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DEFECTIVE BASES OF BANACH SPACES

Abstract

In the paper some methods of establishment of basicity of systems in Banach spaces considered as basic or subspaces with respect to others, are given.

By considering a lot of spectral problems with respect to differential operators when spectral parameter is contained in boundary conditions, either defective bases are usually obtained (for example [1,2]), i.e. systems which after moving off finite number elements that form a basis in closure of linear envelope of the system itself or in the considered Banach space.

Evidently similar idea i.e. basicity of a part of eigen and adjoint elements of differential operators was considered first in the paper [3]. Note at that difficulty in investigation when spectral parameter is contained in boundary conditions is concerned with impossibility of consideration of differential operator corresponding to the given problem in a space of dispatch domain of differential expression. Therefore with the help of the known "linearization" method the operator is constructed in extended (with exit from the considered space) space. The similar method has been applied in the papers [4,5], it was constructed an operator in the space $L_2 \oplus \mathbf{C}$, where $\mathbf{C}$ is a complex plane. The basicity of eigen functions of this operator in $L_2 \oplus \mathbf{C}$ is proved. It naturally arises a question on establishment of basicity of systems of eigen and adjoint functions of original problem using this result. Developing this idea we consider more general problem. Let $B,\ B_i,\ i = \overline{1,m}$ be some Banach spaces, where all spaces are given over the same field of scalars (for example, $\mathbf{C}$) and $B_i, \forall i$ be finite-dimensional. Denote the norms of these spaces by $\|\cdot\|_B, \|\cdot\|_{B_i}, i = \overline{1,m}$ respectively. Consider the Cartesian product of these spaces $\hat{B} = B \oplus B_1 \oplus ... \oplus B_m$ with the norm $\|\hat{x}\|_{\hat{B}} = \|x\|_B + \sum\limits_{i=1}^{m} \|x_i\|_{B_i}$, where $\hat{x} = (x, x_1, ..., x_m) \in \hat{B}$ is an element of the space $\hat{B}$. Let $\{\hat{x}_n\}_{n\geq 1} \subset \hat{B}$ be some basis of the space $\hat{B}$. Thus, there arises a question on basicity of the system $\{P_{r_B}\ \hat{x}_n\}_{n\geq 1}$ in the space B, where P_{r_B} is a projector on $B : P_{r_B} : \hat{B} \ \to B$. Note that earlier not applying the mentioned scheme, the basicity of a system of eigen functions of original problem has been established in the papers [1,2].

Besides the abovementioned scheme the establishment of basicity we can consider from another point of view. Namely let $B_1 \subset B_2$ be Banach spaces, where B_1 is dense in B_2, and the convergence of B_2 follows from convergence in B_1, i.e. $\exists C > 0 :\ \|x\|_{B_2} \leq C \|x\|_{B_1}, \forall x \in B_1$. Let $\{x_n\}_{n\geq 1} \subset B_1$ be some basis in B_1. It is obvious that the system $\{x_n\}_{n\geq 1}$ is complete in B_2. There naturally arises a question on basicity of this system in the space B_2. This method of establishment of basis has been applied in the paper [10]. The present paper is devoted to the studying the abovementioned problems.

1. Auxiliary facts. We first cite some assertions and known facts which will be used in the further statement.

Lemma 1. *Let B be some Banach space and $\{x_i\}_{i\geq 1}$ be basis in it. If after change of the finite number elements of basis the obtained system is minimal or complete in B, then it also forms basis in B isomorphic to original one.*

This elementary lemma is cited in the paper [6].

Denote $X_m \equiv \{x_n\}_{n=m}^{\infty}$.

