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CONVERGENCE OF MULTITYPE CONTINUOUS TIME BRANCHING PROCESSES

Abstract

In the paper the limit theorem on the convergence of multitype continuous time branching processes to the branching process with continuous state space is obtained.

In the present paper the limit theorem on the convergence of multitype continuous time branching processes to multi-dimensional Jiřina process is proved.

Let $\xi_t = \left(\xi_t^1, ..., \xi_t^k\right), \ t \geq 0$ be continuous time branching process with k types of particles where the component ξ_t^j is interpreted as the number of particles of type T_j at the time t.

Denote by $F(t;s) = \left(F^1(t;s), ..., F^k(t;s)\right)$ multi-dimensional generating function corresponding to the process ξ_t, where

$$F^i(t;s) = M_i s_1^{\left(\xi_t^1\right)} ... s_k^{\left(\xi_t^k\right)}, \quad s = (s_1, ..., s_k), \ M_i$$

is a conditional mathematical expectation provided that there was one particle of type T_i at initial time.

If we shall give the branching process by probabilities of the passage $P_\alpha^i(t)$, which equal to the probabilities of the fact that one particle of the type T_i at time t passes into totality of particles corresponding to the vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$, whose corrdinates are non-negative integers, then we can write multi-dimensional generating function in the following form

$$F^i(t;s) = \sum_\alpha P_\alpha^i(t) s^\alpha,$$

where $s^\alpha = s_1^{\alpha_1} s_2^{\alpha_2} ... s_k^{\alpha_k}, \quad s^0 = 1$.

It is always assumed that in the neighborhood of zero the probabilities $P_\alpha^i(t)$ are represented in the following form

$$P_\alpha^i(t) = \delta_\alpha^{e_i} + p_\alpha^i t + o(t),$$

where δ_i^j is Kroneker symbol, $e_i = \left(\underbrace{0, ..., 0}_{i-1}, \underbrace{1}_{i}, \underbrace{0, ..., 0}_{k-i}\right)$. Assume that $\sum\limits_\alpha p_\alpha^i = 0$, and let's introduce the generating functions for densities of passage probabilities of of particle of the type i

$$f^i(s) = \sum_\alpha p_\alpha^i s^\alpha,$$

$$f(s)=\left(f^1(s),...,f^k(s)\right).$$

Another way, if the branching process ξ_t is stochastically continuous, then we can represent it by means of the functions

$$f^i(s)=-p^is_i+\sum_{\alpha\neq e_i}p^i_\alpha s_1^{\alpha_1}...s_k^{\alpha_k},$$

where $p^i_\alpha\geq 0,\ \sum\limits_{\alpha\neq e_i}p^i_\alpha=p^i,\ i=\overline{1,k}$..

The values $p^i,\ p^i_\alpha$ allow the simple probabilistic interpretation that each particle of the type T_i existing at the time t independently of its origin and the availability of other particles with the probability $p^i_\alpha\Delta t+o(\Delta t)\ \ (\alpha\neq e_i)$ after small quantity of time Δt turns into α_1 particles of the type T_1,α_2 particles of type T_2, etc. α_k particles of the type T_k, and with the probability $1-p^i\Delta t+o(\Delta t)$ does not test any changes.

It is known [1, p.119] that the generating function $F(t;s)$ of the continuous time branching process with many types of particles satisfies the system of ordinary differential equations

$$\frac{\partial F(t;s)}{\partial t}=f(F(t;s))\ ,\quad F(0;s)=s, \tag{1}$$

$$\frac{\partial F(t;s)}{\partial t}=\sum_{i=1}^k f^i(s)\frac{\partial F(t;s)}{\partial s^i},\qquad F(0;s)=s. \tag{2}$$

In case with one type particles, at formulation of limit theorems the main conditions were imposed on average number of descendants of one particle and on generating function.

In case of multi types of particles such conditions are imposed on the matrix $A=\|a_{ij}\|_{i,j=1}^k$, where $a_{ij}=\dfrac{\partial f^i(1)}{\partial s^j},\ i,j=1,...,k$ and on the generating function $f(s)$.

Let's now move to the formulation of the theorem on convergence in scheme of series of continuous time branching processes with k type of particles to Jiřina process.

Note that for discrete case the analogous results were obtained in [2].

First of all we give the brief description of Jiřina process (the continuous state space).

Let R^k_+ be the set of k dimensional vectors, $x=\left(x^1,...,x^k\right)$ with non-negative coordinates, C^k_- be a set of complex k-dimensional vectors $\lambda=\left(\lambda^1,...,\lambda^k\right)$ whose coordinates have negative real parts.

