.$$

$$\left.+\frac{C_{11}^p}{\varepsilon_3^p r_0^{2p}}\|u'\|_{L_p(\Omega'\times(0,T))}\right),$$

where $\mathcal{B}=\bigcup\limits_{i=1}^{M}B^+_{\frac{r}{2}}(z^i)$, and $r_0=\min\{r(z_1),...,r(z_M)\}$. Returning to the variables x and noting that pre-image $\mathcal{B}$ contains the set $\Omega\backslash\Omega_{\rho_1}$ with some $\rho_1(\gamma,\sigma,n,\Omega)$ we conclude

$$\|u\|_{W_p^{2,1}((\Omega\backslash\Omega_{\rho_1})\times(0,T))}\le C_{18}\|\mathcal{L}_0u\|_{L_p(Q_T)}+C_{19}\varepsilon_3\|u\|_{W_p^{2,1}(Q_T)}+\frac{C_{20}}{\varepsilon_3}\|u\|_{L_p(Q_T)},$$

where the constants C_{18}, C_{19} and C_{20} depend only on γ,σ,n and the domain Ω. Now it is sufficient to choose $\varepsilon_3=\dfrac{\varepsilon}{C_{19}}$ and the lemma is proved.

It follows the following lemma from lemmas 11 and 12

Lemma 13. *Let relative to coefficients of the operator* $\mathcal{L}_0$ *the conditions (3) and (11) be fulfilled. Then for any funciton* $u(x,t)\in\dot{W}_p^{2,1}(Q_T)$ *at any* $p\in[p_1,2]$ *the estimation*

$$\|u\|_{W_p^{2,1}(Q_T)}\le C_{21}(\gamma,\sigma,n,\Omega)\left(\|\mathcal{L}_0u\|_{L_p(Q_T)}+\|u\|_{L_p(Q_T)}\right)$$

is true.

Now impose the following conditions on minor coefficients of the operator $\mathcal{L}$. For $p\in[p_1,2]$

$$b_i(x,t)\in L_{n+2}(Q_T);\quad i=1,...,n, \tag{20}$$

$$c(x,t)\in L_l(Q_T);\qquad l=\begin{cases}\max\left(p,\frac{n+2}{2}\right), & \text{at}\quad p\ne\frac{n+2}{2},\\ 2+\nu, & \text{at}\quad n=p=2,\end{cases} \tag{21}$$

where ν is some positive constant.

Let $\psi(x,t)\in L_p(Q_T)$, $1<p<\infty$. The quantity

$$\omega_{\psi;p}(\delta)=\sup_{\substack{e\subset Q_T\\ mes\ e\le\delta}}\left(\int\limits_e|\psi|^p\,dxdt\right)^{\frac{1}{p}}$$

is called $\mathcal{AC}$ modulus of the function $\psi(x,t)$. Denote by $\omega_{\mathbb{B};p}(\delta)$ the $\max\limits_{1\le i\le n}\{\omega_{b_i;p}(\delta)\}$.

Further everywhere the symbol $C(\mathcal{L})$ means that the positive constant C depends only on $\gamma,\sigma,\omega_{\mathbb{B};n+2}(\delta),\omega_{c;l}(\delta)$ and ν.

Theorem 1. *Let relative to the coefficients of the operator* $\mathcal{L}$ *the conditions (3), (11), (20) and (21) be fulfilled. Then there exist the constants* $C_{22}(\mathcal{L},n,\Omega)$, $T_0(\mathcal{L},n)$, *such that if* $T\le T_0$ *then for any function* $u(x,t)\in\in\dot{W}_p^{2,1}(Q_T)$ *at every* $p\in[p_1,2]$ *the estimation*

$$\|u\|_{W_p^{2,1}(Q_T)}\le C_{22}\|\mathcal{L}u\|_{L_p(Q_T)}.$$

is true.