Lemma 2. *Let the system $X_0 \subset B_1$ be minimal in B_1, and the system $X_{-n} \subset B_2 \subset B_1$ be complete and minimal in B_2 at some $n \in N$, where convergence in B_1 follows from convergence in B_2, where B_i, $i = 1, 2$ are some Banach spaces. Then in order that the system X_0 be complete in B_1 it is necessary and sufficient the fulfilment of the condition*

$$L\left[\{x_i^*\}_{i=-n}^{-1}\right] \cap B_1^* = \{0\},$$

where $\{x_i^*\}_{i\geq -n} \subset B_2^*$ is a system in X_{-n} biorthogonal to B_2, B_i^* is a space adjoint to B_i, $L[M]$ is a linear envelope of the set M.

The sufficient part of statement of this lemma is proved in the paper [7]. The necessary part is obvious. Since otherwise the system X_0 has nontrivial annihilator as subset B_1.

Lemma 3. *Let $x_1 \notin \overline{L[X_2]}$ and $x_2 \notin \overline{L[X_3]}$. Then $x_2 \in \overline{L[X_1 \setminus \{x_1\}]}$, where $\overline{L[M]}$ is a closure of linear envelope of the set M in some Banach space B.*

This lemma is proved in the paper [8].

2. The first case. We first consider the trivial case $\hat{B} = B \oplus B_1$, where B_1 is one-dimensional. The following lemma is valid

Lemma 4. *Let the system $\{\hat{x}_n\}_{n\geq 1}$ be basis in $\hat{B}$. Then $\exists n^0 \in N$:$\{P_{r_B}\hat{x}_n\}_{n\geq 1, n\neq n^0}$ is complete in B.*

Proof. Let $e^1 \in B_1$ be basis in B_1. Consequently $P_{r_{B_1}}\hat{x}_n = a_n e^1$, $a_n \in \mathbf{C}$, $n \geq 1$. We take an element $\hat{x}_0 : P_{r_B}\,\hat{x}_0 = 0,\ P_{r_{B_1}}\hat{x}_0 = e^1$. Let $\hat{x}_0 = \sum\limits_{i=1}^{\infty} c_n \hat{x}_n$. It is obvious that $\exists n^0 \in N : c_{n^0} \neq 0$. We show that the system $\{P_{r_B}\hat{x}_n\}_{n\geq 1, n\neq n^0}$ is complete in B. For this it is sufficient to prove basicity of the system $\{\hat{x}_0\} \cup \{\hat{x}_n\}_{n\geq 1, n\neq n^0} \equiv \hat{X}_{n^0}$ in the space $\hat{B}$. We show minimality of this system. Let this be not the case. Then by lemma 3 we have $\hat{x}_0 = \sum\limits_{n\neq n^0} d_n \hat{x}_n$. Comparing two expansions for $\hat{x}_0$ we have: $c_{n^0} = 0$. The obtained contradiction proves minimality. As a result by lemma 1 we have basicity. Then it is easy to note that the system $\{P_{r_B}\hat{x}_n\}_{n\geq 1, n\neq n^0}$ is complete in B.

The lemma is proved.

Theorem 1. *Let the system $\{\hat{x}_n\}_{n\geq 1}$ be basis in $\hat{B}$. Then $\exists n^0 \in N$: $\{P_{r_B}\hat{x}_n\}_{n\geq 1, n\neq n^0}$ forms basis in B.*

Proof. Consider the projection operator $P_{r_B} : \hat{B} \to B$. It is a bounded projector and it is easy to note that it is a normally solvable operator. It is clear that $Ker\ P_{r_B}$ is finite-dimensional. It follows from lemma 4 that $CoKer\ P_{r_B}$ is of zero dimension.

In total by results of the paper [9] we have basicity of the system $\{P_{r_B}\hat{x}_n\}_{n\geq 1, n\neq n^0}$ in B.

The theorem is proved.

For clearness of the further line of reasonings we consider the case $\hat{B} = B \oplus B_1 \oplus B_2$, where B_i are one-dimensional Banach spaces.

