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PONTRYAGIN'S MAXIMUM PRINCIPLE IN AN OPTIMAL CONTROL PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS WITH SPECIAL QUALITY TEST

Abstract

In the paper we consider optimal control problems for systems of ordinary differential equations with special quality test that is characteristic for identification problems statement [1, 2].

Let R^n be an n-dimensional Euclidean space, $t > 0$ be a time, $t \in [0, T]$, $x(t) = (x_1(t), x_2(t), ..., x_n(t))$ be an $n-$dimensional vector function,

$$u(t) = (u_1(t), u_2(t), ..., u_m(t))$$

be m-dimensional vector function, $f(t, x, u) = (f_1(t, x, u), f_2(t, x, u) ..., f_n(t, x, u))$ be an n-dimensional vector function determined for all $t \in [0, T]$, $x \in R^n$, $u \in R^n$. Besides, let $x_0 = (x_{01}, x_{02}, ...x_{0n})$, $x_T = (x_{T1}, x_{T2}, ...x_{Tn})$ belong to R^n and the sign $''\overset{\circ}{\forall}''$ mean "almost everywhere". By $L_p^{(m)}(0, T)$ we denote a space of measurable m-dimensional vector-functions summable with the power $p \in [1, \infty]$. In this space a norm is introduced by the formula

$$\|u\|_{m,p} = \|u\|_{L_p^{(m)}(0,T)} = \left(\int_0^T |u(t)|_m^p \, dt \right)^{1/p},$$

where $|u|_m = \|u\|_{R^n} = \left(\sum_{i=1}^m |u_i|^2 \right)^{1/p}$ denotes a norm in the space R^m of the element $u \in R^m$.

Let the motion of the controlled object be described by the equation:

$$x'(t) = f(t, x(t), u(t)), \ 0 \le t \le T \tag{1}$$

and at the fixed time T the boundary conditions

$$x(0) = x_0, \tag{2}$$

$$x(T) = x_T \tag{3}$$

be given.

By $U \equiv \left\{ u = u(t) : \ u \in L_p^{(m)}(0, T), u(t) \in M, \ \overset{\circ}{\forall} t \in [0, T] \right\}$ we denote a set of admissible controls where M is a closed bounded set in R^m.

A wide class of problems of control of the processes described by the system (1) consists in choice of such a control $u = u(t)$ from the field of admissible controls U, that the system pass from the state x_0 to the state x_T.

Let the vectors x_0, x_T be given. Let $x = x_0(t)$ be a solution of the equation (1) under condition (2) and $x = x_T(t)$ be a solution of this equation, under condition(3). The problems on determination of $x_0(t)$ and $x_T(t)$ are the problems (1),(2) and (1),(3), respectively. The problems (1),(2) and (1),(3) are the Cauchy problems for the equation (1).

Under the solution of problem (1),(2) for the given admissible control $u \in U$ we understand an absolutely continuous function $x_0(t)$ satisfying the integral identity:

$$x_0(t) = x_0 + \int_0^t f(\tau, x_0(\tau), u(\tau))\, d\tau \tag{4}$$

for $\forall t \in [0, T]$.

Similarly, under the solution of problem (1),(3) for the given control $u \in U$ we understand an absolutely continuous function $x_T(t)$ satisfying the integral identity:

$$x_T(t) = x_T + \int_t^T f(\tau, x_T(\tau), u(\tau))\, d\tau \tag{5}$$

for $\forall t \in [0, T]$.

Now, let's consider a problem on the minimization of the functional

$$J_\alpha(u) = \|x_0(\cdot, u) - x_T(\cdot, u)\|_{n,2}^2 + \alpha \|u - u_0\|_{m,2}^2, \tag{6}$$

on the set of admissible controls U, where $\alpha \geq 0$ is a number parameter, $u_0 = u_0(t)$ is a given element of the space $L_p^{(m)}(0, T)$, $x_0(t) \equiv x_0(t; u)$, $x_T(t) \equiv x_T(t; u)$ are the solutions of integral equations (4),(5), respectively, for $u \in U$.

