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SOLITARY WAVES IN TWO-LAYER LIQUID WITH REGARD TO SURFACE TENSION

Abstract

We consider stability of hydrodynamic system when a layer of more light liquid with free surface is on the layer of more heavy liquid. It is shown that two types of solitary waves may propagate on the free surface and on interface: classic solitary and generalized-solitary waves, moreover, surface tension coefficient has more considerable effect on motion velocity of generalized-solitary waves, than on velocity of classic solitary waves.

Analysis of complete system of Euler equations for non-linear gravitational-capillary waves may give quantity description [1]. However, up to now this analysis was not made for two-layer liquid because of technical difficulty of the problem.

Existence of generalized-solitary waves in two-layer liquid in the absence of capillary effects on the interface is represented in the paper [2]. Intrinsic solitary waves were experimentally studied in [3.4], the numerical results for classic waves-in [6]. It is assumed that in some cases, the ripple solitary waves can't propagate with velocity C more than some value of $C_{\max}$ [7].

1. Problem statement and dispersion equation. Let the axis $\overline{x}$ coincide with resting liquid separation plane, and the axis $\overline{z}$ be vertical. From below the heavy liquid is bounded by a hard horizontal bottom. All the quantities relating to upper light layer of the liquid will have the index 2, the ones relating to the lower layer-index 1.

Equations and boundary conditions in dimensionless variables, describing plane flow of two-layer ideal incompressible liquid, have the form:

$$\delta\frac{\partial^2\varphi_i}{\partial x^2}+\frac{\partial^2\varphi_i}{\partial z^2}=0,\quad i=1,2;\quad -1<z<\varepsilon\eta_2 \tag{1.1}$$

$$\frac{\partial\varphi_1}{\partial z}=-1;\quad z=-1 \tag{1.2}$$

$$\left\{\begin{array}{l}\dfrac{\partial\eta_1}{\partial t}+\varepsilon\dfrac{\partial\varphi_i}{\partial x}\dfrac{\partial\eta_1}{\partial x}-\dfrac{1}{\delta}\dfrac{\partial\varphi_i}{\partial z}=0,\quad z=\varepsilon\eta_1\\ \dfrac{\partial\varphi_1}{\partial t}-R\dfrac{\partial\varphi_2}{\partial t}+\dfrac{\varepsilon}{2}\left[\left(\dfrac{\partial\varphi_1}{\partial x}\right)^2+\dfrac{1}{\delta}\left(\dfrac{\partial\varphi_1}{\partial z}\right)^2\right]-\\ -\dfrac{\varepsilon R}{2}\left[\left(\dfrac{\partial\varphi_2}{\partial x}\right)^2+\dfrac{1}{\delta}\left(\dfrac{\partial\varphi_2}{\partial z}\right)^2\right]+(1-R)\eta_1-\dfrac{\delta\gamma}{gh_1^2}\dfrac{\partial^2\eta_1}{\partial x^2}=0,\end{array}\right. \tag{1.3}$$

$$\left\{\begin{array}{l}\dfrac{\partial\eta_2}{\partial t}+\varepsilon\dfrac{\partial\varphi_2}{\partial x}\dfrac{\partial\eta_2}{\partial x}-\dfrac{1}{\delta}\dfrac{\partial\varphi_2}{\partial z}=0,\quad z=H+\varepsilon\eta_2\\ \dfrac{\partial\varphi_2}{\partial t}+\dfrac{\varepsilon}{2}\left[\left(\dfrac{\partial\varphi_2}{\partial x}\right)^2+\dfrac{1}{\delta}\left(\dfrac{\partial\varphi_2}{\partial z}\right)^2\right]+\eta_2=0,\end{array}\right. \tag{1.4}$$

where φ_i is a dimensionless velocity potential; η_1, η_2 are dimensionless profiles of surface and free surface, λ, a are characteristic length and wave amplitude, h_i is layer's depth, σ is surface tension coefficient, g is gravity acceleration,

$$x = \overline{x}/\lambda, \quad z = \overline{z}/h_1, \quad t = (gh_1)^{1/2}\frac{\overline{t}}{\lambda}, \quad \varphi_i = \frac{\sqrt{gh_1}}{ga\lambda}\overline{\varphi}_i, \quad \eta_i = \overline{\eta}_i/a, \quad H = h_2/h_1,$$

$$\delta = (h_1/\lambda)^2, \quad \varepsilon = a/h_1, \quad R = \rho_2/\rho_1, \quad \gamma = \sigma/\rho_1.$$

In (1.1)-(1.4) δ and ε are small parameters in long wave theories. The first parameter δ responds for dispersive, the second parameter ε for non-linear effects. For $\varepsilon << \delta$ boundary conditions (1.3) and (1.4) are linearized and take the form

$$\begin{cases} \dfrac{\partial \eta_1}{\partial t} - \dfrac{1}{\delta}\dfrac{\partial \varphi_i}{\partial z} = 0, \quad z = \varepsilon\eta_1 \\ \dfrac{\partial \varphi_1}{\partial t} - R\dfrac{\partial \varphi_2}{\partial t} + (1-R)\eta_1 - \dfrac{\delta\gamma}{gh_1^2}\dfrac{\partial^2 \eta_1}{\partial x^2} = 0, \end{cases} \tag{1.5}$$

$$\begin{cases} \dfrac{\partial \eta_2}{\partial t} - \dfrac{1}{\delta}\dfrac{\partial \varphi_2}{\partial z} = 0, \quad z = H + \varepsilon\eta_2 \\ \dfrac{\partial \varphi_2}{\partial t} + \eta_2 = 0 \end{cases} \tag{1.6}$$

Let's consider harmonic oscillation of linearized problem (1.1),(1.2),(1.5),(1.6): $(\eta_j, \varphi_j) = A_j \exp i(kx - \omega t)$.

Dispersion equation corresponding to this problem

$$\begin{gathered} [1 + RthKth(KH)]\, C^4 - \tfrac{1}{K}\left[(1+Q)thK + th(KH)\right] C^2 + \\ + \frac{1}{K^2}(1 - R + Q)thKth(KH) = 0 \end{gathered} \tag{1.7}$$

consists of two branches responding to so-called surface and internal modes.

