Yakub N. ALIYEV

ON THE BASIS PROPERTIES OF STURM-LIOUVILLE PROBLEMS WITH DECREASING AFFINE BOUNDARY CONDITIONS

Abstract

We consider Sturm-Liouville problems with a boundary condition linearly dependent on the eigenparameter. We study the decreasing affine case where non-real or non-simple (multiple) eigenvalues are possible. We prove that the system of root (i.e. eigen and associated) functions of the corresponding operator, with arbitrary function removed, form a basis in $L_2(0,1)$, except some cases where this system is neither complete nor minimal. The method used is based on the determination of the explicit form of the biorthogonal system. For the basisness in L_2 we prove that the part of the system of root functions is quadratically close to sine or cosine systems. We also consider these basis properties in the context of general L_p. For this we use F. Riesz's theorem.

Consider the following spectral problem

$$-y'' + q(x)y = \lambda y,\ 0 < x < 1, \tag{0.1}$$

$$y'(0)\sin\beta = y(0)\cos\beta,\ 0 \le \beta < \pi, \tag{0.2}$$

$$y'(1) = (a\lambda + b)y(1), \tag{0.3}$$

where a, b are real constants and $a < 0$, λ is the spectral parameter, $q(x)$ is a real valued and continuous function over the interval $[0,1]$.

The present article is about the basis properties in $L_p(0,1)$, $1 < p < \infty$ of the system of root functions of the boundary value problem (0.1)-(0.3).

It was proved in [2] (see also [1]) that the eigenvalues of the boundary value problem (0.1)-(0.3) form an infinite sequence accumulating only at $+\infty$ and only following cases are possible: **(a)** all the eigenvalues are real and simple; **(b)** all the eigenvalues are real and all, except one double, are simple; **(c)** all the eigenvalues are real and all, except one triple, are simple; **(d)** all the eigenvalues are simple and all, except a conjugate pair of non-real, are real.

The eigenvalues λ_n $(n \ge 0)$ will be considered to be listed according to non-decreasing real part and repeated according to algebraic multiplicity. Asymptotics of eigenvalues and oscillation of eigenfunctions of the boundary value problem (0.1)-(0.3), with linear function in the boundary condition, replaced by general rational function, were studied in a recent paper [3]. For affine (linear) decreasing case this asymptotics is as follows [2]:

$$\lambda_n = \begin{cases} \left(n - \frac{1}{2}\right)^2 \pi^2 + O(1),\ \beta \ne 0, \\ n^2\pi^2 + O(1),\ \beta = 0. \end{cases} \tag{0.4}$$

The case $a > 0$ in our problem is considerably simpler and can be found as a special case in papers [10,11]. In paper [15] the following boundary value problem was considered:

$$-y'' = \lambda y,\ 0 < x < 1, \tag{0.5}$$

$$y'(0) = 0,\ y'(1) = a\lambda y(1),\ a \neq 0. \tag{0.6}$$

For this problem only cases (a) and (b) are possible and in [15] a complete solution of the problem on the basis properties in $L_p(0,1)$ $(1 < p < \infty)$ of the system of root functions, was given. We shall discuss this problem further in the last section. The situation in problem (0.1)-(0.3) is much more complicated, with the possibility of non-real eigenvalues and of eigenvalues with algebraic multiplicity 3 being present.

There is a vast literature on the boundary value problems with a spectral parameter in the boundary conditions (see e.g. [7,17]). We mention also [4,12] as recent contributions in this area.

1. Inner products and norms of eigenfunctions. Let $y(x,\lambda)$ be a non-zero solution of (0.1), (0.2), and we write the characteristic equation

$$\varpi(\lambda) = y'(1,\lambda) - (a\lambda + b)y(1,\lambda). \tag{1.1}$$

By (0.3), λ_n is an eigenvalue of (0.1)-(0.3) if $\varpi(\lambda_n) = 0$. λ_n is a simple eigenvalue if $\varpi(\lambda_n) = 0 \neq \varpi'(\lambda_n)$. λ_k is a double eigenvalue if

$$\varpi(\lambda_k) = \varpi'(\lambda_k) = 0 \neq \varpi''(\lambda_k), \tag{1.2}$$

and triple eigenvalue if

$$\varpi(\lambda_k) = \varpi'(\lambda_k) = \varpi''(\lambda_k) = 0 \neq \varpi'''(\lambda_k). \tag{1.3}$$

We note also that $y(x,\lambda) \to y(x,\lambda_n)$, uniformly, according to $x \in [0,1]$, as $\lambda \to \lambda_n$, and the function $y_n(x) = y(x,\lambda_n)$ is an eigenfunction of (0.1)-(0.3) corresponding to eigenvalue λ_n (see [5, Sect. 10.72]). By (0.1)-(0.3) we have,

$$-y_n'' + q(x)y_n = \lambda_n y_n,$$

$$y_n'(0)\sin\beta = y_n(0)\cos\beta,$$

$$y_n'(1) = (a\lambda_n + b)y_n(1).$$

Throughout this paper we denote by $(\cdot,\cdot)$ the scalar product in $L_2(0,1)$.

Lemma 1.1. *Let y_n, y_m be eigenfunctions corresponding to eigenvalues λ_n, λ_m $(\lambda_n \neq \overline{\lambda_m})$. Then following equality holds:*

$$(y_n, y_m) = -a y_n(1)\overline{y_m(1)}. \tag{1.4}$$

Proof. To begin we note that

$$\frac{d}{dx}(y(x,\lambda)\overline{y'(x,\mu)} - y'(x,\lambda)\overline{y(x,\mu)}) = (\lambda - \overline{\mu})y(x,\lambda)\overline{y(x,\mu)}.$$

By integrating this identity from 0 to 1, we obtain

$$(\lambda - \overline{\mu})(y(\cdot,\lambda), y(\cdot,\mu)) = \left.(y(x,\lambda)\overline{y'(x,\mu)} - y'(x,\lambda)\overline{y(x,\mu)})\right|_0^1. \tag{1.5}$$

From (0.2), we obtain

$$y(0,\lambda)\overline{y'(0,\mu)} - y'(0,\lambda)\overline{y(0,\mu)} = 0. \tag{1.6}$$

By (1.1),

$$y(1,\lambda)\overline{y'(1,\mu)} - y'(1,\lambda)\overline{y(1,\mu)} = -a(\lambda-\overline{\mu})y(1,\lambda)\overline{y(1,\mu)}$$
$$+y(1,\lambda)\overline{\varpi(\mu)} - \overline{y(1,\mu)}\varpi(\lambda). \tag{1.7}$$

From (1.5)-(1.7), it follows that for $\lambda \neq \overline{\mu}$,

$$(y(\cdot,\lambda),y(\cdot,\mu)) = -ay(1,\lambda)\overline{y(1,\mu)} + y(1,\lambda)\frac{\overline{\varpi(\mu)}}{\lambda-\overline{\mu}} - \overline{y(1,\mu)}\frac{\varpi(\lambda)}{\lambda-\overline{\mu}}, \tag{1.8}$$

which is the generalization of analogous formula in [5, Sect. 10.72].

We obtain (1.4) from (1.8) by substituting parameters λ,μ respectively by λ_n,λ_m, where we use the fact that $\varpi(\lambda_n) = \varpi(\lambda_m) = 0$.

Since λ_n,λ_m are eigenvalues of (0.1)-(0.3) then $\varpi(\lambda_n) = \varpi(\lambda_m) = 0$, hence by tending $\lambda \to \lambda_n$ ($\overline{\mu} \neq \lambda_n$) and then tending $\mu \to \lambda_m$ we obtain (1.4). □

Now we collect some easy facts about inner products of eigenfunctions.