Lemma 5. *Let $\{\hat{x}_n\}_{n\geq 1} \subset B$ be basis in $\hat{B}$. Then $\exists n^0, n^1 \in N$ $\{P_{r_B}\hat{x}_n\}_{n\geq 1, n\neq n^0, n^1}$ is complete in B.*

Proof. Denote: $\hat{x}_0 = \left(0, e^1, 0\right)$, $\hat{x}_{-1} = \left(0, 0, e^2\right)$, $e^i \in B_i$, $i = 1, 2$ is basis in B_i. It is obvious that $\hat{x}_0$ and $\hat{x}_{-1}$ are linearly independent. Analogously to the proof of lemma 4 it easy to establish that $\exists n^0 \in N$: the system $\{\hat{x}_0\} \cup \{\hat{x}_n\}_{n\geq 1, n\neq n^0}$ forms basis in $\hat{B}$. Expand $\hat{x}_{-1}$ by this basis: $\hat{x}_{-1} = c_0\hat{x}_0 + \sum\limits_{n\neq n^0} c_n\hat{x}_n$.

It follows from this expansion that $\exists n' \in N \setminus \left\{n^0\right\} : c_{n'} \neq 0$. Following lemma 4 we have basicity of the system $\{\hat{x}_n\}_{n\geq 1, n\neq n^0, n'}$ in $\hat{B}$ and as a result completeness of the system $\{P_{r_B}\hat{x}_n\}_{n\geq 1, n\neq n^0, n'}$ in B.

The lemma is proved.

Continuing this reasoning and granting that m-dimensional space B_i is an isometric space $\underbrace{C \oplus C \oplus ... \oplus C}_{m}$we obtain the following theorem.

Theorem 2. *Let $\hat{B} = B \oplus B_1 \oplus ... \oplus B_m$ and the system $\{\hat{x}_n\}_{n\geq 1}$ be basis in $\hat{B}$, where B_i are finite-dimensional spaces with the dimension r_i. Then there exists a set of indices $J \equiv \{n_1, n_2, ..., n_l\}$, where $n_i \neq n_j$ at $i \neq j$, and $l = \sum\limits_{i=1}^{m} r_i$ for which the system $\{P_{r_B}\hat{x}_n\}_{n\in N\setminus J}$ forms basis in B.*

3. The second case. For facilitating the statement we consider more concrete case. Let $D : W_p^m \longmapsto L_p$ be some differential operator (ordinary) of m-th order with definition domain $D_f \subset W_p^m$. We can consider D as a closed operator $D : L_p \longmapsto L_p$ with definition domain dense in L_p. As usual the operator D is contracted by giving of functionals $f_i : D_f \to C$, $i = \overline{1, l}$. At that all the functionals are linearly independent and bounded from W_p^m to C. For contraction of D_0 we have $D_0 = D/_\Gamma$, $\Gamma \equiv \left\{U \in D_f : f_i(u) = 0,\ i = \overline{1, l}\right\}$.

Thus let the system $\{u_n\}_{n\in N}$ formed from eigen and adjoint elements of the operator D_0 form basis in $D_{0f} \equiv D_f \cap \Gamma$. It is absolutely obvious that this system is incomplete in D_f and $dimCo\ker L\left[\{u_n\}_{n\in N}\right] = l$. Denoting by $\{\nu_i\}_{i=1}^{l}$ a system biorthogonal to $\{f_i\}_1^l$ we get that the system $\{\nu_i\}_1^l \cup \{u_n\}_{n\in N}$ forms a basis in $D_f \equiv W_{p_1}^m$ moreover, for biorthogonal system $\{f_i^*\}_{i\in N}$ we have $f_i^* = f_i$, $i = \overline{1, l}$. We obtain from lemma 2 immediately that the system $\{u_n\}_{n\in N}$ is complete in L_p. Thus, after throwing out l elements from the system $\{\nu_i\}_1^l \cup \{u_n\}_{n\in N} \subset D_f$, the obtained system remains complete in L_p.Usually by investigating eigen values problem and functions for differential operators line of reasoning is conducted in such reverse order. In conclusion we come to the following deduction: for establishment of basis properties of systems of eigen and adjoint elements of differential operators the natural spaces are either their definition domains or extended spaces in terms of case 2.

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