Here

$$Q = \gamma k^2/gh_1^2, \quad C^2 = c^2/gh_1, \quad K = kh_1, \quad c = \omega/k \tag{1.8}$$

For small K, expansion in series of (1.8) gives

$$\begin{gathered} \left[1 + RK^2H\right] C^4 - \left[1 + Q + H - \tfrac{1}{3}K^2(1 + Q + H^3)\right] C^2 + \\ +(1 - R + Q)H\left[1 - \frac{1}{3}K^2(1 + H^2)\right] = 0 \end{gathered} \tag{1.9}$$

As $K \to 0$ in (1.9) phase velocities for infinitely long waves satisfy the relation

$$C^4 - (1 + Q + H)C^2 + (1 - R + Q)H = 0 \tag{1.10}$$

The large root of equation (1.10) satisfies inequalities $C_+^2 > 1 + Q$ and $C_+^2 > 1 - R + Q$, when the smallest root satisfies the inequalities $C_-^2 < H$ and $C_-^2 < H(1 - R + Q)/(1 + Q)$

Internal mode responding to velocity C_-, is in resonance with surface mode corresponding to wave number K_{res}, belonging to the branch of dispersion relation (1.8), submitted in C_+. Therefore, it should be expected that slow waves are generalized-solitary [4.9];

In the range $R \to 0$ it follows from (1.10) that

$$C_+^2 = 1 + Q, \quad C_-^2 = H$$

for small K (long waves) only phase velocity C_+ depends on surface tension coefficient.

In the range $R \to 1$ the resonance number K_{res} in ocean conditions is sufficiently large, but corresponding length of wave is completely real, about one meter.

2. Evolution equation of long waves. For deriving model equations of small amplitude waves we use expansion of potentials in the plane $z = 0$ [8,9]

$$\varphi_i = \varphi_i^0 + z\varphi_{i,z}^0 + \frac{z^2}{2}\varphi_{i,zz}^0 + \frac{z^3}{6}\varphi_{i,zzz}^0 + \frac{z^4}{24}\varphi_{i,zzzz}^0 + \frac{z^5}{120}\varphi_{i,zzzzz}^0 + ... \tag{2.1}$$

In (2.1) the lower index z means differentiation with respect to this variable. Substituting (2.1) into (1.1) and equating the coefficients y^0, y, y^2, y^3 to zero, we have

$$\varphi_{i,zz}^0 = -\delta\varphi_{i,xx}^0, \quad \varphi_{i,zzz}^0 = -\delta(\varphi_{i,z}^0)_{xx},$$
$$\varphi_{i,zzzz}^0 = \delta^2\varphi_{i,xxxx}^0, \quad \varphi_{i,zzzzz}^0 = \delta^2(\varphi_{i,z}^0)_{xxxx}, \tag{2.2}$$

Using relations (2.2) in (2.1) and then substituting it into boundary condition (1.2) with retaining the terms up to order δ^3, we get

$$\varphi_{1,z}^0 = -\delta\varphi_{1,xx}^0 - \frac{\delta^2}{3}\varphi_{1,xxxx}^0 \tag{2.3}$$

Again, we use (2.1) and (2.2) in the first equation of (1.4) and performing the similar procedure, we get an expression for $\varphi_{2,z}^0$ with retaining the terms of order ε and δ^2

$$\varphi_{2,z}^0 = \delta\eta_{2,t} + \delta H\varphi_{2,xx}^0 + \varepsilon\delta(\eta_2\varphi_{2,x}^0)_x + \frac{\delta^2 H^2}{2}\eta_{2,txx} + \frac{\delta^2 H^3}{3}\varphi_{2,xxxx}^0, \tag{2.4}$$

Substituting (2.3) and (2.4) into equation (1.3) and second equation of (1.4), we get a closed system with respect to the unknown variables φ_i and η_i. If in this system we neglect the terms of order $0(\varepsilon, \delta)$ and omit the index "0", we reduce it to the form

$$\eta_{1,t} + \varepsilon(\eta_1\varphi_{1,x})_x + \varphi_{1,xx} + \frac{\delta}{3}\varphi_{1,xxxx} = 0, \tag{2.5}$$

$$\begin{gathered}\eta_{2,t} - \eta_{1,t} + \varepsilon(\eta_2\varphi_{2,x})_x + H\varphi_{2,xx} - \varepsilon(\eta_1\varphi_{2,x})_x + \\ +\delta\frac{H^2}{2}\eta_{2,txx} + \delta\frac{H^3}{3}\varphi_{2,xxxx} = 0,\end{gathered} \tag{2.6}$$

$$R\varphi_{2,t} - \varphi_{1,t} + \frac{\delta R}{2}(\varphi_{2,x})^2 - \frac{\varepsilon}{2}(\varphi_{1,x})^2 - (1-R)\eta_1 + \delta\gamma_1\eta_{1,xx} = 0, \tag{2.7}$$

$$\varphi_{2,t} + \delta H\eta_{2,tt} + \delta\frac{H^2}{2}\varphi_{2,txx} + \frac{\varepsilon}{2}\varphi_{2,x}^2 + \eta_2 = 0, \quad \gamma_1 = \frac{\gamma}{gh_1^2}. \tag{2.8}$$

We twice differentiate (2.7) with respect to x, and (2.8) with respect to t, again take the obtained expression into account in these equations, retain the terms to within the order ε and δ, and have

$$\begin{gathered}
\eta_1=\frac{R}{1-R}\varphi_{2,t}-\frac{1}{1-R}\varphi_{1,t}+\varepsilon\frac{R}{2\,(1-R)}\varphi_{2,x}^2-\varepsilon\frac{1}{2\,(1-R)}\varphi_{1,x}^2+ \\
+\delta\frac{\gamma_1 R}{(1-R)^2}\varphi_{2,txx}-\delta\frac{\gamma_1 R}{(1-R)^2}\varphi_{1,txx}, \\
\eta_2=-\varphi_{2,t}+\delta H\varphi_{2,ttt}-\delta\frac{H^2}{2}\varphi_{2,txx}-\frac{\varepsilon}{2}\varphi_{2,x}^2.
\end{gathered} \tag{2.9}$$