Lemma 1.2. *If λ_n is a real eigenvalue then*

$$\|y_n\|_2^2 = (y_n,y_n) = -ay_n(1)^2 - y_n(1)\varpi'(\lambda_n). \tag{1.9}$$

Proof. Since $\varpi(\lambda_n) = 0$ then $\varpi(\lambda)/(\lambda-\lambda_n) \to \varpi'(\lambda_n)$ as $\lambda \to \lambda_n$. Therefore, by tending $\mu \to \lambda_n$ ($\lambda \neq \lambda_n$) and then tending $\lambda \to \lambda_n$ in (1.8) we obtain (1.9). □

Corollary 1.1. *If λ_k is a multiple eigenvalue then*

$$\|y_k\|_2^2 = (y_k,y_k) = -ay_k(1)^2. \tag{1.10}$$

An immediate corollary of (1.4) is the following.

Corollary 1.2. *If λ_r is a non-real eigenvalue then*

$$\|y_r\|_2^2 = -a|y_r(1)|^2. \tag{1.11}$$

Proof. Since $\lambda_r \neq \overline{\lambda_r}$ then (1.11) follows at once from (1.4) by replacing λ_n,λ_m by λ_r. □

For the eigenfunction y_n define

$$B_n = \|y_n\|_2^2 + a|y_n(1)|^2. \tag{1.12}$$

The following corollary of (1.9) and (1.11) will be useful (cf. [1, Theorem 4.3]).

Corollary 1.3. *$B_n \neq 0$ if and only if the corresponding eigenvalue λ_n is real and simple.*

We conclude this section with the following

Lemma 1.3. *If λ_r and $\lambda_s = \overline{\lambda_r}$ are a conjugate pair of non-real eigenvalues then*

$$(y_r,y_s) = -ay_r(1)^2 - y_r(1)\varpi'(\lambda_r). \tag{1.13}$$

The proof is similar to the proof of (1.9).

We note also that since all non-real eigenvalues of (0.1)-(0.3) are simple then $\varpi'(\lambda_r) \neq 0$ in (1.13).

2. Inner products and norms of associated functions. We need to establish some facts about eigenfunctions and associated functions corresponding to real eigenvalues. So in this and subsequent sections we consider only real eigenvalues.

If λ_k is a multiple eigenvalue ($\lambda_k = \lambda_{k+1}$) then for the first order associated function y_{k+1} corresponding to eigenfunction y_k, following relations hold [16, p. 28]:

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k,$$

$$y'_{k+1}(0)\sin\beta = y_{k+1}(0)\cos\beta,$$
$$y'_{k+1}(1) = (a\lambda_k + b)y_{k+1}(1) + ay_k(1).$$

If λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) then together with the first order associated function y_{k+1} there exists the second order associated function y_{k+2} for which following relations hold:

$$-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1},$$

$$y'_{k+2}(0)\sin\beta = y_{k+2}(0)\cos\beta,$$
$$y'_{k+2}(1) = (a\lambda_k + b)y_{k+2}(1) + ay_{k+1}(1).$$

Following well known properties of associated functions play an important role in our investigation. The functions $y_{k+1} + cy_k$ and $y_{k+2} + dy_k$, where c and d are arbitrary constants, are also associated functions of the first and second order respectively. Next we observe that replacing the associated function y_{k+1} by $y_{k+1} + cy_k$, the associated function y_{k+2} changes to $y_{k+2} + cy_{k+1}$.

By differentiating (0.1), (0.2) and (1.1) with respect to λ we obtain

$$-y''_{\lambda}(x,\lambda) + q(x)y_{\lambda}(x,\lambda) = \lambda y_{\lambda}(x,\lambda) + y(x,\lambda),$$

$$y'_{\lambda}(0,\lambda)\sin\beta = y_{\lambda}(0,\lambda)\cos\beta,$$
$$\varpi'(\lambda) = y'_{\lambda}(1,\lambda) - (a\lambda + b)y_{\lambda}(1,\lambda) - ay(1,\lambda),$$

where the suffix denotes differentiation with respect to λ.

Let λ_k be a multiple (double or triple) eigenvalue of (0.1)-(0.3). Since $\varpi(\lambda_k) = \varpi'(\lambda_k) = 0$ then $y(x,\lambda) \to y_k$, $y_{\lambda}(x,\lambda) \to \tilde{y}_{k+1}$, uniformly according to $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\tilde{y}_{k+1}$ is one of the associated functions of the first order, and it is obvious that $\tilde{y}_{k+1} = y_{k+1} + \tilde{c}y_k$, where $\tilde{c} = (\tilde{y}_{k+1}(1) - y_{k+1}(1))/y_k(1)$.

Similarly, with the index notation introduced above, we may write

$$-y''_{\lambda\lambda}(x,\lambda) + q(x)y_{\lambda\lambda}(x,\lambda) = \lambda y_{\lambda\lambda}(x,\lambda) + 2y_{\lambda}(x,\lambda),$$

$$y'_{\lambda\lambda}(0,\lambda)\sin\beta = y_{\lambda\lambda}(0,\lambda)\cos\beta,$$
$$\varpi''(\lambda) = y'_{\lambda\lambda}(1,\lambda) - (a\lambda + b)y_{\lambda\lambda}(1,\lambda) - 2ay_{\lambda}(1,\lambda).$$

We note again that if λ_k is a triple eigenvalue of (0.1)-(0.3) then $\varpi''(\lambda_k) = 0$, hence $y_{\lambda\lambda} \to 2\tilde{y}_{k+2}$, uniformly according to $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\tilde{y}_{k+2}$ is one of the associated functions of the second order corresponding to the first associated function $\tilde{y}_{k+1}$, and it is obvious that $\tilde{y}_{k+2} = y_{k+2} + \tilde{c}y_{k+1} + \tilde{d}y_k$, and $\tilde{d} = (\tilde{y}_{k+2}(1) - y_{k+2}(1) - \tilde{c}y_{k+1}(1))/y_k(1)$.

Lemma 2.1. *If λ_k is a multiple eigenvalue and $\lambda_n \neq \lambda_k$ then*

$$(y_{k+1}, y_n) = -a y_{k+1}(1) y_n(1). \tag{2.1}$$

Proof. Differentiating (1.8) with respect to λ we obtain

$$(y_\lambda(\cdot,\lambda), y(\cdot,\mu)) = -a y_\lambda(1,\lambda) y(1,\mu) + y_\lambda(1,\lambda)\frac{\varpi(\mu)}{\lambda-\mu}$$

$$-y(1,\lambda)\frac{\varpi(\mu)}{(\lambda-\mu)^2} - y(1,\mu)\frac{\varpi'(\lambda)}{\lambda-\mu} + y(1,\mu)\frac{\varpi(\lambda)}{(\lambda-\mu)^2}. \tag{2.2}$$

Tending $\mu \to \lambda_n$ $(\lambda \neq \lambda_n)$ and then tending $\lambda \to \lambda_k$ in (2.2) we obtain $(\tilde{y}_{k+1}, y_n) = -a\tilde{y}_{k+1}(1) y_n(1)$. We note that $\tilde{y}_{k+1} = y_{k+1} + \tilde{c} y_k$. Therefore,

$$(y_{k+1}, y_n) + \tilde{c}(y_k, y_n) = -a y_{k+1}(1) y_n(1) - a\tilde{c} y_k(1) y_n(1).$$

From this and the equality $(y_k, y_n) = -a y_k(1) y_n(1)$ it follows that (2.1) is true. We note that legality of differentiation and subsequent passage to limit within integrals is based on [8, Ch.3, §4, Theorems 1, 2]. □