Let the function $f(t, x, u)$ be determined on the domain $[0, T] \times R^n \times M$ and be continuous in totality of variables, satisfy the Lipschitz condition with respect to variables $x \in R^n, u \in M$ for all $t \in [0, T]$:

$$|f(t, x, u) - f(t, y, v)|_n \leq L[|x - y|_n + |u - v|_m], \tag{7}$$

where $L > 0$ is a Lipschitz constant.

Then it follows from the results of the papers [3,4] that for any function $u = u(t)$ from the set U the problems (4)-(5) have unique solutions determined and continuous on the interval $[0, T]$. Almost everywhere these solutions have derivatives belonging to the space $L_\infty^{(n)}(0, T)$.

We denote a space of continuous vector-functions

$$x(t) = (x_1(t), x_2(t), ..., x_n(t)), \ 0 \leq t \leq T$$

with the norm

$$\|x\|_{C^{(n)}[0,T]} = \max_{0 \leq t \leq T} |x(t)|_n$$

by $C^{(n)}[0,T]$. As is known $C^{(n)}[0,T]$ is a complete normed space. Then we can show that for any function $x = x(t)$ from $C^{(n)}[0,T]$ (see.[3],[4]) and for any control $u = u(t)$ from the set U the function $f(t,x(t),u(t))$ will be a boundedly-measurable function on the interval $[0,T]$ i.e.

$$|f(t,x(t),u(t))|_n \le f_0(t), \quad \overset{\circ}{\forall} t \in [0,T], \tag{8}$$

for $\forall x \in C^{(n)}[0,T]$, $\forall u \in U$, where $f_0(t) > 0$, $f_0 \in L_\infty(0,T)$ is some function.

By the inequality (8) and integral relations (4),(5) we can easily establish the validity of the estimations:

$$|x_0(t)|_n \le c_0 \left(|x_0|_n + \|f_0\|_{L_\infty(0,T)} \right), \tag{9}$$

$$|x_T(t)|_n \le c_1 \left(|x_T|_n + \|f_0\|_{L_\infty(0,T)} \right) \tag{10}$$

for $\forall t \in [0,T]$, where $c_0 > 0$, $c_1 > 0$ are some constants.

In this paper we study necessary condition of optimality in the form of Pontryagin's maximum principle. To this end we introduce the Hamilton-Pontryagin function for the problem (6) in the form:

$$H(t,x_0(t),x_T(t),u(t),\psi(t),\psi_T(t)) = <\psi_0(t), f(t,x_0(t),u(t))>_{R^n} +$$

$$+ <\psi_T(t), f(t,x_T(t),u(t))>_{R^n} - |x_0(t) - x_T(t)|_n^2 - \alpha |u(t) - u_0(t)|_m^2, \tag{11}$$

where $x_0(t)$, $x_T(t)$ are the solutions of the problems (4),(5), respectively, and the functions $\psi_0(t)$, $\psi_T(t)$ are the solutions of the following Conjugate system

$$\psi_0'(t) = -\frac{\partial H(t,x_0(t),x_T(t),u(t),\psi_0(t),\psi_T(t))}{\partial x_0}, \ 0 \le t \le T, \tag{12}$$

$$\psi_T'(t) = -\frac{\partial H(t,x_0(t),x_T(t),u(t),\psi_0(t),\psi_T(t))}{\partial x_T}, \ 0 \le t \le T, \tag{13}$$

$$\psi_0(t) = 0, \ \psi_T(t) = 0. \tag{14}$$

Using the form of the Hamilton-Pontryagin function, we can write this system in the explicit form:

$$\psi_0^1(t) = 2(x_0(t) - x_T(t)) - (f_{x_0}(t,x_0(t),u(t)))^T \psi_0(t), \ 0 \le t \le T, \tag{15}$$

$$\psi_0(t) = 0, \tag{16}$$

$$\psi_T^1(t) = -2(x_0(t) - x_T(t)) - (f_{x_T}(t,x_0(t),u(t)))^T \psi_T(t), \ 0 \le t \le T, \tag{17}$$

$$\psi_T(t) = 0, \tag{18}$$

where $f_x(t,x,u)$ is a matrix of partial derivatives of the vector-function $f(t,x,u)$ with respect to x, $(f_x(t,x,u))^T$ is a transposition of the matrix $f_x(t,x,u)$.