In new variables $\xi=x-ct$ and $\vartheta_i=\dot{\varphi}_{i,x}=\dot{\varphi}_{i,\xi}$ the profiles of interface and free surface take the form

$$\begin{gathered}
\eta_1=\frac{C}{1-R}\vartheta_1-\frac{CR}{1-R}\vartheta_2-\frac{\varepsilon}{2(1-R)}\vartheta_1^2+\frac{\varepsilon R}{2(1-R)}\vartheta_2^2+ \\
+\delta\frac{\gamma_1 R}{(1-R)^2}\ddot{\vartheta}_1-\delta\frac{C\gamma_1 R}{(1-R)^2}\ddot{\vartheta}_2, \\
\eta_2=C\vartheta_2-\delta CH\left(C^2-\frac{H}{2}\right)\ddot{\vartheta}_2-\frac{\varepsilon}{2}\vartheta_2^2.
\end{gathered} \tag{2.10}$$

In the similar way, we make change of variables, integrate the equations (2.5), (2.6) and by means of (2.10), eliminate η_i from them, and to within the numbers of order $0(\delta,\varepsilon^2)$ we get

$$\frac{\delta}{3}\left(1-\frac{3\gamma_1 C^2}{(1-R)^2}\right)\ddot{\vartheta}_1+\delta\frac{3\gamma_1 C^2 R}{(1-R)^2}\ddot{\vartheta}_2=-\left(1-\frac{C^2}{1-R}\right)\vartheta_1-\frac{C^2 R}{1-R}\vartheta_2-$$

$$\begin{gathered}
-\varepsilon\frac{3C}{2(1-R)}\vartheta_1^2+\varepsilon\frac{CR}{2(1-R)}\vartheta_2^2+\varepsilon\frac{CR}{1-R}\vartheta_1\vartheta_2+\varepsilon^2\frac{\vartheta_1^3}{2\,(1-R)}-\varepsilon^2\frac{R}{2\,(1-R)}\vartheta_1\vartheta_2^2, \\
\delta\frac{\gamma_1 C^2}{(1-R)^2}\ddot{\vartheta}_1+\delta\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{C^2\gamma_1 R}{(1-R)^2}\right]\ddot{\vartheta}_2= \\
=-\frac{C^2}{1-R}\vartheta_1+\left(\frac{C^2}{1-R}-H\right)\vartheta_2+\varepsilon\frac{C}{2(1-R)}\vartheta_1^2-\varepsilon\frac{3C}{2(1-R)}\vartheta_2^2+ \\
+\varepsilon\frac{C}{1-R}\vartheta_1\vartheta_2+\varepsilon^2\frac{\vartheta_2^3}{2\,(1-R)}-\varepsilon^2\frac{\vartheta_1^2\vartheta_2}{2\,(1-R)},
\end{gathered} \tag{2.11}$$

or

$$\ddot{\vartheta}_1=\frac{1}{D\delta}\left\{\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1 C^2 R}{(1-R)^2}\right]f_1(\vartheta_1,\vartheta_2)-\frac{\gamma_1 C^2 R}{(1-R)^2}f_2(\vartheta_1,\vartheta_2)\right\},$$

$$\ddot{\vartheta}_2=\frac{1}{D\delta}\left[\left(\frac{1}{3}-\frac{\gamma_1 C^2}{(1-R)^2}\right)f_2(\vartheta_1,\vartheta_2)-\frac{\gamma_1 C^2}{(1-R)^2}f_1(\vartheta_1,\vartheta_2)\right],$$

where

$$D=\left(\frac{1}{3}-\frac{\gamma_1 C^2}{(1-R)^2}\right)\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1 C^2 R}{(1-R)^2}\right]-\frac{\gamma_1^2 C^4 R}{(1-R)^4},$$

$$f_1(\vartheta_1,\vartheta_2)=-\left(1-\frac{C^2}{1-R}\right)\vartheta_1-\frac{C^2 R}{1-R}\vartheta_2-\varepsilon\frac{3C}{2(1-R)}\vartheta_1^2+$$

$$+\varepsilon\frac{CR}{2(1-R)}\vartheta_2^2+\varepsilon\frac{CR}{1-R}\vartheta_1\vartheta_2+\varepsilon^2\frac{\vartheta_1^3}{2\,(1-R)}-\varepsilon^2\frac{R}{2\,(1-R)}\vartheta_1\vartheta_2^2,$$

$$f_2\left(\vartheta_1,\vartheta_2\right)=-\frac{C^2}{1-R}\vartheta_1+\left(\frac{C^2}{1-R}-H\right)\vartheta_2+\varepsilon\frac{C}{2(1-R)}\vartheta_1^2-$$

$$-\varepsilon\frac{3C}{2(1-R)}\vartheta_2^2+\varepsilon\frac{C}{1-R}\vartheta_1\vartheta_2+\varepsilon^2\frac{1}{2\,(1-R)}\vartheta_2^3-\varepsilon^2\frac{\vartheta_1^2\vartheta_2}{2\,(1-R)};$$

In (2.11) we pass to new independent and dependent variables: $\xi=\sqrt{(1-R)\delta}\zeta$, $\vartheta_i=\varepsilon^{-1}u_i,\ \eta_i=\varepsilon^{-1}\eta_i^{'},\ i=1,2.$

$$\ddot{u}_1=-\alpha_1(C)u_1+\alpha_2(C)u_2-\alpha_3(C)u_1^2+\alpha_4(C)u_2^2+\alpha_5(C)u_1u_2+$$

$$+\alpha_6(C)u_1^3-R\alpha_6(C)u_1u_2^2-\alpha_7(C)u_2^3+\alpha_7(C)u_1^2u_2;$$

$$\begin{gathered}\ddot{u}_2=\beta_1(C)u_1+\beta_2(C)u_2+\beta_3(C)u_1^2-\beta_4(C)u_2^2+\beta_5(C)u_1u_2-\\-\beta_6(C)u_1^3+\beta_7(C)u_2^3+\beta_6(C)u_1u_2^2-\beta_7(C)u_1^2u_2,\end{gathered}\tag{2.12}$$