Lemma 2.2. *If λ_k is a multiple eigenvalue then*

$$(y_{k+1}, y_k) = -a y_{k+1}(1) y_k(1) - y_k(1)\frac{\varpi''(\lambda_k)}{2}. \tag{2.3}$$

Proof. Tending $\mu \to \lambda_k$ $(\lambda \neq \lambda_k)$ and then tending $\lambda \to \lambda_k$ in (2.2) we obtain

$$(\tilde{y}_{k+1}, y_k) = -a\tilde{y}_{k+1}(1) y_k(1) - y_k(1)\frac{\varpi''(\lambda_k)}{2}.$$

In analogy with previous lemma, noting (1.10) we obtain (2.3). □

Lemma 2.3. *If λ_k is a multiple eigenvalue then*

$$\|y_{k+1}\|_2^2 = (y_{k+1}, y_{k+1}) = -a y_{k+1}(1)^2$$

$$-\widehat{y}_{k+1}(1)\frac{\varpi''(\lambda_k)}{2} - y_k(1)\frac{\varpi'''(\lambda_k)}{6}. \tag{2.4}$$

where $\widehat{y}_{k+1} = y_{k+1} - \tilde{c} y_k$.

Proof. Differentiating (2.2) with respect to μ we obtain

$$(y_\lambda(\cdot,\lambda), y_\mu(\cdot,\mu)) = -a y_\lambda(1,\lambda) y_\mu(1,\mu) + y_\lambda(1,\lambda)\frac{\varpi'(\mu)}{\lambda-\mu}$$

$$+y_\lambda(1,\lambda)\frac{\varpi(\mu)}{(\lambda-\mu)^2} - y(1,\lambda)\frac{\varpi'(\mu)}{(\lambda-\mu)^2} - y(1,\lambda)\frac{2\varpi(\mu)}{(\lambda-\mu)^3} - y_\mu(1,\mu)\frac{\varpi'(\lambda)}{\lambda-\mu}$$

$$-y(1,\mu)\frac{\varpi'(\lambda)}{(\lambda-\mu)^2} + y_\mu(1,\mu)\frac{\varpi(\lambda)}{(\lambda-\mu)^2} + y(1,\mu)\frac{2\varpi(\lambda)}{(\lambda-\mu)^3}. \tag{2.5}$$

Tending $\mu \to \lambda_k$ $(\lambda \neq \lambda_k)$ and then tending $\lambda \to \lambda_k$ we obtain

$$(\tilde{y}_{k+1}, \tilde{y}_{k+1}) = -a\tilde{y}_{k+1}(1)^2 - \tilde{y}_{k+1}(1)\frac{\varpi''(\lambda_k)}{2} - y_k(1)\frac{\varpi'''(\lambda_k)}{6}.$$

As in previous lemmas substituting $\tilde{y}_{k+1} = y_{k+1} + \tilde{c}y_k$, after some computations we prove (2.4). □

Lemma 2.4. *If λ_k is a triple eigenvalue and $\lambda_n \neq \lambda_k$ then*

$$(y_{k+2}, y_n) = -ay_{k+2}(1)y_n(1). \tag{2.6}$$

Proof. Differentiating (2.2) with respect to λ we obtain

$$(y_{\lambda\lambda}(\cdot,\lambda), y(\cdot,\mu)) = -ay_{\lambda\lambda}(1,\lambda)y(1,\mu) + y_{\lambda\lambda}(1,\lambda)\frac{\varpi(\mu)}{\lambda-\mu} - y_\lambda(1,\lambda)\frac{2\varpi(\mu)}{(\lambda-\mu)^2}$$

$$+y(1,\lambda)\frac{2\varpi(\mu)}{(\lambda-\mu)^3} - y(1,\mu)\frac{\varpi''(\lambda)}{\lambda-\mu} + y(1,\mu)\frac{2\varpi'(\lambda)}{(\lambda-\mu)^2} - y(1,\mu)\frac{2\varpi(\lambda)}{(\lambda-\mu)^3}.$$

Tending $\lambda \to \lambda_k$ $(\mu \neq \lambda_k)$ we obtain

$$(\tilde{y}_{k+2}, y(\cdot,\mu)) = -a\tilde{y}_{k+2}(1)y(1,\mu) + \tilde{y}_{k+2}(1)\frac{\varpi(\mu)}{\lambda_k-\mu}$$

$$-\tilde{y}_{k+1}(1)\frac{\varpi(\mu)}{(\lambda_k-\mu)^2} + y_k(1)\frac{\varpi(\mu)}{(\lambda_k-\mu)^3}. \tag{2.7}$$

Tending $\mu \to \lambda_n$ we obtain $(\tilde{y}_{k+2}, y_n) = -a\tilde{y}_{k+2}(1)y_n(1)$, from which noting $\tilde{y}_{k+2} = y_{k+2} + \tilde{c}y_{k+1} + \tilde{d}y_k$, $(y_k, y_n) = -ay_k(1)y_n(1)$ and (2.1) we obtain (2.6). □

Lemma 2.5. *If λ_k is a triple eigenvalue then*

$$(y_{k+2}, y_k) = -ay_{k+2}(1)y_k(1) - y_k(1)\frac{\varpi'''(\lambda_k)}{6}. \tag{2.8}$$

Proof. Tending $\mu \to \lambda_k$ in (2.7) and noting (1.3) we obtain

$$(\tilde{y}_{k+2}, y_k) = -a\tilde{y}_{k+2}(1)y_k(1) - y_k(1)\frac{\varpi'''(\lambda_k)}{6}.$$

Similar to previous lemma, using (2.3) and noting (1.3) we prove (2.8). □

Lemma 2.6. *If λ_k is a triple eigenvalue then*

$$(y_{k+2}, y_{k+1}) = -ay_{k+2}(1)y_{k+1}(1) - \widehat{y}_{k+1}(1)\frac{\varpi'''(\lambda_k)}{6} - y_k(1)\frac{\varpi^{IV}(\lambda_k)}{24}. \tag{2.9}$$

Proof. Differentiating (2.7) with respect to μ we obtain

$$(\tilde{y}_{k+2}, y_\mu(\cdot,\mu)) = -a\tilde{y}_{k+2}(1)y_\mu(1,\mu)$$

$$+\tilde{y}_{k+2}(1)\frac{\varpi'(\mu)}{\lambda_k-\mu} + \tilde{y}_{k+2}(1)\frac{\varpi(\mu)}{(\lambda_k-\mu)^2}$$

$$-\tilde{y}_{k+1}(1)\frac{\varpi'(\mu)}{(\lambda_k-\mu)^2} - \tilde{y}_{k+1}(1)\frac{2\varpi(\mu)}{(\lambda_k-\mu)^3}$$

$$+y_k(1)\frac{\varpi'(\mu)}{(\lambda_k-\mu)^3} + y_k(1)\frac{3\varpi(\mu)}{(\lambda_k-\mu)^4}. \tag{2.10}$$

Tending $\mu \to \lambda_k$, after simplifications we obtain (2.9). □

Lemma 2.7. *If λ_k is a triple eigenvalue then*

$$\|y_{k+2}\|_2^2 = (y_{k+2}, y_{k+2}) = -ay_{k+2}(1)^2$$

$$-\widehat{y}_{k+2}(1)\frac{\varpi'''(\lambda_k)}{6} - \widehat{y}_{k+1}(1)\frac{\varpi^{IV}(\lambda_k)}{24} - y_k(1)\frac{\varpi^{V}(\lambda_k)}{120}, \tag{2.11}$$

where $\widehat{y}_{k+2} = y_{k+2} - \tilde{c}\widehat{y}_{k+1} - \tilde{d}y_k$.