Proof. We'll use the following embedding theorems [3]: for any function $u(x,t) \in \dot{W}_q^{2,1}(Q_T)$ it hold the estimations

$$\|u_i\|_{L_{\frac{q(n+2)}{n+2-q}}(Q_T)} \le C_{23}(q,n)\,\|u\|_{W_q^{2,1}(Q_T)}, \quad \text{if} \quad 1 \le q < n+2, \tag{22}$$

$$\|u\|_{L_{\frac{q(n+2)}{n+2-2q}}(Q_T)} \le C_{24}(q,n)\,\|u\|_{W_q^{2,1}(Q_T)}, \quad \text{if} \quad 1 \le q < \frac{n+2}{2}. \tag{23}$$

According to lemma 13

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_{21}\,\|\mathcal{L}u\|_{L_p(Q_T)} + C_{21}\,\|(\mathcal{L}-\mathcal{L}_0)\,u\|_{L_p(Q_T)} + C_{21}\,\|u\|_{L_p(Q_T)} \le$$

$$\le C_{21}\,\|\mathcal{L}u\|_{L_p(Q_T)} + C_{21}\sum_{i=1}^{n}\|b_iu_i\|_{L_p(Q_T)} + C_{21}\,\|cu\|_{L_p(Q_T)} + C_{21}\,\|u\|_{L_p(Q_T)}. \tag{24}$$

Let's fix an arbitrary $i,\ 1 \le i \le n$ and assume in (22) $q = p$. We obtain

$$\|b_iu_i\|_{L_p(Q_T)} \le \|b_i\|_{L_{n+2}(Q_T)}\,\|u_i\|_{L_{\frac{(n+2)p}{n+2-p}}(Q_T)} \le C_{23}\,\|b_i\|_{L_{n+2}(Q_T)}\,\|u\|_{W_p^{2,1}(Q_T)}.$$

Thus

$$\sum_{i=1}^{n}\|b_iu_i\|_{L_p(Q_T)} \le C_{23}\sum_{i=1}^{n}\|b_i\|_{L_{n+2}(Q_T)}\,\|u\|_{W_p^{2,1}(Q_T)} \le$$

$$\le C_{25}(n)\,\omega_{\mathbb{B};n+2}(\delta)\,\|u\|_{W_p^{2,1}(Q_T)}, \tag{25}$$

where $\delta = T\ mes\Omega,\ C_{25} = \sup\limits_{p\in[p_1,2]} C_{23}(p,n)$.

Analogously by virtue of (23) at $n \ge 3$ we have

$$\|cu\|_{L_p(Q_T)} \le \|c\|_{L_{\frac{n+2}{2}}(Q_T)}\,\|u\|_{L_{\frac{p(n+2)}{n+2-2p}}(Q_T)} \le C_{24}\,\|c\|_{L^{\frac{n+2}{2}}(Q_T)}\,\|u\|_{W_p^{2,1}(Q_T)} \le$$

$$\le C_{26}(n)\,\omega_{c;\frac{n+2}{2}}(\delta)\,\|u\|_{W_p^{2,1}(Q_T)},$$

where $C_{26} = \sup\limits_{p\in[p_1,2]} C_{24}(p,n)$.

It is easy to see that the analogous estimation holds at $n = 2,\ p \ne 2$. Let now $n = p = 2$. Then according to embedding theorem [3] for any function $u(x,t) \in W_2^{2,1}(Q_T)$ at every $q \in [1,\infty)$ it holds the estimation

$$\|u\|_{L_q(Q_T)} \le C_{27}(q,n)\,\|u\|_{W_2^{2,1}(Q_T)}.$$

Therefore if $c(x,t) \in L_{2+\nu}(Q_T)$, then

$$\|cu\|_{L_2(Q_T)} \le \|c\|_{L_{2+\nu}(Q_T)}\,\|u\|_{L_{\frac{2(2+\nu)}{\nu}}(Q_T)} \le C_{28}(\nu)\,\omega_{c;2+\nu}(\delta)\,\|u\|_{W_2^{2,1}(Q_T)}.$$

Finally let $n = 1$. Then according to embedding theorem [3] for any function $u(x,t) \in W_p^{2,1}(Q_T)$ the estimation

$$\sup_{Q_T}|u| \le C_{28}\,\|u\|_{W_p^{2,1}(Q_T)}$$

is true.

By that

$$\|cu\|_{L_p(Q_T)} \leq \sup_{Q_T} |u| \cdot \|c\|_{L_p(Q_T)} \leq C_{28}\omega_{c;p}(\delta) \|u\|_{W_p^{2,1}(Q_T)}.$$

Thus in any case it holds the inequality

$$\|cu\|_{L_p(Q_T)} \leq C_{29}(n)\,\omega_{c;l}(\delta) \|u\|_{W_p^{2,1}(Q_T)}. \qquad (26)$$