where

$$\alpha_1(C)D=-\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1C^2R}{(1-R)^2}\right](C^2-1+R)-\frac{\gamma_1C^4R}{(1-R)^2},$$

$$\alpha_2(C)D=-C^2R\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1C^2R}{(1-R)^2}\right]-\frac{\gamma_1C^2R}{(1-R)^2}(C^2-H\,(1-R)),$$

$$\alpha_3(C)D=\frac{C}{2}\left\{3\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1C^2R}{(1-R)^2}\right]+\frac{\gamma_1C^2R}{(1-R)^2}\right\},$$

$$\alpha_4(C)D=\frac{C}{2}\left\{R\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1C^2R}{(1-R)^2}\right]+\frac{3\gamma_1C^2R}{(1-R)^2}\right\},$$

$$\alpha_5(C)D=C\left\{R\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1C^2R}{(1-R)^2}\right]-\frac{\gamma_1C^2R}{(1-R)^2}\right\},$$

$$\alpha_6(C)D=\frac{1}{2}\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1C^2R}{(1-R)^2}\right],\quad \alpha_7(C)=\frac{\gamma_1C^2R}{2(1-R)^2}.$$

$$\beta_1(C)D=-C^2\left(\frac{1}{3}-\frac{\gamma_1C^2}{1-R}\right),$$

$$\beta_2(C)D=\left(\frac{1}{3}-\frac{\gamma_1C^2}{(1-R)^2}\right)\left[C^2-H(1-R)\right]+\frac{\gamma_1C^4R}{(1-R)^2},$$

$$\beta_3(C)D=\frac{C}{2}\left(\frac{1}{3}+\frac{2\gamma_1C^2}{(1-R)^2}\right),$$

$$\beta_4(C)D=\frac{C}{2}\left[3\left(\frac{1}{3}-\frac{\gamma_1C^2}{(1-R)^2}\right)+\frac{\gamma_1C^2R}{(1-R)^2}\right],$$

$$\beta_5(C)D=C\left[\left(\frac{1}{3}-\frac{\gamma_1C^2}{(1-R)^2}\right)-\frac{\gamma_1C^2R}{(1-R)^2}\right],$$

$$\beta_6(C)D=\frac{\gamma_1C^2}{2\,(1-R)^2}=\frac{\alpha_7(C)}{R};\quad \beta_7(C)=\frac{1}{2}\left(\frac{1}{3}-\frac{\gamma_1C^2}{(1-R)^2}\right),\tag{2.13}$$

$$D=\left(\frac{1}{3}-\frac{\gamma_1 C^2}{(1-R)^2}\right)\left[H\left(C^4-HC^2+\frac{H^2}{3}\right)-\frac{\gamma_1 C^2 R}{(1-R)^2}\right]-\frac{\gamma_1^2 C^4 R}{(1-R)^4}.$$

For simplicity we'll denote

$$\alpha_i(C_\pm)=\alpha_i,\quad \beta_i(C_\pm)=\beta_i.$$

The coefficients of equations (2.12) for $\gamma_1=0$ are significantly simplified and take the form

$$\alpha_1(C)=-3(C^2-1+R),\ \alpha_2(C)=-3RC^2,\ \alpha_3(C)=\frac{9}{2}C,\ \alpha_4(C)=\frac{3}{2}CR,$$

$$\alpha_5(C)=3CR,\ \alpha_6=\frac{3}{2},\ \alpha_7=0;$$

$$\beta_1(C)=-\frac{C^2}{D},\ \beta_2(C)=\frac{C^2-H(1-R)}{D},\ \beta_3(C)=\frac{C}{2D},\ \beta_4(C)=\frac{3C}{2D},$$

$$\beta_5(C)=\frac{C}{D},\ \beta_6=0,\ \beta_7=\frac{1}{2D},$$

$$D=\frac{H}{3}\left(C^4-HC^2+\frac{H^2}{3}\right)>0\,. \tag{2.14}$$

Equations (2.12) may be written in the form of dynamic system

$$\begin{cases}\dot{u}_1=\theta_1,\quad \dot{u}_2=\theta_2,\\ \dot{\theta}_1=-\alpha_1(C)u_1+\alpha_2(C)u_2+f_1(u_1,u_2),\\ \dot{\theta}_2=\beta_1(C)u_1+\beta_2(C)u_2+f_2(u_1,u_2),\end{cases} \tag{2.15}$$

$$f_1(u_1,u_2)=-\alpha_3(C)u_1^2+\alpha_4(C)u_2^2+\alpha_5(C)u_1u_2+\alpha_6(C)u_1^3-$$
$$-R\alpha_6(C)u_1u_2^2-\alpha_7(C)u_2^3+\alpha_7(C)u_1^2u_2,$$
$$f_2(u_1,u_2)=\beta_3(C)u_1^2-\beta_4(C)u_2^2+\beta_5(C)u_1u_2-\beta_6(C)u_1^3+$$
$$+\beta_7(C)u_2^3+\beta_6(C)u_1u_2^2-\beta_7(C)u_1^2u_2.$$

System of equations (2.15) has the form of invertible dynamic system, where $w(u_1,u_2,\theta_1,\theta_2)$, $\quad A=A(C_\pm)$,

$$\dot{w}=Aw+F(\mu,w),\quad A(C)=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-\alpha_1(C)&\alpha_2(C)&0&0\\\beta_1(C)&\beta_2(C)&0&0\end{pmatrix},$$
$$F=\begin{pmatrix}0\\0\\f_1\\f_2\end{pmatrix} \tag{2.16}$$

Here, the dot means differentiation with respect to ζ. We'll consider solutions of (2.16) of traveling waves type for $C=C_\pm+\mu$ where μ is a small positive number. Reversibility in (2.16) means that a set of solutions must contain even functions u_1 and u_2.