Proof. Differentiating (2.10) with respect to μ we obtain

$$(\tilde{y}_{k+2}, y_{\mu\mu}(\cdot,\mu)) = -a\tilde{y}_{k+2}(1)y_{\mu\mu}(1,\mu)$$

$$+\tilde{y}_{k+2}(1)\frac{\varpi''(\mu)}{\lambda_k-\mu} + \tilde{y}_{k+2}(1)\frac{2\varpi'(\mu)}{(\lambda_k-\mu)^2} + \tilde{y}_{k+2}(1)\frac{2\varpi(\mu)}{(\lambda_k-\mu)^3}$$

$$-\tilde{y}_{k+1}(1)\frac{\varpi''(\mu)}{(\lambda_k-\mu)^2} - \tilde{y}_{k+1}(1)\frac{4\varpi'(\mu)}{(\lambda_k-\mu)^3} - \tilde{y}_{k+1}(1)\frac{6\varpi(\mu)}{(\lambda_k-\mu)^4}$$

$$+y_k(1)\frac{\varpi''(\mu)}{(\lambda_k-\mu)^3} + y_k(1)\frac{6\varpi'(\mu)}{(\lambda_k-\mu)^4} + y_k(1)\frac{12\varpi(\mu)}{(\lambda_k-\mu)^5}.$$

Tending $\mu \to \lambda_k$, after elementary but lengthy computations we obtain (2.11). □

3. Existence of auxiliary associated functions. In this section we shall prove the existence of some special associated functions which have the properties of an eigenfunction in inner products with original associated functions. In the proof of these results we shall require some facts about the inner products of root functions, which have been gathered in sections 1, 2.

Lemma 3.1. *If λ_k is a double eigenvalue then there exists associated function $y^*_{k+1} = y_{k+1} + c_1 y_k$, where c_1 is a constant, for which*

$$(y^*_{k+1}, y_{k+1}) = -ay^*_{k+1}(1)y_{k+1}(1). \tag{3.1}$$

Proof. Summing (2.4) with (2.3) multiplied by c_1 we obtain

$$(y_{k+1} + c_1 y_k, y_{k+1}) = -a(y_{k+1}(1) + c_1 y_k(1))y_{k+1}(1)$$

$$-(\widehat{y}_{k+1}(1) + c_1 y_k(1))\frac{\varpi''(\lambda_k)}{2} - y_k(1)\frac{\varpi'''(\lambda_k)}{6}.$$

The equality (3.1) holds exactly if we take

$$c_1 = -\frac{y_k(1)\varpi'''(\lambda_k) + 3\widehat{y}_{k+1}(1)\varpi''(\lambda_k)}{3y_k(1)\varpi''(\lambda_k)}. \ \square$$

It should be pointed out here that $y^*_{k+1}(1) = 0$ if and only if $\varpi'''(\lambda_k) = 3\tilde{c}\varpi''(\lambda_k)$. Before proceeding, we note also that for $\lambda_n \neq \lambda_k$,

$$(y^*_{k+1}, y_n) = -ay^*_{k+1}(1)y_n(1), \tag{3.2}$$

$$(y^*_{k+1}, y_k) = -ay^*_{k+1}(1)y_k(1) - y_k(1)\frac{\varpi''(\lambda_k)}{2}. \tag{3.3}$$

We shall now concentrate more on the study of triple eigenvalue case.

Lemma 3.2. *If λ_k is a triple eigenvalue then there exist associated functions $y^{**}_{k+1} = y_{k+1} + c_2 y_k$, $y^{**}_{k+2} = y_{k+2} + c_2 y_{k+1}$, where c_2 is a constant, for which*

$$(y^{**}_{k+1}, y_{k+2}) = -a y^{**}_{k+1}(1) y_{k+2}(1), \tag{3.4}$$

$$(y^{**}_{k+2}, y_{k+1}) = -a y^{**}_{k+2}(1) y_{k+1}(1). \tag{3.5}$$

Proof. The proof in this case is very similar to the proof of Lemma 3.1, so we only indicate the main steps. Summing (2.9) with (2.8) multiplied by c_2, and (2.9) with (2.4) multiplied by c_2, where

$$c_2 = -\frac{y_k(1)\varpi^{IV}(\lambda_k) + 4\widehat{y}_{k+1}(1)\varpi'''(\lambda_k)}{4y_k(1)\varpi'''(\lambda_k)},$$

we obtain (3.4) and (3.5) correspondingly. □

We shall now indicate some relations between y^{**}_{k+1}, y^{**}_{k+2} and other root functions:

$$(y^{**}_{k+1}, y_n) = -a y^{**}_{k+1}(1) y_n(1), \ (n \neq k+1, \ k+2); \tag{3.6}$$

$$(y^{**}_{k+1}, y_{k+1}) = -a y^{**}_{k+1}(1) y_{k+1}(1) - y_k(1)\frac{\varpi'''(\lambda_k)}{6}; \tag{3.7}$$

$$(y^{**}_{k+2}, y_n) = -a y^{**}_{k+2}(1) y_n(1), \ (n \neq k, \ k+1, \ k+2); \tag{3.8}$$

$$(y^{**}_{k+2}, y_k) = -a y^{**}_{k+2}(1) y_k(1) - y_k(1)\frac{\varpi'''(\lambda_k)}{6}; \tag{3.9}$$

$$(y^{**}_{k+2}, y_{k+2}) = -a y^{**}_{k+2}(1) y_{k+2}(1) - Q_k,$$

where

$$Q_k = \widehat{y}_{k+2}(1)\frac{\varpi'''(\lambda_k)}{6} + \widehat{y}_{k+1}(1)\frac{\varpi^{IV}(\lambda_k)}{24} + y_k(1)\frac{\varpi^{V}(\lambda_k)}{120}$$
$$+c_2\left(\widehat{y}_{k+1}(1)\frac{\varpi'''(\lambda_k)}{6} + y_k(1)\frac{\varpi^{IV}(\lambda_k)}{24}\right).$$

It is worthwhile to note that $y^{**}_{k+1}(1) = 0$ if and only if $\varpi^{IV}(\lambda_k) = 4\tilde{c}\varpi'''(\lambda_k)$.

Lemma 3.3. *If λ_k is a triple eigenvalue then there exists associated function $y^{\#}_{k+2} = y_{k+2} + d_1 y_k$, where d_1 is a constant, for which*

$$(y^{\#}_{k+2}, y_{k+2}) = -a y^{\#}_{k+2}(1) y_{k+2}(1). \tag{3.10}$$

Proof. Summing (2.11) with (2.8) multiplied by d_1, where

$$d_1 = -\frac{y_k(1)\varpi^{V}(\lambda_k) + 5\widehat{y}_{k+1}(1)\varpi^{IV}(\lambda_k) + 20\widehat{y}_{k+2}(1)\varpi'''(\lambda_k)}{20y_k(1)\varpi'''(\lambda_k)},$$

we obtain (3.10). □

With the above notations we also have

$$(y^{\#}_{k+2}, y_n) = -a y^{\#}_{k+2}(1) y_n(1), \ (n \neq k, \ k+1, \ k+2); \tag{3.11}$$

$$(y^{\#}_{k+2}, y_k) = -a y^{\#}_{k+2}(1) y_k(1) - y_k(1)\frac{\varpi'''(\lambda_k)}{6}; \tag{3.12}$$

$$(y^{\#}_{k+2}, y_{k+1}) = -ay^{\#}_{k+2}(1)y_{k+1}(1) - \widehat{y}_{k+1}(1)\frac{\varpi'''(\lambda_k)}{6} - y_k(1)\frac{\varpi^{IV}(\lambda_k)}{24}.$$