Let now $t \in (0,T)$. We have

$$u(x,t) = \int_0^t u_t(x,\tau)\,d\tau.$$

Using the Hölder inequality we obtain

$$u(x,t) \leq T^{\frac{p-1}{p}} \left(\int_0^T |u_t(x,\tau)|^p \,d\tau \right)^{\frac{1}{p}},$$

and consequently

$$|u(x,t)|^p \leq T^{p-1} \int_0^T |u_t(x,\tau)|^p \,d\tau.$$

Integrating the both sides of this inequality by Q_T and raising to power $\frac{1}{p}$ we have

$$\|u\|_{L_p(Q_T)} \leq T \|u_t\|_{L_p(Q_T)}. \qquad (27)$$

Subject to (25), (26) and (27) in (24) we come to the estimation

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C_{21} \|\mathcal{L}u\|_{L_p(Q_T)} + C_{21}\left(C_{25}\omega_{\mathbb{B};n+2}(\delta) + C_{29}\omega_{c;l}(\delta) + T\right) \times$$
$$\times \|u\|_{W_p^{2,1}(Q_T)}.$$

Then there exists the constant $T_0(\mathcal{L}, n)$ such that at $T \leq T_0$

$$C_{25}\omega_{\mathbb{B};n+2}(\delta) + C_{29}\omega_{c;\frac{n+2}{2}}(\delta) + T < \frac{1}{2C_{21}}.$$

The lemma is proved.

4^0. Case $p > 2$. Let $p \in \left[2, \frac{7}{3}\right]$, and $K(p)$ have the same meaning as in lemma 3. By Riesz-Thorin theorm for any $p \in \left[2, \frac{7}{3}\right]$

$$K(p) \leq (K(2))^{1-\theta} \left(K\left(\frac{7}{3}\right)\right)^{\theta} \leq \left(K\left(\frac{7}{3}\right)\right)^{\theta},$$

where $\theta = \dfrac{2(p-2)}{p\left(\frac{7}{3}-2\right)}$. Denoting by $a_1(n)$ the $\left\{ \left(\frac{7}{3}\right)^2, \left(K\left(\frac{7}{3}\right)\right)^2 \right\}$ we obtain

$$K(p) \leq a_1^{p-2}.$$

Thus the following analogue of lemma 3 is true.

Lemma 14. *If* $u(x,t) \in \dot{W}_p^{2,1}\left(Q_R^T\right)$ *then for any* $p \in \left[2, \frac{7}{3}\right]$

$$\|u\|_{\mathring{W}_p^{2,1}(Q_R^T)} \le a_1^{p-2} \|u\|_{\mathring{V}_p^{2,1}(Q_R^T)} .$$

is true.

The analogy of lemmas 4 and 5 is proved absolutely analogously.

Lemma 15. *For* $p \in \left[2, \frac{7}{3}\right]$ *it holds the estimation*

$$\delta_p \le h^{\frac{p-2}{p}} \delta.$$

Lemma 16. *Let* $\delta < 1$*. Then there exists* $p_2(\gamma, \delta, n) \in \left(2, \frac{7}{3}\right]$ *such that for all* $p \in [2, p_2]$

$$a_1^{2-p} \delta_p \le \delta^{\frac{1}{3}}.$$

Impose now the following restrictions on minor coefficients of the operator $\mathcal{L}$ for p$\in (2, p_2]$

$$b_i(x,t) \in L_{n+2}(Q_T); \quad i = 1, ..., n, \tag{28}$$

$$c(x,t) \in L_{l'}(Q_T); \quad l' = \max\left(p, \frac{n+2}{2}\right). \tag{29}$$

Using the scheme conducted in lemmas 6-12, and subject to lemmas 14-16 we are sure in validity of theorem 1 for $p \in (2, p_2]$ and $u(x,t) \in \dot{W}_p^{2,1}(Q_T)$ if only relative to the coefficients of the operator $\mathcal{L}$ the conditions (3), (11), (18) and (29) are fulfilled. Let unify the conditions (21) and (29) asssuming that $p \in [p_1, p_2]$ namely we'll suppose that

$$c(x,t) \in L_m(Q_T), \qquad where \qquad m = \begin{cases} l, & if \quad p \in [p_1, 2], \\ l', & if \quad p \in (2, p_2]. \end{cases} \tag{30}$$

Theorem 2. *Let relative to coefficients of the operator* $\mathcal{L}$ *the conditions (3), (11), (28) and (30) be fulfilled. Then there exist the positive constants* $T_0(\mathcal{L}, n)$ *and* $C_{30}(\mathcal{L}, n, \Omega)$ *such that for any function* $u(x,t) \in \dot{W}_p^{2,1}(Q_T)$ *at* $T \le T_0$ *and at every* $p \in [p_1, p_2]$ *the estimation*

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_{30} \|\mathcal{L}u\|_{L_p(Q_T)} .$$

is true.