Eigen values λ of the matrix A satisfy the equation

$$\lambda^4 + [\alpha_1(C) - \beta_2(C)]\lambda^2 - \beta_1(C)\alpha_2(C) - \beta_2(C)\alpha_1(C) = 0\ . \qquad (2.17)$$

If $\beta_1(C)\alpha_2(C) + \beta_2(C)\alpha_1(C) = 0$, we get (1.10) for infinitely long waves $C_\pm$. At real values of parameters of the problem in (2.17) the coefficient $\alpha_1(C_-) - -\beta_2(C_-) > 0$ and at $C = C_\pm$ is negative. In the second case ($C = C_-$) a unique imaginary eigen value of the operator A is a double zero eigen value, i.e. equation (2.17) has a root on an imaginary axis of two multiplicity zero. However, for $C = C_-$ -four roots: double zero and pairs of simple imaginary values, $\pm iq$, $q^2 = \alpha_1(C_-) - \beta_2(C_-)$, $q > 0$.

It is turned out to be that for $C = C_+ + \mu$ with $0 < \mu << 1$ there hold two real eigen values arising from zero for $\mu = 0$, and also two real eigen values. For $C = C_- + \mu$ two eigen values arise from zero $\mu = 0$, and the other pair remains imaginary. In the solution of the system of equations (2.16) there may arise classic type solitary waves in the vicinity C_+, and generalized-solitary waves in the vicinity C_-. We'll consider the case, when in (2.16) cubic non-linearity is omitted and quadratic non-linearity is dominated.

Using the theorem on central manifold we reduce the equations (2.16) to the mentioned form [10]. We assume that the operator $A{:}D(A) \to X$ is a closed unbounded operator acting from its domain of definition $D(A)$ to real Hilbert space $X, D(A) \in X$. Let $F{:}I \times D(A) \to X$ be a non-linear mapping for all $C \in \in I$, $I \in R$, $F(\mu, w) = O\left(\mu, \|w\|^2\right)$ where $\mu = C - C_\pm$, $\|\cdot\|$ norm in X. Dynamical system (2.16) for $\mu = 0$ has a trivial solution $w = 0$. Under reduction we'll mean construction of the function $h(\mu, w_0)$, $w_0 \in X_0$ such that the manifold $M_\mu = \{(w_0, h(\mu, w_0)) \in X,\ \ \|w_0\| < \varepsilon << 1\}$is an invariant manifold of dynamic system (2.16) and contains all small bounded solutions of this system. Invariant manifold M_μ is said to be a central manifold [1, 2, 7, 9].

Dimension of the space X_0 is finite that is equivalent to the finiteness of the number of imaginary eigen values of the operator A. The projection of the system of equations on the space X_0 and its complement in X accepts the form

$$\begin{aligned} \dot{w}_0 &= A_0 w_0 + F_0(\mu, w_0 + w_1) \\ \dot{w}_1 &= A_1 w_1 + F_1(\mu, w_0 + w_1), \end{aligned} \qquad (2.18)$$

where

$$w(w_0, w_1) \in X = X_0 \times X_1, \quad A_0 = A\,|_{X_0}\ , \quad A_1 = A\,\big|_{X_1 \cap D(A)}\ .$$

The solution $w(w_0, w_1)$ of system (2.18) belongs to M_μ and therefore $w_1 = = h(\mu, w_0)$. Thus, a set of all small bounded solutions satisfies the finite dimensional dynamical system of equations

$$\dot{w}_0 = A_0 w_0 + F_0(\mu, w_0 + h(\mu, w_0)) = A_0 w_0 + f_0(\mu, w_0) \qquad (2.19)$$

The use of the theory of normal forms for approximation of finite dimensional dynamical systems of equations is more convenient for finding the solutions of the mentioned system (2.19). In canonical basis in finite dimensional space X_0 (identified with $\mathbb{C}^m$) w_0 is represented by a vector in the complex space $\mathbb{C}^m$, where m

equals the number of imaginary eigen values of A.

A theorem on reduction to quasinormal form [10]. *For the given* $n \in N, n > 2$ *there exist polynomial vector functions* $T(\mu = C - C_{\pm}, \alpha)$, $N(\mu = = C - C_{\pm}, \alpha)$, *of power* $n-1$

$$T : I' \times X_0 \to X_0, \quad N : I' \times X_0 \to X_0,$$

$$T(\mu, 0)\,|_{\mu=0} = \frac{\partial}{\partial \alpha} T(\mu, 0)\,|_{\mu=0} = 0, \quad N(\mu, 0)\,|_{\mu=0} = \frac{\partial}{\partial \alpha} N(\mu, 0)\,|_{\mu=0} = 0$$

such that the transformation

$$w_0 = \overline{a} + T(\mu, \overline{a}) \tag{2.20}$$

transfers (2.19) to the equation

$$\frac{d\overline{a}}{d\zeta} = A_0 \overline{a} + N(\mu, \overline{a}) + o\left((\mu + |a|)^{n-1}\right) \tag{2.21}$$

for μ *and* $|\overline{a}|$ *close to zero* ($\|\cdot\|$ *is any norm in* $\mathbb{C}^m$). *The vector-function* $N(\mu, \overline{a})$ *satisfies the equation*

$$N\left(\mu, e^{tA_0^*}\overline{a}\right) = e^{tA_0^*} N(\mu, \overline{a}) \tag{2.22}$$

for $t \in R$. *Besides* $R_0 N(\mu, \overline{a}) = -N(\mu, R_0 \overline{a})$ *linear isometry. Here* $*$ *denotes hermitian conjugation for operators and matrices and complex conjugation of vectors and scalars.*

If we omit $o\left((\mu + |a|)^{n-1}\right)$, we get a quasi normal form of reduced equations (2.21)

$$\frac{d\overline{a}}{d\zeta} = A_0 \overline{a} + N(\mu, \overline{a}) \,. \tag{2.23}$$

Substituting (2.20) into (2.19) and taking (2.23) into account, we get an equation for determining $T(\mu, \overline{a})$

$$\begin{gathered} \frac{\partial T(\mu, \overline{a})}{\partial \overline{a}} A_0 \overline{a} - A_0 T(\mu, \overline{a}) = f_0(\mu, \overline{a}) - N(\mu, \overline{a}), \\ f_0(\mu, \overline{a}) = F_0(\mu, \overline{a} + h(\mu, \overline{a})). \end{gathered} \tag{2.24}$$

Differentiating (2.22) with respect to t for $t = 0$, we have

$$\frac{\partial T(\mu, \overline{a})}{\partial \overline{a}} A_0^* \overline{a} = A_0^* N(\mu, \overline{a}) \,. \tag{2.25}$$

Polynomial vector functions $T(\mu, \overline{a})$ and $N(\mu, \overline{a})$ determine the form of the functions F_0.