As is seen, the conjugate system consists of two Cauchy problems on definition of the functions $\psi_0(t)$ and $\psi_T(t)$ from the conditions (15),(16) and (17),(18), respectively. Under the solution of these problems we understand the functions $\psi_0(t)$,$\psi_T(t)$, satisfying we following integral identities:

$$\psi_0(t) = \int_t^T \left[f_{x_0}(\tau, x_0(\tau), u(\tau))^T \psi_0(\tau) - 2(x_0(\tau) - x_T(\tau)) \right] d\tau, \tag{19}$$

$$\psi_T(t) = -\int_0^t \left[f_{x_T}(\tau, x_T(\tau), u(\tau))^T \psi_T(\tau) + 2(x_0(\tau) - x_T(\tau)) \right] d\tau \tag{20}$$

for $\forall t \in [0, T]$.

Alongside with the conditions adopted above we'll assume that the vector-function has partial derivatives with respect to the variables $f_x = \{f_{1k}, f_{2k}, ..., f_{nk},\}$ that are continuous in totality of $x \in R$ is their arguments $t \in [0, T], x \in R^n,\ u \in M$ and moreover, $f_x(t, x, u)$ satisfies the Lipschitz condition with respect to variables (x, u):

$$\|f_x(t, x + \Delta x, u + \Delta u) - f_x(t, x, u)\| \le \widetilde{L}\left(|\Delta x|_n + |\Delta u|_m\right) \tag{21}$$

for all $(t, x, u)(t, x + \Delta x, u + \Delta u) \in [0, T] \times R^n \times M$ where $\|\cdot\|$ means a norm of the matrix $f_x(t, x, u), \widetilde{L} > 0$ is a Lipschitz constant.

Arguing similarly as in obtaining the inequality (8), we can establish

$$\|f_x(t, x(t) + u(t))\| \le \widetilde{f}_0(t),\ \ \overset{\circ}{\forall} t \in [0, T] \tag{22}$$

for $\forall x \in C^\tau[0, T],\ \forall u \in U$ where $\widetilde{f}_0(t) > 00,\ \widetilde{f}_0 \in L_\infty(0; T)$ is some function, i.e. the elements of the matrix $f_x(t, x(t), u(t))$ will be bounded measurable functions on the interval $[0, T]$.

In view of the fact that $x_0(t),\ x_T(t)$ from $C^n[0, T],\ x_0'(t),\ x_T'(t)$ from $L_\infty^{(n)}(0; T)$, and $f_x(t, x, u)$ satisfy the conditions (21),(22) we can affirm that the Cauchy problems (15),(16) and (17),(18) have continuous solutions continuous on the interval $[0, T]$ and $\psi_0' \in L_\infty^{(n)}(0; T),\ \psi_T' \in L_\infty^{(n)}(0; T)$.

Let's establish estimation for the solutions of the Cauchy problems (15),(16) and (17),(18). Really, using (19) and applying the Gronwall lemma we get the validity of the estimation:

$$|\psi_0(t)|_n \le C_2\left(|x_0|_n + |x_T|_n + \|f_0\|_{L_\infty(0;T)}\right) \tag{23}$$

for $\forall t \in [0, T]$, where $C_2 > 0$ is a constant independent of t.

Similarly, using the equality (20) we can establish the validity of the estimation:

$$|\psi_T(\tau)|_n \le C_3\left(|x_0|_n + |x_T|_n + \|f_0\|_{L_\infty(0;T)}\right) \tag{24}$$

for $\forall t \in [0, T]$, where $C_3 > 0$ is a constant independent of t.

Theorem 1. *Let the function $f(t, x, u)$ and its partial derivatives $f_x(t, x, u)$ with respect to the variable x be continuous in the domain $[0, T] \times R^n \times M$ and satisfy the conditions (7),(21). If $u = u(t)$ from U is an optimal control in the* *problem (6) and* $x_0(t)$, $x_T(t)$ *is an appropriate solution of problems (4),(5) for* $u \in U$, *then*

1) there exist n-dimensional vector-functions $\psi_0(t)$,$\psi_T(t)$ *that are the solutions of the Cauchy problems (15),(16) and (17),(18), respectively.*

2) for all $t \in [0,T]$ *being the continuity points of the optimal control* $u(t)$, *the function* $H(t, x_0(t), x_T(t), u, \psi_0(t), \psi_T(t))$ *as a variable* $u \in R^m$ *achieves its upper bound on the set* M *for* $u = u(t)$ *i.e.*

$$H(t, x_0(t), x_T(t), u(t), \psi_0(t), \psi_T(t)) =$$

$$= \sup_{v \in M} H(t, x_0(t), x_T(t), v, \psi_0(t), \psi_T(t)) \tag{25}$$

for $\overset{\circ}{\forall} t \in [0,T]$.