5^0. Solvability of the first boundary value problem. Now consider the first boundary value problem (1)-(2).

Theorem 4. *Let in domain* Q_T *be given the coefficients of the operator* $\mathcal{L}$ *satisfying the conditions (3), (11), (28) and (30). Then if* $T \le T_0$ *and* $\partial\Omega \in C^2$ *then the first boundary value problem (1)-(2) is identically strongly solvable in the space* $\dot{W}_p^{2,1}(Q_T)$ *at every* $f(x,t) \in L_p(Q_T)$, $p \in [p_1, p_2]$*. At this for solution* $u(x,t)$ *the estimation*

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_{30} \|f\|_{L_p(Q_T)} \tag{31}$$

is true.

Proof. Let's prove the theorem by the method of continuation by parameter. Introduce for $s \in [0,1]$ the family of the operator

$$\mathcal{L}_s = s\mathcal{L} + (1-s)\, M_0.$$

It is easy to see that the conditions (3) and (11) are fulfilled for the operator $\mathcal{L}_s$ with the constants γ and σ respectively. Show this on the example of condition (11). According to lemma 8 the last to within non-lingular linear transformation coincides with the condition $\delta < 1$. Let $a^s_{ij}(x,t)$ be leading coefficinets of the operator $\mathcal{L}_s$; $i,j = 1,...,n$ and

$$\delta^s = \sup_{Q_T} \left(\sum_{i,j=1}^{n} \left(a^s_{ij}(x,t) - \delta_{ij} \right)^2 \right)^{\frac{1}{2}}.$$

We have

$$\delta^s = \sup_{Q_T} \left(\sum_{i,j=1}^{n} \left(s a_{ij}(x,t) + (1-s)\,\delta_{ij} - \delta_{ij} \right)^2 \right)^{\frac{1}{2}} =$$

$$= s \sup_{Q_T} \left(\sum_{i,j=1}^{n} \left(a_{ij}(x,t) - \delta_{ij} \right)^2 \right)^{\frac{1}{2}} = s\delta \le \delta.$$

Besides if $b^s_i(x,t)$ and $c^s(x,t)$; $i = 1,...,n$ are minor coefficients of the operator $\mathcal{L}_s$, then the quantity $\sum_{i=1}^{n} \|b^s_i(x,t)\|_{L_{n+2}(Q_T)} + \|c^s(x,t)\|_{L_m(Q_T)}$ is by majorized by the constant, depending only on $\sum_{i=1}^{n} \|b_i\|_{L_{n+2}(Q_T)} + \|c\|_{L_m(Q_T)}$. Hence it follows that the assertion of theorem 2 is true for the operator $\mathcal{L}_s$ with the constant C'_{30} not depending on s. Denote by E the set of points of the section $[0,1]$ for which the problem

$$\mathcal{L}_s u = f(x,t); \quad (x,t) \in Q_T, \quad u \in \dot{W}^{2,1}_p(Q_T), \tag{32}$$

has solution. Note that by virtue of theorem 2 this solution is unique. Now show that the set E is nonempty and it is open and closed simultaneously relative to $[0,1]$. Then E coincides with the segment $[0,1]$ and in particular the problem (32) is identically solvable at $s = 1$ when $\mathcal{L}_1 = \mathcal{L}$. At this the estimation (31) follows from theorem 2. Nonemptiness of the set E follows from that the problem (32) is solvable at $s = 0$ (see: [3]). Show that the set E is open relative to $[0,1]$. Let $s^0 \in E$, $s \in [0,1]$ be such that $\left|s - s^0\right| < \alpha$ where $\alpha > 0$ will be choosen later. Represent the problem (32) in the form

$$\mathcal{L}_{s^0} u = f(x,t) + (\mathcal{L}_{s^0} - \mathcal{L}_s)\, u; \quad (x,t) \in Q_T, \quad u \in \dot{W}^{2,1}_p(Q_T). \tag{33}$$