3. Classic solitary waves. In this case a unique imaginary eigen value of the operator A is a double zeroth eigen value, and dimension of a central manifold (X_0) equals two. Canonical basis in X_0 consists of eigen Φ_0 and adjoint Φ_1 vectors A responding to zeroth eigen number: $A\Phi_0 = 0$, $A\Phi_1 = \Phi_0$. Let's consider the isometry operator R: $R\Phi_0 = \Phi_0$. Then from the equality $RA\Phi_1 = R\Phi_0 = -AR\Phi_1$ it follows $R\Phi_1 = -\Phi_1$ and consequently

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

Adjoint operator A^* has one eigen vector ψ_1 and one adjoint vector ψ_0

$$A^*\psi_1 = 0, \quad A^*\psi_0 = \psi_1, \quad \langle \Phi_i, \psi_j \rangle = \delta_i^j, \quad A_0^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

Here $\langle \cdot, \cdot \rangle$ denotes a scalar product in some Hilbert space X, δ_i^j is a Kronecker symbol. The projection on the central manifold R_0 is realized by means of the vectors ψ_1 and ψ_0.

$$w_0 = p_0 w = \sum_{j=0}^{1} \langle w, \psi_j \rangle \Phi_j.$$

Equations (2.19) in the present case represent a dynamic system of second order, $f_0(\mu, w_0)$ has the components $\langle F, \psi_j \rangle$, $j = 0, 1$

$$N(\alpha) = (N_1, N_2)(\overline{a}), \quad \overline{a} = (a_0, a_1)(\zeta). \tag{3.3}$$

By means of (3.3) we reduce equation (2.25) to the form

$$\begin{pmatrix} \partial_{a_0} N_1 & \partial_{a_1} N_1 \\ \partial_{a_0} N_2 & \partial_{a_1} N_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

Hence, it follows

$$a_0 \frac{\partial N_1}{\partial a_1} = 0, \quad a_0 \frac{\partial N_2}{\partial a_1} = N_1, \quad N_1 = N_1(a_0).$$

Further, we use the invertibility N:$N(R_0\overline{a}) = -R_0 N(\overline{a})$ and get $N_1 = 0$, $N_2 = N_2(a_0)$. Finally, equations (2.23) accept the form

$$\frac{da_0}{d\zeta} = a_1, \quad \frac{da_1}{d\zeta} = \sum_{j=1}^{2} c_j(\mu) a_0^j + o\left(|a_0|^2\right) = \Phi(\mu, a_0) \tag{3.4}$$

$$o_1(\mu) = c_1(\mu) + o(\mu), \quad c_2(\mu) = c_2 + o(\mu).$$

The coefficient c_j are calculated from (2.23) where $N = (0, \Phi)$. The system of equations is integrable and has the first integral. Multiplying the second equation of (3.4) by a_1 and transforming it by means of the first equation, we get quasinormal form of reduced equations (2.18)

$$\left(\frac{da_0}{d\zeta}\right)^2 = \mu c_1 a_0^2 + \frac{2}{3} c_2 a_0^3 + o(\mu). \tag{3.5}$$

Let's conduct scale transformation in equation (3.5) and this equation in the lowest approximation with respect to μ has solitone-like solutions [1, 2, 5-9]

$$a_0 = \frac{3}{2} \frac{|c_1|}{|c_2|} \mu P_0(\overline{\zeta}), \quad \overline{\zeta} = \nu J, \quad \nu = |c_1 \mu|^{1/2}.$$

The types of solutions (3.5) are determined by the coefficients c_i and $\mu > 0$. For example, if $signc_1 = 1$, then for $signc_2 = 1$ we have solitary waves of "pot" type

$$P_0 = -ch^{-2}\frac{\overline{\zeta}}{2},$$

If $signc_2 = -1$ solitary elevation waves

$$P_0 = ch^{-2}\frac{\overline{\zeta}}{2}\ . \tag{3.6}$$

4. Generalized-solitary waves. As we have already noted, in this case the phase velocity of wave C exceeds C_- very little, and eigen values of the matrix A now are multiple zero and

$$\pm iq, q^2 = \alpha_1 - \gamma_2, \mu > 0.$$

Make change of the unknown functions

$$w = a_0\Phi_0 + a_1\Phi_1 + a_+\Phi_+ + a_-\Phi_-, \quad \Phi_+ = \Phi_-^*, \ \ a_+ = a_-^*\ , \tag{4.1}$$

where $A\Phi_0 = 0,\ \ A\Phi_1 = \Phi_0,\ \ A\Phi_\pm = \pm iq\psi_\pm$.

Eigen and adjoint vectors of the matrix (2.16) (A) are of the form:

$$\Phi_0 = \begin{pmatrix} -\beta_2/\beta_1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} 0 \\ 0 \\ -\beta_2/\beta_1 \\ 1 \end{pmatrix}, \quad \Phi_\pm = \begin{pmatrix} -\alpha_1/\beta_1 \\ 1 \\ \pm iq\alpha_1/\beta_1 \\ \pm iq \end{pmatrix}. \tag{4.2}$$

Eigen and adjoint vectors of transposed matrix A^* are given by the expressions

$$\psi_1 = \frac{1}{q^2}\begin{pmatrix} 0 \\ 0 \\ \beta_1 \\ \alpha_1 \end{pmatrix}, \quad \psi_0 = \frac{1}{q^2}\begin{pmatrix} \beta_1 \\ \alpha_1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_\pm = -\frac{1}{2q^2}\begin{pmatrix} \beta_1 q \\ \beta_2 q \\ \pm i\beta_1 \\ \pm i\beta_2 \end{pmatrix}. \tag{4.3}$$

With the help (4.2), (4.3) the expressions for the unknown functions u_i by the new (4.1) are expressed by the formulae

$$u_1 = -\frac{\beta_2}{\beta_1}a_0 - \frac{\alpha_1}{\beta_1}(a_+ + a_-), \quad u_2 = a_0 + a_+ + a_-\ . \tag{4.4}$$