The homogeneous Markov process $\mu(\tau)=\left(\mu^1(\tau),...,\mu^k(\tau)\right)$ with the values in R^k_+ is called Jiřina process if at $x\in R^k_+,\ \lambda\in C^k_-$

$$M\left[e^{(\lambda,\mu(\tau))}|\mu(0)=x\right]=e^{(x,K(\tau,\lambda))},$$

where $(x,\lambda) = \sum_{i=1}^{k} x^i\lambda^i$, $K(\tau,\lambda) = K^1(\tau,\lambda),...,K^k(\tau;\lambda)$ and $K^j(\tau;\lambda)$ at any $\tau \geq 0$, $j = \overline{1,k}$ is logarithm of Laplace transformation of some infinitely divisible distribution in R_+^k.

If the process $\mu(\tau)$ is stochastically continuous, we give it by means of the functions

$$H^j(\lambda) = (c_j,\lambda) + d^j\left(\lambda^j\right)^2 + \int\limits_{R_+^k}\left(e^{(\lambda;z)} - 1 - \frac{\lambda^j z^j}{1+(z,z)}\right)\Pi_j(dz),$$

where d^j is a non-negative number, the vector $c_j = \left(c_j^1,...,c_j^k\right)$ is such that $c_j^i \geq 0$ when $i \neq j$ and the measure $\Pi_j(dz)$ is such that

$$\int\limits_{0<\sum_{i=1}^{k} z^i<1}\left(\sum_{i\neq j} z^i + \left(z^j\right)^2\right)\Pi_j(dz) < \infty.$$

In this case $K(\tau;\lambda)$ is a solution of the system of ordinary differential equations

$$\frac{\partial K(\tau;\lambda)}{\partial\tau} = H(K(\tau;\lambda));\ K(0;\lambda) = \lambda, \tag{3}$$

where $H(\lambda) = \left(H^1(\lambda),...,H^k(\lambda)\right)$ is an accumulant of Irjini process.

Let now $\xi_t(\varepsilon) = \left(\xi_t^1(\varepsilon),...,\xi_t^k(\varepsilon)\right)$, $t \geq 0$ be a continuous time branching process with k types of particles, ε be a parameter of series. This process is given by means of the generating functions $F_\varepsilon(t;s)$ and $f_\varepsilon(s)$.

For the process $\xi_t(\varepsilon)$ the matrix of average numbers of descendents of the type T_j from one particle of the type T_i will be the following

$$A_\varepsilon = \|a_{ij}(\varepsilon)\|_{i,j=1}^{k}, \qquad \text{where} \qquad a_{ij}(\varepsilon) = \frac{\partial f_\varepsilon^i(1)}{\partial s^j}.$$

The corresponding generating function of the numbers of particles at the time t will be $F_\varepsilon(t;s)$.

Assume that the matrix A_ε allows the following decomposition

$$A_\varepsilon = \varepsilon C + o(\varepsilon), \tag{4}$$

where $C = \|c_{ij}\|_{i,j=1}^{k}$ is a matrix of constants c_{ij}.

Consider the random processes $\mu_\varepsilon(\tau) = \left(\mu_\varepsilon^1(\tau),...,\mu_\varepsilon^k(\tau)\right)$, where $\mu_\varepsilon^j(\tau) = = \dfrac{\xi_{\tau/\varepsilon}^j(\varepsilon)}{b_\varepsilon^j}$, $\xi_0^j(\varepsilon) = \left[x^j b_\varepsilon^j\right]$, $0 < x = \left(x^1,...,x^k\right)$.

Here $[b]$ is an entire part of b, and the normalizing constants b_ε^j are such that $b_\varepsilon^j \to \infty$ when $\varepsilon \to 0$.

The following is true

Theorem. *Let the decomposition (4) hold. Then if for* $0 \geq \lambda = \left(\lambda^1, ..., \lambda^k\right)$ *the conditions*

$$\frac{1}{\varepsilon} b_\varepsilon^j f_\varepsilon^j \left(e^{\lambda^1/b_\varepsilon^1}, ..., e^{\lambda^k/b_\varepsilon^k}\right) \underset{\varepsilon\to 0}{\longrightarrow} H^j(\lambda) \tag{5}$$

and

$$\left.\frac{\partial H^j(\lambda)}{\partial \lambda^i}\right|_{\lambda=0} = c_{ij},$$

are fulfilled, then the finite dimensional distribution of the processes $\mu_\varepsilon(\tau)$ *weakly converge when* $\varepsilon \to 0$ *to the finite dimensional distributions of stochastically continuous Jiřina process* $\mu(\tau)$, $\mu(0) = x$ *with cumulant* $H(\lambda)$.