It is easy to see that $\mathcal{L}_{s^0} - \mathcal{L}_s = \left(s^0 - s\right)(\mathcal{L} - M_0)$. Consider auxiliary problem

$$\mathcal{L}_{s^0} u = f(x,t) + \left(s^0 - s\right)(\mathcal{L} - M_0)\,\vartheta; \quad (x,t) \in Q_T, \quad u \in \dot{W}^{2,1}_p(Q_T), \tag{34}$$

where $\vartheta(x,t) \in \dot{W}^{2,1}_p(Q_T)$. Acting as in theorem 1 we can show that

$$\|(\mathcal{L} - M_0)\,\vartheta\|_{L_p(Q_T)} \le C_{31}(\mathcal{L}, n)\, \|\vartheta\|_{W^{2,1}_p(Q_T)}.$$

Thus the operator M associating to every function $\vartheta(x,t) \in \dot{W}_p^{2,1}(Q_T)$ the solution $u(x,t)$ of the problem (34) is determined, i.e. $u = M\vartheta$. Show that at corresponding way choosen by α the operator M is contractive. Let $u^1 = M\vartheta^1$, $u^2 = M\vartheta^2$. We have

$$\mathcal{L}_{s^0}\left(u^1 - u^2\right) = \left(s^0 - s\right)(\mathcal{L} - M_0)\left(\vartheta^1 - \vartheta^2\right); \quad u^1 - u^2 \in \dot{W}_p^{2,1}(Q_T).$$

Then according to theorem 2

$$\left\|u^1 - u^2\right\|_{W_p^{2,1}(Q_T)} \le C_{30}\alpha C_{31}\left\|\vartheta^1 - \vartheta^2\right\|_{W_p^{2,1}(Q_T)},$$

and it is sufficient to choose $\alpha = \dfrac{1}{2C_{30}C_{31}}$. Then the operator M has a fixed point $u = Mu$. But at $\vartheta = u$ the problem (34) coincides with the problem (33), i.e. with (32). The openness of the set E is proved. Now prove its closure. Let $s^m \in E$; $m = 1, 2, ...,$ $s^0 = \lim\limits_{m\to\infty} s^m$. Show that $s^0 \in E$. Denote by $u^m(x,t)$ the solution of the boundary value problem

$$\mathcal{L}_{s^m}u^m = f(x,t); \quad (x,t) \in Q_T, \quad u^m \in \dot{W}_p^{2,1}(Q_T).$$

According to theorem 2

$$\|u^m\|_{W_p^{2,1}(Q_T)} \le C_{30}\|f\|_{L_p(Q_T)}.$$

Thus the sequence $\{u^m(x,t)\}$ is bounded by the norm $W_p^{2,1}(Q_T)$. Hence it follows that it is weakly compact, i.e. there exist subsequence $m_k \to \infty$ at $k \to \infty$ and the function $u(x,t) \in \dot{W}_p^{2,1}(Q_T)$ such that $u(x,t)$ is weak limit in $\dot{W}_p^{2,1}(Q_T)$ of the subsequence $\{u^{m_k}(x,t)\}$ at $k \to \infty$. Hence in particular it follows that for any function $\dot{W}_p^{2,1}(Q_T)$

$$< \mathcal{L}_{s^0}u^{m_k}, \varphi > \to < \mathcal{L}_{s^0}, \varphi >; \quad k \to \infty$$

where $< u, \vartheta > = \int\limits_{Q_T} u\vartheta dxdt$. But

$$< L_{s_0}u^{m_k}, \varphi > = < (\mathcal{L}_{s^0} - \mathcal{L}_{s^{m_k}})u^{m_k}, \varphi > + < \mathcal{L}_{s^{m_k}}u^{m_k}, \varphi > = i_1 + i_2.$$

We have

$$|i_1| \le \left|s^0 - s^{m_k}\right| |< (\mathcal{L} - M_0)u^{m_k}, \varphi >| \le \left|s^0 - s^{m_k}\right| C_{32}(\varphi, p)\, C_{31}\|u^{m_k}\|_{W_p^{2,1}(Q_T)} \le$$

$$\le C_{30}C_{32}C_{31}\left|s^0 - s^{m_k}\right| \|f\|_{L_p(Q_T)}.$$

Thus $i_1 \to 0$ at $k \to \infty$. On the other hand $i_2 = < f, \varphi >$. So for any function $\varphi(x,t) \in \overset{\circ}{W}{}_p^{2,1}(Q_T)$

$$< \mathcal{L}_{s^0}u, \varphi > = < f, \varphi >.$$

It means that $\mathcal{L}_{s^0}u = f(x,t)$ almost everywhere in Q_T, i.e. $s^0 \in E$. The theorem is proved.

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