In a similar, way from (2.10) and (4.4) it follows

$$\begin{gathered} \eta_1' = -\frac{C}{1-R}\left(R + \frac{\beta_2}{\beta_1}\right)a_0 - \frac{\gamma_1 R}{(1-R)^3}\left(C + \frac{\beta_2}{\beta_1}\right)\ddot{a}_0, \\ \eta_2' = Ca_0 - \frac{2CH(2C^2 - H)}{1-R}\ddot{a}_0. \end{gathered} \tag{4.5}$$

In the system of equations (2.16) we represent the vector function F in the form: $F = F_{01}(a_0) + F_{02}(a_0, a_+, a_-)$, $\quad F_{02}(a_0, 0, 0) = 0$,

$$F_{01} = \begin{pmatrix} 0 \\ 0 \\ \left[\alpha_2(c) + \alpha_1(c)\dfrac{\beta_2}{\beta_1}\right] a_0 + \left[\alpha_4 - \alpha_3\dfrac{\beta_2^2}{\beta_1^2} - \alpha_5\dfrac{\beta_2}{\beta_1}\right] a_0^2 \\ \left[\beta_2(c) - \beta_1(c)\dfrac{\beta_2}{\beta_1}\right] a_0 + \left[-\beta_4 + \beta_3\dfrac{\beta_2^2}{\beta_1^2} - \beta_5\dfrac{\beta_2}{\beta_1}\right] a_0^2 \end{pmatrix}. \tag{4.6}$$

Then (2.16) in new variables will be transformed to the form

$$\begin{aligned} \dot{a}_0 &= a_1, \\ \dot{a}_1 &= f_{01} + (F_{02}, \psi_1), \\ \dot{a}_+ &= iqa_+ + if_{02} + (F_{02}, \psi_+), \\ \dot{a}_- &= -iqa_- - if_{02} + (F_{02}, \psi_-), \end{aligned} \tag{4.7}$$

where $f_{01} = \langle F_{01}, \psi_1 \rangle$, $\quad f_{02} = -i \langle F_{01}, \psi_+ \rangle$. Because of awkwardness of the expressions f_{01} and f_{02} we write them only in the case $\gamma_1 = 0$. Therewith by means of (2.14) they are significantly simplified

$$\begin{aligned} f_{01} &= \mu\frac{6(1-R)}{C_-^3}\frac{C_-^2(1+H) - 2H(1-R)}{C_-^2(1+H^2+3HR) - H(1+H)}a_0 - \\ &- \frac{9(1-R)}{2C_-}\frac{C_-^4 + C_-^2(1-2H) + H^2 - 1}{C_-^2(1+H^2+3HR) - H(1+H)}a_0^2, \\ f_{02} &= -\frac{\beta_2}{2q^3}\left\{\left[\frac{\beta_2\beta_1(C)}{\beta_1} - \beta_2(C) + \frac{\alpha_1}{\alpha_2}\left(\alpha_2(C) + \frac{\beta_2}{\beta_1}\alpha_1(C)\right)\right] a_0 + \right. \\ &\left. + \left[\frac{C_-}{D}\left(\frac{3}{2} - \frac{\beta_2^2}{2\beta_1^2} + \frac{\beta_2}{\beta_1}\right) + \frac{C_-\alpha_1}{\alpha_2}\left(\frac{3}{2}R - \frac{9\beta_2^2}{2\beta_1^2} - 3R\frac{\beta_2}{\beta_1}\right)\right] a_0^2\right\}. \end{aligned} \tag{4.8}$$

Perform scale transformations considering smallness of the shortwave constituent $a_\pm$ of the considered solution in comparison with its long wave part a_0

$$a_0 = \mu P_0, \quad a_1 = \mu^{3/2} P_1, \quad a_\pm = \mu^2 P_\pm, \quad \bar{\zeta} = \mu^{1/2}\zeta.$$

Writing (4.7) in new variables, we get $a_0 = o(\mu)$, $a_\pm = o(\mu^2)$ and consequently, F_{02} has order higher than μ^2

$$\begin{aligned} \frac{dP_0}{d\bar{\zeta}} &= P_1, \\ \frac{dP_1}{d\bar{\zeta}} &= c_1 P_0 - c_2 P_0^2 + o(\mu), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} c_1 &= \frac{6(1-R)}{C_-^3}\frac{C_-^2(1+H) - 2H(1-R)}{C_-^2(1+H^2+3HR) - H(1+H)}, \\ c_2 &= \frac{9(1-R)}{2C_-}\frac{C_-^4 + C_-^2(1-2H) + H^2 - 1}{C_-^2(1+H^2+3HR) - H(1+H)}. \end{aligned} \tag{4.10}$$

It follows from (1.10) that for $Q = 0$ the value c_1 is always non-negative quantity. Then the system of equations (4.7) possess solitone-like solutions (3.6) and in the lowerst order μ the solutions in old variables (4.4) have the form

$$u_1 = -\frac{3\beta_2}{2\beta_1}\frac{\mu c_1}{c_2} ch^{-2}\frac{\sqrt{\mu c_1}}{2}\zeta + o(\mu^2),$$
$$u_2 = \frac{3}{2}\frac{\mu c_1}{c_2} ch^{-2}\frac{\sqrt{\mu c_1}}{2}\zeta + o(\mu^2). \tag{4.11}$$

Substituting (4.10) into (4.5) we find profiles of solitary waves

$$\eta_1' = -\frac{3}{2}\frac{C}{1-R}\frac{\mu c_1}{c_2}\left(R + \frac{\beta_2}{\beta_1}\right) ch^{-2}\frac{\sqrt{\mu c_1}}{2}\zeta -$$
$$-\frac{3}{4}\frac{\mu^2 c_1^2}{c_2}\frac{\gamma_1 R}{(1-R)^3}\left(C + \frac{\beta_2}{\beta_1}\right)\left(3th^2\frac{\sqrt{\mu c_1}}{2}\zeta - 1\right) ch^{-2}\frac{\sqrt{\mu c_1}}{2}\zeta + o(\mu^3),$$
$$\eta_2' = \frac{3}{2}\frac{C\mu c_1}{c_2} ch^{-2}\frac{\sqrt{\mu c_1}}{2}\zeta - \frac{3CH(2C^2 - H)\mu^2 c_1^2}{2(1-R)c_2}\times$$
$$\times\left(3th^2\frac{\sqrt{\mu c_1}}{2}\zeta - 1\right) ch^{-2}\frac{\sqrt{\mu c_1}}{2}\zeta + o(\mu^3). \tag{4.12}$$