We observe also that for the function $y^{\#\#}_{k+2} = y^{**}_{k+2} + d_2 y_k$, where

$$d_2 = -\frac{Q_k}{y_k(1)\varpi'''(\lambda_k)/6},$$

both of the following equalities hold:

$$(y^{\#\#}_{k+2}, y_{k+1}) = -ay^{\#\#}_{k+2}(1)y_{k+1}(1), \tag{3.13}$$

$$(y^{\#\#}_{k+2}, y_{k+2}) = -ay^{\#\#}_{k+2}(1)y_{k+2}(1). \tag{3.14}$$

Note also that for the function $y^{\#\#}_{k+2}$, equalities like (3.11), (3.12) are also true:

$$(y^{\#\#}_{k+2}, y_n) = -ay^{\#\#}_{k+2}(1)y_n(1), \ (n \neq k, \ k+1, \ k+2); \tag{3.15}$$

$$(y^{\#\#}_{k+2}, y_k) = -ay^{\#\#}_{k+2}(1)y_k(1) - y_k(1)\frac{\varpi'''(\lambda_k)}{6}. \tag{3.16}$$

We remark that $y^{\#\#}_{k+2}(1) = 0$ if and only if

$$\frac{\varpi^{IV}(\lambda_k)}{4!}\left(\frac{\varpi^{IV}(\lambda_k)}{4!} - \tilde{c}\frac{\varpi'''(\lambda_k)}{3!}\right) = \frac{\varpi'''(\lambda_k)}{3!}\left(\frac{\varpi^{V}(\lambda_k)}{5!} - \tilde{d}\frac{\varpi'''(\lambda_k)}{3!}\right).$$

4. Asymptotic formulas for eigenfunctions.

Lemma 4.1. *Following asymptotic formula is true:*

$$y_n = \begin{cases} \sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi x + O(1/n), \ \beta \neq 0, \\ \sqrt{2}\sin n\pi x + O(1/n), \ \beta = 0. \end{cases} \tag{4.1}$$

Proof. From (0.4) it follows that

$$\sqrt{\lambda_n} = \begin{cases} \pi\left(n - \frac{1}{2}\right) + O(1/n), \ \beta \neq 0, \\ \pi n + O(1/n), \ \beta = 0. \end{cases} \tag{4.2}$$

Denote by $\psi_1(x,\mu)$ and $\psi_2(x,\mu)$ a fundamental system of solutions of the differential equation $u'' - q(x)u + \mu^2 u = 0$, with initial conditions

$$\psi_1(0,\mu) = \psi_2(0,\mu) = 1, \ \psi_1'(0,\mu) = i\mu, \ \psi_2'(0,\mu) = -i\mu. \tag{4.3}$$

As is well known (see [13] or [16, Ch.II, §4.5]), for sufficiently large μ,

$$\psi_j(x,\mu) = \exp(\mu\theta_j x)(1 + O(1/\mu)) \quad (j = 1, 2), \tag{4.4}$$

where $\theta_1 = -\theta_2 = i$.

We seek the eigenfunction $y_n(x)$ corresponding to the eigenvalue λ_n in the form

$$y_n(x) = P_n\left\| \begin{matrix} \psi_1(x,\sqrt{\lambda_n}) & \psi_2(x,\sqrt{\lambda_n}) \\ U(\psi_1(x,\sqrt{\lambda_n})) & U(\psi_2(x,\sqrt{\lambda_n})) \end{matrix} \right\|, \tag{4.5}$$

where

$$P_n = \begin{cases} \left(i\sqrt{2\lambda_n}\sin\beta\right)^{-1}, & \text{if } \beta \neq 0; \\ \left(i\sqrt{2}\right)^{-1}, & \text{if } \beta = 0, \end{cases} \tag{4.6}$$

and

$$U(\psi(x)) = \psi(0)\cos\beta - \psi'(0)\sin\beta, \tag{4.7}$$

for arbitrary function $\psi(x) \in C^1[0,1]$. From relations (4.2)-(4.7) we easily obtain (4.1). □

We shall need later the following estimates, which follows at once from (4.1).

Lemma 4.2. *Following asymptotic formulas are true:*

$$\|y_n\|_2 = 1 + O(1/n), \tag{4.8}$$

$$y_n(x) = O(1), \tag{4.9}$$

$$y_n(1) = O(1/n), \tag{4.10}$$

$$B_n = 1 + O(1/n). \tag{4.11}$$

After these preliminaries, we may study the basis properties of root functions. For this purpose we need to establish theorems on minimality.

5. Minimality of the system of root functions. We discuss the various cases. In each case we determine the explicit form of the biorthogonal system.

Case (a).

Theorem 5.1. *If all the eigenvalues of (0.1)-(0.3) are real and simple then the system*

$$\{y_n\} \ \ (n = 0,\ 1, \ldots; n \neq l), \tag{5.1}$$

where l is a non-negative integer, is minimal in $L_p(0,1)$, $1 < p < \infty$.

Proof. It suffices to show the existence of the system

$$\{u_n\} \ \ (n = 0,\ 1, \ldots; n \neq l), \tag{5.2}$$

biorthogonal to the system (4.1). Noting the relation $B_n \neq 0$ we define elements of the system (5.2) by

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y_l(1)} y_l(x)}{B_n}. \tag{5.3}$$

It remains to see, noting (1.4), (1.9) and (1.12), that

$$(u_n, y_m) = \delta_{nm}, \tag{5.4}$$

where δ_{nm} $(n,\ m = 0,\ 1, \ldots;\ n,\ m \neq l)$ denotes as usually, Kronecker's symbol: $\delta_{nm} = 0$ if $n \neq m$ and $\delta_{nn} = 1$. □

Case (b).

Theorem 5.2. *If λ_k is a double eigenvalue then the system*

$$\{y_n\} \ \ (n = 0,\ 1, \ldots; n \neq k+1), \tag{5.5}$$

is minimal in $L_p(0,1)$, $1 < p < \infty$.

Proof. In this case the biorthogonal system is defined by

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y_k(1)} y_k(x)}{B_n}, \ (n \neq k, \ k+1); \tag{5.6}$$

$$u_k(x) = \frac{y_{k+1}(x) - \frac{y_{k+1}(1)}{y_k(1)} y_k(x)}{-y_k(1)\varpi''(\lambda_k)/2}. \tag{5.7}$$

Using formulas (1.4), (1.9), (1.10), (1.12), (2.1), (2.3) the relation (5.4) for $n, \ m = 0, \ 1, \ldots; \ n, \ m \neq k+1$ can easily be verified. □

Theorem 5.3. *If λ_k is a double eigenvalue, and if $y^*_{k+1}(1) \neq 0$ then the system*

$$\{y_n\} \ \ (n = 0, \ 1, \ldots; n \neq k), \tag{5.8}$$

is minimal in $L_p(0,1)$, $1 < p < \infty$.

Proof. The elements of the biorthogonal system are defined as follows

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y^*_{k+1}(1)} y^*_{k+1}(x)}{B_n}, \ (n \neq k, \ k+1); \tag{5.9}$$

$$u_{k+1}(x) = \frac{y_k(x) - \frac{y_k(1)}{y^*_{k+1}(1)} y^*_{k+1}(x)}{-y_k(1)\varpi''(\lambda_k)/2}. \tag{5.10}$$

The relation (5.4) for $n, \ m = 0, \ 1, \ldots; \ n, \ m \neq k$ follows from (1.4), (1.9), (1.12), (2.1), (2.3), (3.1), (3.2). □

Before proceeding we comment on the above condition $y^*_{k+1}(1) \neq 0$. Let $y^*_{k+1}(1) = 0$, then by (3.1), (3.2) the function y^*_{k+1} is orthogonal to all the elements of the system (5.8). Therefore the system (5.8) is not complete (cf. [15, Theorem 3]) in $L_p(0.1)$. In the next section we shall prove that the condition $y^*_{k+1}(1) \neq 0$ is also necessary for the minimality of the system (5.8).