Proof. Let $u = u(t) \in U$ be an optimal control in the problem (6). We give to the control $u = u(t)$ such increment $\Delta u = \Delta u(t)$, $0 \le t \le T$ that $u + u\Delta \in U$. In this case $u(t) + \Delta u(t) \in M$, $\overset{\circ}{\forall} t \in [0,T]$. If $x_0(t) = x_0(t;u)$, $x_T(t) = x_T(t;u)$ are the solutions of problems (4),(5) for $u \in U$ and $x_{0\Delta}(t) \equiv x_0(t; u + \Delta u)$, $x_{T\Delta}(t) \equiv x_T(t; u + \Delta u)$ are the solutions of problems (4),(5) for $u + u\Delta \in U$.

Now, let's consider an increment of the functional $J_\alpha(u)$ on the element $u = u(t)$ of the set U. Using the formula for the functional $J_\alpha(u)$ itself, we get:

$$\Delta J_\alpha(u) = J_\alpha(u + \Delta u) - J_\alpha(u) =$$

$$= 2\int_t^T < (x_0(t) - x_T(t)), \Delta x_0(t) - \Delta x_T(t) >_{R^n} dt +$$

$$+2\alpha \int_t^T < u(t) - u_0(t),\ \Delta u(t) >_{R^m} dt + \|\Delta x_0\|^2_{L_2^{(m)}(0,T)} + \|\Delta x_T\|^2_{L_2^{(n)}(0,T)} +$$

$$+\alpha \|\Delta u\|^2_{L_2^{(m)}(0,T)} - 2\int_t^T < \Delta x_0(t), \Delta x_T(t) >_{R^m} dt. \tag{26}$$

For the Hamilton-Pontryagin function we can represent the increment of the function given by the formula (26) in the form:

$$\Delta J(u) = -\int_t^T [H(t, x_0(t), x_T(t), u(t) + \Delta u \psi_0(t), \psi_T(t))] -$$

$$- [H(t, x_0(t), x_T(t), u(t), \psi_0(t), \psi_T(t))]\, dt + R, \tag{27}$$

where R is determined by the formula

$$R = -\int_t^T < \psi_0(t), f(t, x_0(t), x_0(t) + \Delta x_0(t), u(t) + \Delta u(t)) -$$

$$-f\left(t,x_0\left(t\right),u\left(t\right)+\Delta u\left(t\right)\right)>_{R^n} dt-\int\limits_t^T<\psi_0\left(t\right),f\left(t,x_T\left(t\right)+\Delta\left(t\right),u\left(t\right)+\Delta u\left(t\right)\right)-$$

$$-f\left(t,x_T\left(t\right),u\left(t\right)+\Delta u\left(t\right)\right)>_{R^n} dt+$$

$$+\int\limits_t^T<f_{x_T}\left(t,x_T\left(t\right),u\left(t\right)\right)^T\psi_T\left(t\right),\Delta x_T\left(t\right)>_{R^n} dt+$$

$$+\left\|\Delta x_0\right\|^2_{L_2^{(n)}(0,T)}+\left\|\Delta x_T\right\|^2_{L_2^{(m)}(0,T)}-2\int\limits_t^T<\Delta x_0\left(t\right),\Delta x_T\left(t\right)>_{R^n} dt. \qquad (28)$$

Now, let's estimate the remainder term R. To this end we apply the finite increments method, and by the estimations (24) and (25), from the last inequality we get:

$$|R|\le C_{13}\left(\int\limits_t^T\left|\Delta x_0\left(t\right)\right|_n\left|\Delta u\left(t\right)\right|_m dt+\left|\Delta x_T\left(t\right)\right|_n\left|\Delta u\left(t\right)\right|_m dt+\right.$$

$$\left.+2\left\|\Delta x_0\right\|^2_{L_2^{(n)}(0,T)}+2\left\|\Delta x_T\right\|^2_{L_2^{(n)}(0,T)}\right).$$

Hence we get the validity of the estimation (22).