Remind that equivalent expressions (4.11) and (4.12) in physical variables are obtained by substitution (see denotation after formulae (1.4) and (2.12)): $\vartheta_i = \sqrt{gh_1}u_i$, $\overline{\eta}_i = h_1\eta_i$, $\overline{x} - C\overline{t} = h_1\sqrt{1-R}\xi$. Expressions for velocities (4.11) and profiles of surfaces (4.12) for $\gamma_1 = 0$ pass to the expressions [1, 2, 7, 9]. Free surface has the form of pit for $\Lambda < 0$, $\gamma_1 = 0$ and elevation for $\Lambda > 0$, $\gamma_1 = 0$ and vice versa for the interface. However, for $\gamma_1 \neq 0$ distortion of the form of a pit and elevation of surfaces insignificantly diminish at the expense of surface tension.

Let's again consider waves with velocities $C = C_+ + \mu$, $\mu > 0$. For $C = C_+$ dispersive equation (2.17) has a second order unique zero on an imaginary axis. In this case

$$w = a_0(\zeta)\Phi_0 + a_1(\zeta)\Phi_1.$$

Eigen Φ_0, Φ_1 and adjoint vectors ψ_0, ψ_1 of the matrices A and A^* correspond to the zeroth eigen value. The system of equations (2.16) in the variables a_0, a_1 is transformed to the form (3.4)

$$\frac{da_0}{d\zeta} = a_1,\quad \frac{da_0}{d\zeta} = \mu c_1 a_0 - c_2 a_0^2 + o\left(\mu\,|a| + |a|^2\right). \tag{4.13}$$

The expressions for c_1 and c_2 coincide with formulae (4.10) while substituting C_- for C_+. In the lowest order with respect to μ the solution of the reduced equations (4.13) approximate the solutions (4.10). For small values of $1 - R$ we estimate c_2

$$c_2 \sim \frac{9(1-R)}{2(1+H)^2\sqrt{1+R}}.$$

In the considered case the elevation waves ($C_+^2 > H$) will be profile of solitary waves η_1' and η_2' (4.12) and as $h_2 \to 0$ internal wave amplitude tends to surface wave amplitude.

5. Asymptotic solutions. Omitting the small terms $(F_{02}, \psi_\pm)$ we write the last pair of equations (4.7) in the form

$$\frac{\partial a_+}{\partial \zeta} - iqa_+ = if_{02}(a_0), \quad \frac{\partial a_-}{\partial \zeta} + iqa_- = -if_{02}(a_0)\ , \tag{5.1}$$

where $f_{02}(a_0) = \mu d_1 a_0 + d_2 a_0^2$, the coefficients d_1 and d_2 are defined by the same expressions (4.8) for $C = C_- + \mu,\ \ \mu > 0$. To (5.1) we apply the Fourier transform

$$a_+^F = -\frac{f_{02}^F}{s+q}, \quad a_-^F = \frac{f_{02}^F}{s-q}, \tag{5.2}$$

where

$$a_\pm^F = \frac{1}{2\pi}\int_{-\infty}^{\infty} a_\pm \exp(-is\zeta)d\zeta, \quad f_{02}^F = (d_1 S + d_2 S^2) sh^{-1}\left(-\frac{\pi S}{\sqrt{\mu d_1}}\right). \tag{5.3}$$

We'll look for the solution of system (4.7) expressed by the invertibility singularity. Precisely speaking, the considered solution has a principal part (4.11) and the same asymptotics on both infinities. The solution is given by the formula

$$a_\pm = \frac{1}{2}\int_\Gamma a_\pm^F \exp(is\zeta)d\zeta\ , \tag{5.4}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$, Γ_1 integration contour passes over the poles $a_\pm^F$ on a real axis, and the contour Γ_2 below these poles. The contours Γ_1 and Γ_2 give contribution to asymptotics in minus and plus infinities. In the lowest order with respect to μ, from (5.2)-(5.4) we have

$$\mathrm{Re}\, a_\pm \to \pm d \exp\left(-\frac{\pi q}{\sqrt{\mu d_1}}\right)\sin q\zeta, \quad for\ \ \zeta \to \pm\infty\ . \tag{5.5}$$

Here the value of the constant d may be determined only while analyzing the complete system (4.7). Asymptotics (5.5) qualitatively defines exponential small asymptotics of periodic constituent of generalized solitary waves, responding to (4.11).

Appropriate velocities as $\zeta \to \pm\infty$ in the lowest order with respect to μ are given by the expressions (4.4)

$$u_1 \to \pm\frac{\alpha_1}{\beta_1} d \exp\left(-\frac{\pi q}{\sqrt{\mu d_1}}\right)\sin q\zeta, \quad u_2 \to \pm d \exp\left(-\frac{\pi q}{\sqrt{\mu d_1}}\right)\sin q\zeta. \tag{5.6}$$

We considered the solutions, where a_+ is of order μ^2. We can consider the solutions of self-agreed system of equations (4.7) where a_+ is of order μ. In this case solitary waves with ripples exist [7].

Conclusion. Classic solitary and generalized-solitary waves and also solitary wave packets are found among the solutions of generalized equations KdV [7]. As examples of wave motions, where additional dispersion effects for long waves should be taken into account in propagation of gravitational-capillary waves in the interface of two-layer liquid [1,2], the present model with high accuracy gives long wave part of solitone-like structures. From formula (4.12) it follows the known fact that internal solitone is always a wave in which thin layer thickens, i.e. deviation of interface is downwards directed, if the upper layer is thin and vice versa [11]. Shortwave oscillations imposed on solitary waves are described in qualitative level within the given model.

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