Theorem 5.4. *If λ_k is a double eigenvalue then the system*

$$\{y_n\} \ \ (n = 0, \ 1, \ldots; n \neq l), \tag{5.11}$$

where $l \neq k, \ k+1$ is a non-negative integer, is minimal in $L_p(0,1)$, $1 < p < \infty$.

Proof. The biorthogonal system is given by the formula (5.3) for $n \neq k, \ k+1$, and

$$u_{k+1}(x) = \frac{y_k(x) - \frac{y_k(1)}{y_l(1)} y_l(x)}{-y_k(1)\varpi''(\lambda_k)/2}, \tag{5.12}$$

$$u_k(x) = \frac{y^*_{k+1}(x) - \frac{y^*_{k+1}(1)}{y_l(1)} y_l(x)}{-y_k(1)\varpi''(\lambda_k)/2}. \tag{5.13}$$

The relation (5.4) for $n, \ m = 0, \ 1, \ldots; \ n, \ m \neq l$ follows from (1.4), (1.9), (1.12), (2.1), (2.3), (3.1)-(3.3). □

Case (c).

Theorem 5.5. *If λ_k is a triple eigenvalue then the system*

$$\{y_n\} \ \ (n = 0, \ 1, \ldots; n \neq k+2), \tag{5.14}$$

is minimal in $L_p(0,1)$, $1<p<\infty$.

Proof. The biorthogonal system is given by the formula (5.6) for $n\neq k,\ k+1,\ k+2$, and

$$u_{k+1}(x)=\frac{y_{k+1}(x)-\frac{y_{k+1}(1)}{y_k(1)}y_k(x)}{-y_k(1)\varpi'''(\lambda_k)/6},\tag{5.15}$$

$$u_k(x)=\frac{y^{**}_{k+2}(x)-\frac{y^{**}_{k+2}(1)}{y_k(1)}y_k(x)}{-y_k(1)\varpi'''(\lambda_k)/6}.\tag{5.16}$$

The relation (5.4) for $n,\ m=0,\ 1,\ \ldots;\ n,\ m\neq k+2$ follows from the mentioned results of sections 1, 2 and formulas (3.5), (3.8), (3.9). □

Theorem 5.6. *If* λ_k *is a triple eigenvalue, and if* $y^{**}_{k+1}(1)\neq 0$ *then the system*

$$\{y_n\}\ \ (n=0,\ 1,\ldots;n\neq k+1),\tag{5.17}$$

is minimal in $L_p(0,1)$, $1<p<\infty$.

Proof. In this case the elements of the biorthogonal system are

$$u_n(x)=\frac{y_n(x)-\frac{y_n(1)}{y^{**}_{k+1}(1)}y^{**}_{k+1}(x)}{B_n},\ (n\neq k,\ k+1,\ k+2);\tag{5.18}$$

$$u_{k+2}(x)=\frac{y_k(x)-\frac{y_k(1)}{y^{**}_{k+1}(1)}y^{**}_{k+1}(x)}{-y_k(1)\varpi'''(\lambda_k)/6},\tag{5.19}$$

$$u_k(x)=\frac{y^{\#}_{k+2}(x)-\frac{y^{\#}_{k+2}(1)}{y^{**}_{k+1}(1)}y^{**}_{k+1}(x)}{-y_k(1)\varpi'''(\lambda_k)/6}.\tag{5.20}$$

The relation (5.4) for $n,\ m=0,\ 1,\ldots;\ n,\ m\neq k+1$ can be verified using the mentioned results of sections 1, 2 and formulas (3.4), (3.6), (3.10)-(3.12). □

In analogy with theorem 5.3, we may show that if $y^{**}_{k+1}(1)=0$ then the function $y^{**}_{k+1}(x)$ is orthogonal to all the elements of the system (5.17); hence the system (5.17) is not complete.

Theorem 5.7. *If* λ_k *is a triple eigenvalue, and if* $y^{\#\#}_{k+2}(1)\neq 0$ *then the system*

$$\{y_n\}\ \ (n=0,\ 1,\ldots;n\neq k),\tag{5.21}$$

is minimal in $L_p(0,1)$, $1<p<\infty$.

Proof. We define the elements of the biorthogonal system by

$$u_n(x)=\frac{y_n(x)-\frac{y_n(1)}{y^{\#\#}_{k+2}(1)}y^{\#\#}_{k+2}(x)}{B_n},\ (n\neq k,\ k+1,\ k+2);\tag{5.22}$$

$$u_{k+2}(x)=\frac{y_k(x)-\frac{y_k(1)}{y^{\#\#}_{k+2}(1)}y^{\#\#}_{k+2}(x)}{-y_k(1)\varpi'''(\lambda_k)/6},\tag{5.23}$$

$$u_{k+1}(x)=\frac{y_{k+1}(x)-\frac{y_{k+1}(1)}{y^{\#\#}_{k+2}(1)}y^{\#\#}_{k+2}(x)}{-y_k(1)\varpi'''(\lambda_k)/6}.\tag{5.24}$$

The relation (5.4) for $n,\ m=0,\ 1,\ldots;\ n,\ m\neq k$ follows from the mentioned results of sections 1, 2 and formulas (3.13)-(3.15). □

If $y^{\#\#}_{k+2}(1)=0$ then the system (4.21) is not complete.

Theorem 5.8. *If λ_k is a triple eigenvalue then the system*

$$\{y_n\}\ \ (n=0,\ 1,\ldots;n\neq l), \tag{5.25}$$

where $l\neq k,\ k+1,\ k+2$ is a non-negative integer, is minimal in $L_p(0,1)$, $1<p<\infty$.

Proof. The elements of the biorthogonal system can be represented in the form

$$u_n(x)=\frac{y_n(x)-\frac{y_n(1)}{y_l(1)}y_l(x)}{B_n},\ \ (n\neq k,\ \ k+1,\ \ k+2,\ \ l); \tag{5.26}$$

$$u_{k+2}(x)=\frac{y_k(x)-\frac{y_k(1)}{y_l(1)}y_l(x)}{-y_k(1)\varpi'''(\lambda_k)/6}, \tag{5.27}$$

$$u_{k+1}(x)=\frac{y^{**}_{k+1}(x)-\frac{y^{**}_{k+1}(1)}{y_l(1)}y_l(x)}{-y_k(1)\varpi'''(\lambda_k)/6}, \tag{5.28}$$

$$u_k(x)=\frac{y^{\#\#}_{k+2}(x)-\frac{y^{\#\#}_{k+2}(1)}{y_l(1)}y_l(x)}{-y_k(1)\varpi'''(\lambda_k)/6}. \tag{5.29}$$

The relation (5.4) for $n,\ m=0,\ 1,\ldots;\ n,\ m\neq l$ follows from the mentioned results of sections 1, 2 and formulas (3.4), (3.6), (3.7), (3.13)-(3.16). □

Case (d).