Proof. Denote

$$H_\varepsilon^j(\lambda) = \frac{1}{\varepsilon} b_\varepsilon^j f_\varepsilon^j \left(e^{\lambda^1/b_\varepsilon^1}, ..., e^{\lambda^k/b_\varepsilon^k}\right) e^{-\lambda^j/b_\varepsilon^j},$$

$$K_\varepsilon^j(\tau;\lambda) = b_\varepsilon^j \log F_\varepsilon^j \left(\frac{\tau}{\varepsilon}; e^{\lambda^1/b_\varepsilon^1}, ..., e^{\lambda^k/b_\varepsilon^k}\right).$$

If we take into account the equation (1) in scheme of series, we obtain

$$\frac{\partial K_\varepsilon^j(\tau;\lambda)}{\partial \tau} = \frac{1}{\varepsilon} b_\varepsilon^j \frac{\partial F_\varepsilon^j\left(\frac{\tau}{\varepsilon}; e^{\lambda^1/b_\varepsilon^1}, ..., e^{\lambda^k/b_\varepsilon^k}\right)}{\partial \tau} \frac{1}{F_\varepsilon^j\left(\frac{\tau}{\varepsilon}; e^{\lambda^1/b_\varepsilon^1}, ..., e^{\lambda^k/b_\varepsilon^k}\right)} =$$

$$= \frac{1}{\varepsilon} b_\varepsilon^j f_\varepsilon^j \left(e^{K_\varepsilon^1(\tau;\lambda)/b_\varepsilon^1}, ..., e^{K_\varepsilon^k(\tau;\lambda)/b_\varepsilon^k}\right) e^{-K_\varepsilon^j(\tau;\lambda)/b_\varepsilon^j} = H_\varepsilon^j\left(K_\varepsilon(\tau;\lambda)\right).$$

Since $K_\varepsilon^j(0;\lambda) = \lambda^j$, hence it follows

$$K_\varepsilon^j(\tau;\lambda) = \lambda^j + \int_0^\tau H_\varepsilon^j\left(K_\varepsilon(\tau;\lambda)\right) dt.$$

On the other hand from the equation (3) we obtain

$$K^j(\tau;\lambda) = \lambda^j + \int_0^\tau H^j\left(K(t;\lambda)\right) dt.$$

Denote

$$\Delta_\varepsilon(\tau;\lambda) = \left|K_\varepsilon(\tau;\lambda) - K(\tau;\lambda)\right|.$$

Then from the two last relations we obtain

$$\Delta_\varepsilon(\tau;\lambda) \leq \int_0^\tau \left|H_\varepsilon^j\left(K(t;\lambda)\right) - H^j\left(K(t;\lambda)\right)\right| +$$

$$+ \int_0^\tau \left|H_\varepsilon^j\left(K_\varepsilon(t;\lambda)\right) - H_\varepsilon^j\left(K(t;\lambda)\right)\right| dt.$$

By condition (5) the first integral does not exceed $\delta_\varepsilon(\tau) \to 0,\ 0 \le \tau \le T$. Consider the second integral

$$\left| H_\varepsilon^j \left(K_\varepsilon (t;\lambda)\right) - H_\varepsilon^j \left(K (t;\lambda)\right) \right| \le \Delta_\varepsilon (\tau;\lambda) \left| \frac{\partial H_\varepsilon (\lambda)}{\partial \lambda} \right| .$$

It follows from the condition (4) that $\sup\limits_{|\lambda| \ge \theta} \left| \frac{\partial H_\varepsilon (\lambda)}{\partial \lambda} \right| < \infty$ for any $\theta < 0$. Therefore the right hand side in the last relation does not exceed $c\Delta_\varepsilon (\tau;\lambda)$ for all λ from the interval $\theta < 0$. Consequently

$$\Delta_\varepsilon (\tau;\lambda) \le \delta_\varepsilon (\tau) + c \int\limits_0^\tau \Delta_\varepsilon (t;\lambda)\, dt.$$

By Gronwall-Bellman lemma [3] hence we obtain

$$\Delta_\varepsilon (\tau;\lambda) \le \delta_\varepsilon (\tau)\, e^{c\tau} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

By the determination it means that

$$K_\varepsilon^j (\tau;\lambda) \underset{\varepsilon \to 0}{\longrightarrow} K^j (\tau;\lambda) .$$

Consequently

$$b_\varepsilon^j \log F_\varepsilon^j \left(\frac{\tau}{\varepsilon}; e^{\lambda^1 / b_\varepsilon^1}, ..., e^{\lambda^k / b_\varepsilon^k} \right) \underset{\varepsilon \to 0}{\longrightarrow} K^j (\tau;\lambda) ,$$

or the same

$$\left\{ F_\varepsilon^j \left(\frac{\tau}{\varepsilon}; e^{\lambda^1 / b_\varepsilon^1}, ..., e^{\lambda^k / b_\varepsilon^k} \right) \right\}^{\left[x^j b_\varepsilon^j\right]} \underset{\varepsilon \to 0}{\longrightarrow} e^{x^j K^j (\tau;\lambda)}.$$

Since $F_\varepsilon (t;s)$ is a generating function of the number of particles at the time t of the process $\xi_t (\varepsilon)$ then from the last relation it is obtained by the induction the convergence of finite dimensional distributions.

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