As $u=u\left(t\right)$ from U is an optimal control in the problem (6) we have:

$$\Delta J_\alpha\left(u\right)=\Delta J_\alpha\left(u+\Delta u\right)-J_\alpha\left(u\right)\ge 0 \qquad (29)$$

for all $\Delta u\in L_2^{(m)}\left(0,T\right)$ that $u+\Delta u\in V$.

Take any continuity point θ from the interval $[0,T]$ of the optimal control $u=u\left(t\right)$ i.e. the Lebesgue point of the interval $[0,T]$ and $\varepsilon<0$ that $\theta,\theta+\varepsilon\in[0,T]$ .Besides, we consider an increment of the functional $u\left(t\right)$ in the form:

$$\Delta u_\varepsilon\left(t\right)=\begin{cases} v-u\left(t\right), & \theta-\dfrac{\varepsilon}{2}\le t\le\theta+\dfrac{\varepsilon}{2}, \\ 0, & t\in[0,T]/\left[\theta-\dfrac{\varepsilon}{2},\ \theta+\dfrac{\varepsilon}{2}\right], \end{cases}$$

where $V\in M$ is any vector. This increment is said to be a needle-shaped variation of the optimal control $u=u\left(t\right)$. By this name we underline the fact that for small $\varepsilon>0$ the increment $\Delta u_\varepsilon\left(t\right)$ will differ from zero only on the small length interval. It is easy to verify that $u\left(t\right)+\Delta u_\varepsilon\left(t\right)$ will be an element of the set U, i.e. an admissible control.

Now, let's consider an increment of the functional $J_\alpha\left(v\right)$. Using the formula (27) we have:

$$\Delta J_\alpha\left(u\right)=-\int\limits_t^T g\left(t\right)dt+R=-\int\limits_{\theta-\frac{\varepsilon}{2}}^{\theta+\frac{\varepsilon}{2}} g\left(t\right)dt+R, \qquad (30)$$

where

$$g(t) = H(t, x_0(t), x_T(t), v, \psi_0(t), \psi_T(t)) -$$
$$-H(t, x_0(t), x_T(t), u(t), \psi_0(t), \psi_T(t)),$$
$$\theta - \frac{\varepsilon}{2} \le t \le \theta + \frac{\varepsilon}{2}. \tag{31}$$

By means of the estimation (29) we have:

$$|R| = |R(\varepsilon)| \le C_{12} \left(\int_{\theta-\frac{\varepsilon}{2}}^{\theta+\frac{\varepsilon}{2}} |\Delta u_\varepsilon(t)|\, dt \right)^2 \le \varepsilon C_{12} \int_{\theta-\frac{\varepsilon}{2}}^{\theta+\frac{\varepsilon}{2}} |\Delta u_\varepsilon(t)|\, dt. \tag{32}$$

By (31) from (32) we get the validity of the inequality:

$$\Delta J_\alpha(u) = - \int_{\theta-\frac{\varepsilon}{2}}^{\theta+\frac{\varepsilon}{2}} g(t)\, dt + R(\varepsilon) \ge 0.$$

Applying the definition of the Lebesque point from the last inequality we have:

$$-\varepsilon g(\theta) + o(\varepsilon) + R(\varepsilon) \ge 0,$$

where $\lim\limits_{\varepsilon\to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$.

If we divide the both hand sides of this inequality into $\varepsilon > 0$ and allowing to (32) pass to the limit as $\varepsilon \to +0$ we get the validity of the inequality

$$g(\theta) \ge 0 \tag{33}$$

for all θ continuity point from the interval $[0, T]$ of optimal control. Taking into account (31), from this inequality we have

$$H(\theta, x_0(\theta), x_T(\theta), v, \psi_0(\theta), \psi_T(\theta)) \le H(\theta, x_0(\theta), x_T(\theta), u(\theta), \psi_0(\theta), \psi_T(\theta))$$

for all the Lebesque θ-points, i.e. continuity points from the interval $[0, T]$ of the optimal control $u(t)$. Hence, by the arbitrariness of the vector $v \in M$ we get the affirmation of the theorem. Theorem 1 is proved.

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