Theorem 5.9. *If λ_r and $\lambda_s=\overline{\lambda_r}$ are a conjugate pair of non-real eigenvalues then each of the systems*

$$\{y_n\}\ \ (n=0,\ 1,\ldots;n\neq r), \tag{5.30}$$

$$\{y_n\}\ \ (n=0,\ 1,\ldots;n\neq l), \tag{5.31}$$

where $l\neq r,s$ is a non-negative integer, is minimal in $L_p(0,1)$, $1<p<\infty$.

Proof. The biorthogonal system of (5.30) is as follows

$$u_n(x)=\frac{y_n(x)-\frac{y_n(1)}{y_s(1)}y_s(x)}{B_n}.\ (n\neq r,\ \ s); \tag{5.32}$$

$$u_s(x)=\frac{y_r(x)-\frac{y_r(1)}{y_s(1)}y_s(x)}{-y_r(1)\varpi'(\lambda_r)}. \tag{5.33}$$

The equality (5.4) for $n,\ m=0,\ 1,\ldots;\ n,\ m\neq r$ can be verified using (1.4), (1.9), (1.11)-(1.13).

The biorthogonal system of (5.31) is defined by (5.3) for $n\neq r,s$; (5.33) and

$$u_r(x)=\frac{y_s(x)-\frac{y_s(1)}{y_r(1)}y_r(x)}{-y_s(1)\varpi'(\lambda_s)}.\square \tag{5.34}$$

6. Basisness of the system of root functions. We shall use a method somewhat similar to the one of [10].

Theorem 6.1. *Each of the systems in theorems 5.1-5.9 is a basis of* $L_p(0,1)$, $1<p<\infty$. *Moreover if* $p=2$ *then this basis is unconditional.*

Proof. We shall give the details only for the system (5.1). The proofs for other systems are similar. First we prove that the system (5.1) is an unconditional basis of $L_2(0,1)$. For this we compare the system (5.1) with the system

$$\{\varphi_n(x)\}(n=1,2,3,...), \tag{6.1}$$

where

$$\varphi_n=\begin{cases}\sqrt{2}\cos\left(n-\frac{1}{2}\right)\pi x, & \beta\neq 0,\\ \sqrt{2}\sin n\pi x, & \beta=0.\end{cases} \tag{6.2}$$

This system is a basis of $L_p(0,1)$ $1<p<\infty$ (see e.g. [14]), and in particular an orthonormal basis of $L_2(0,1)$. By (4.1), for sufficiently large n we have

$$\|y_n(x)-\varphi_n(x)\|_2\le const\cdot n^{-1}. \tag{6.3}$$

This inequality implies that the series

$$\sum_{n=1}^{l}\|y_{n-1}(x)-\varphi_n(x)\|_2^2+\sum_{n=l+1}^{\infty}\|y_n(x)-\varphi_n(x)\|_2^2, \tag{6.4}$$

is convergent, hence the minimal system (5.1) is quadratically close to the system (6.1), which is an orthonormal basis of $L_2(0,1)$ as mentioned above. Therefore the system (5.1) is a basis of $L_2(0,1)$ (see Section 9.9.8 of the Russian transl. of [6]).

By (4.8)-(4.11) and (5.3), we have

$$u_n(x)=y_n(x)+O(1/n). \tag{6.5}$$

From (6.5) and (4.1) it follows that

$$y_n(x)=\varphi_n(x)+O(1/n), \tag{6.6}$$

$$u_n(x)=\varphi_n(x)+O(1/n). \tag{6.7}$$

Let $1<p<2$ and p is fixed. Below we denote by $\|\cdot\|_p$ the norm in $L_p(0,1)$. It was proved above that the system (5.1) is a basis of $L_2(0,1)$. Consequently, this system is complete in $L_p(0,1)$. Then for basisness of the system (5.1) in $L_p(0,1)$ it is sufficient to prove the existence of a constant $M>0$, for which the inequality

$$\left\|\sum_{n=1;\,n\neq l}^{N}(f,u_n)y_n\right\|_p\le M\cdot\|f\|_p,\quad (N=1,2,\dots), \tag{6.8}$$

is true for arbitrary function f from $L_p(0,1)$ (see [9, Chapter I]).

By (6.6)-(6.7) (below Σ' denotes $\sum\limits_{n=1;\,n\neq l}^{N}$),

$$\left\|\Sigma'(f,u_n)y_n\right\|_p\le\left\|\Sigma'(f,\varphi_n)\varphi_n\right\|_p+$$

$$+\left\|\Sigma'(f,u_n)O(1/n)\right\|_p+\left\|\Sigma'(f,O(1/n))\varphi_n\right\|_p. \qquad (6.9)$$

Since the system (6.1) is a basis of $L_p(0,1)$ then

$$\left\|\Sigma'(f,\varphi_n)\varphi_n\right\|_p\le const\cdot\|f\|_p. \qquad (6.10)$$

Applying in succession Holder's and Minkowsky's inequalities, and by (6.7),

$$\left\|\Sigma'(f,u_n)O(1/n)\right\|_p\le const\cdot\Sigma'\left|(f,u_n)n^{-1}\right|$$

$$\le const\cdot\left(\Sigma'|(f,u_n)|^q\right)^{1/q}\cdot\left(\Sigma'n^{-p}\right)^{1/p}$$

$$\le const\cdot[\left(\Sigma'|(f,\varphi_n)|^q\right)^{1/q}+\left(\Sigma'|(f,O(1/n))|^q\right)^{1/q}], \qquad (6.11)$$

where $1/p+1/q=1$.

Since (6.1) is an orthonormal uniformly bounded function system then by F. Riesz's theorem (see Theorem 2.8, Ch.XII from [18]),

$$\left(\Sigma'|(f,\varphi_n)|^q\right)^{1/q}\le const\cdot\|f\|_p. \qquad (6.12)$$

Using the well known fact that $\|f\|_p$ is a non-decreasing function of p, we have

$$\left(\Sigma'|(f,O(1/n))|^q\right)^{1/q}\le const\cdot\|f\|_1\cdot\left(\Sigma'n^{-q}\right)^{1/q}\le const\cdot\|f\|_p. \qquad (6.13)$$

Similarly, using Parseval's equality we have,

$$\left\|\Sigma'(f,O(1/n))\varphi_n\right\|_p\le\left\|\Sigma'(f,O(1/n))\varphi_n\right\|_2=\left(\Sigma'|(f,O(1/n))|^2\right)^{1/2}\le$$

$$\le const\cdot\|f\|_1\cdot\left(\Sigma'n^{-2}\right)^{1/2}\le const\cdot\|f\|_p. \qquad (6.14)$$

Finally, (6.8) follows from (6.9)-(6.14). Hence the system (5.1) is a basis of $L_p(0,1)$, $1<p<2$.

Consider now the case when $2<p<\infty$. It is obvious that the system (5.2) is a basis of $L_p(0,1)$. Then this system is complete in $L_q(0,1)$, where $1/p+1/q=1$. Note that $1<q<2$.

To complete the proof, we note that using the same kind of arguments as above, one can prove that the system (5.2) is a basis of $L_q(0,1)$. From this it follows that the system (5.1) is a basis of $L_p(0,1)$, $2<p<\infty$. □

We recall from Theorem 5.3 and subsequent notes that if $y^*_{k+1}(1)\neq 0$ then the system (5.8) is minimal in $L_p(0,1)$, $1<p<\infty$. Moreover, we have seen in Theorem 6.1 that this system is a basis of $L_p(0,1)$. But if $y^*_{k+1}(1)=0$ then this system is not complete in $L_p(0,1)$. As we shall now see the condition $y^*_{k+1}(1)=0$ also implies that this system is not minimal in $L_2(0,1)$. Indeed, if this system is minimal then using the method of the Theorem 6.1 we can prove basisness of this system in $L_2(0,1)$, which contradicts with incompleteness of this system in $L_2(0,1)$. It may be shown in the same way that the system (5.17) (the system (5.21)) is not minimal if $y^{**}_{k+1}(1)=0$ (respectively, if $y^{\#\#}_{k+1}(1)=0$).

In conclusion, we note that in some cases it is possible to define the elements of the biorthogonal system in a different way. For example the element (5.20) of the biorthogonal system of (5.17) can be replaced by

$$u_k(x) = \frac{y_{k+2}^{\#\#}(x) - \frac{y_{k+2}^{\#\#}(1)}{y_{k+1}^{**}(1)} y_{k+1}^{**}(x)}{-y_k(1)\varpi'''(\lambda_k)/6}.$$

But using the equality $d_2 = d_1 + c_2^2$, which is easily verified, we can show that this representation of the element $u_k(x)$ coincide with (5.20). This observation agrees with the well known fact that the biorthogonal system of a basis is unique.

6. Example. As a complement, we note a special result for the problem (0.5), (0.6). In particular, as was noted in [15], if $a = -1$ then $\lambda_0 = \lambda_1 = 0$ is a double eigenvalue and the eigenvalues $0 < \lambda_2 < \lambda_3 < \ldots$ are solutions of the equation $\tan\sqrt{\lambda} = \sqrt{\lambda}$. Eigenfunctions are $y_0 = 1$, $y_n = \cos\sqrt{\lambda_n}x$ $(n \geq 2)$ and associated function corresponding to eigenfunction y_0 is $y_1 = -\frac{1}{2}x^2 + c$, where c is an arbitrary constant. We seek the auxiliary associated function in the form $y_1^* = -\frac{1}{2}x^2 + c'$. That is $c_1 = c' - c$. By (3.1),

$$\int_0^1 \left(-\frac{1}{2}x^2 + c\right)\left(-\frac{1}{2}x^2 + c'\right)dx = \left(-\frac{1}{2} + c\right)\left(-\frac{1}{2} + c'\right).$$

From this equality we obtain that $c' = -c + \frac{3}{5}$, so $y_1^*(1) = c - \frac{1}{10}$. Therefore the above condition $y_1^*(1) = 0$ in Theorem 5.3 is equivalent to $c = \frac{1}{10}$. This result coincide with [15, Theorem 3] if we note that the definition of the first associated function in [15] differs from ours by its sign.

We shall now indicate different approach for this problem. Note that for this problem $y(x,\lambda) = \cos\sqrt{\lambda}x$, then $y_\lambda(x,\lambda) = -\frac{x\sin\sqrt{\lambda}x}{2\sqrt{\lambda}}$. We obtain that $\tilde{y}_1 = \lim_{\lambda\to 0} y_\lambda(x,\lambda) = -\frac{x^2}{2}$. Let $y_1 = -\frac{1}{2}x^2 + c$. Then $\tilde{c} = -c$. Note also that $\varpi(\lambda) = \lambda\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}$, then

$$\varpi''(0) = \lim_{\lambda\to 0}\varpi''(\lambda) = -2/3,$$

$$\varpi'''(0) = \lim_{\lambda\to 0}\varpi'''(\lambda) = 1/5.$$

As was pointed out in subsequent remarks of Lemma 3.1, the condition $y_1^*(1) = 0$ is equivalent to $\varpi'''(\lambda_k) = 3\tilde{c}\varpi''(\lambda_k)$, from which we obtain, once again, $c = \frac{1}{10}$. These calculations confirm our results.

Acknowledgement. I thank Professor N.B. Kerimov for proposal of the problem and his attention to this work.

References

[1]. Binding P. A. and Browne P. J., *Application of two parameter eigencurves to Sturm-Liouville problems with eigenparameter-dependent boundary conditions*, Proc. Roy. Soc. Edinburgh 125A (1995), 1205-1218.

[2]. Binding P. A., Browne P. J., Code W. J. and Watson B. A., *Transformation of Sturm-Liouville problems with decreasing affine boundary conditions*, Proc. Edinb. Math. Soc. 47 (2004), 533-552.

[3]. Binding P. A., Browne P. J. and Watson B. A., *Equivalence of inverse Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter*, J. Math. Anal. Appl. 291 (2004), 246-261.

[4]. Coşkun H. and Bayram N., *Asymptotics of eigenvalues for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition*, J. Math. Anal. Appl. 306 (2005), 548-566.

[5]. Ince E. L., *Ordinary differential equations*, Dover, New York, 1956; Russian transl. of 1st ed., GONTI-NKTP-DNTVU, Kharkov, 1939.

[6]. Kaczmarz S. and Steinhaus H., *Theorie der Orthogonalreihen*, Warszawa-Lwów, 1935; Russian transl., Fizmatgiz, Moscow, 1958.

[7]. Kapustin N. Yu. and Moiseev E. I., *The basis property in L_p of the systems of eigenfunctions corresponding to two problems with a spectral parameter in the boundary conditions*, Diferentsial'nye Uravneniya 36 (2000), 1357-1360 (in Russian); English transl.: Differential Equations 36 (2000), 1498-1501.

[8]. Kartashev A. P. and Rojdestvenskiy B. L., *Ordinary differential equations and foundations of calculus of variation*, Nauka, Moscow, 1980. (in Russian)

[9]. Kashin B. S. and Saakian A. A., *Orthogonal Series*, Nauka, Moscow, 1984 (in Russian); English transl., Amer. Math. Soc., 1989.

[10]. Kerimov N. B. and Aliyev Y. N., *The basis property in L_p of the boundary value problem rationally dependent on the eigenparameter*, Studia Math. 174 (2006), 201-212.

[11]. Kerimov N. B. and Mirzoev V. S., *On the basis properties of one spectral problem with a spectral parameter in boundary conditions*, Siberian Math. J. 44 (2003), 813-816.

[12]. Kerimov N. B. and Poladov R. G., *On basicity in L_p $(1 < p < \infty)$ of the system of eigenfunctions of one boundary value problem*, I, II, Proc. of Institute of Math. and Mech. of NAS of Azerbaijan 22 (2005), 53-64; 23 (2005), 65-76.

[13]. Makin A. S., *On the class of boundary value problems for the Sturm-Liouville operator*, Diferentsial'nye Uravneniya 35 (1999), 1058-1066 (in Russian); English transl.: Differential Equations 35 (1999), 1067-1076.

[14]. Moiseev E. I., *On the basis properties of sine and cosine systems*, Dokl. Akad. Nauk SSSR 275 (1984), 794-798 (in Russian).

[15]. Moiseev E. I. and Kapustin N. Yu., *On peculiar properties of the set of root functions of one spectral problem with a spectral parameter in boundary condition*, Dokl. Akad. Nauk, 385 (2002), 20-24. (in Russian)

[16]. Naimark M. A., *Linear differential operators*, 2nd edn, Nauka, Moscow, 1969 (in Russian); English trans. of 1st edn, Parts I,II, Ungar, New York, 1967, 1968.

[17]. Tretter C., *Boundary eigenvalue problems for differential equations $N\eta = \lambda P\eta$ with λ-polynomial boundary conditions*, J. Differential Equations 170 (2001), 408-471.

[18]. Zygmund A., *Trigonometric Series*, Vol. 2, Cambridge Univ. Press, Cambridge, 1959; Russian transl., Mir, Moscow, 1965.

Yakub N. Aliyev,
Baku State University
23, Z.Khalilov str., AZ1148, Baku, Azerbaijan
E-mail: yakubaliyev@yahoo.com

Received December 01, 2005; Revised February 06, 2006.
Translated by Author