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STEADY-STATE DISTRIBUTION OF PROBABILITIES FOR THE $M/G/1/m$ QUEUE WITH THE FORCED VACATIONS AFTER n CUSTOMERS SERVED IN SUCCESSION

Abstract

For the $M/G/1/m$ queue with the forced vacations in work after n customers served in succession probabilities of states of limiting steady-state process are found.

1. Introduction. In case of arbitraryly distributed service time and exponentially distributed interarrival time existence of limiting steady-state process for the single-server queueing system with unlimited queue is proved by a method of embedded Markov chains [1, p. 98]. For $M/G/1/m$ queue probabilities of states of limiting steady-state process are found in [2, p. 235].

Below we study $M/G/1/m$ queue which feature consists that after n customers served in succession the forced vacations in work of system occur. Necessity of such pause can be caused by technological or other reasons. For example, the technical device which consistently carries out homogeneous operations, at designing can be calculated on the limited number of operations which are carried out without interruption between operations.

2. Formulation of a problem. The general formulas for steady-state probabilities. We study single-server queue for which the number of places in queue cannot exceed number m. Let time intervals T_λ between the moments of arrival of customers in system are independent random variables exponentially distributed with parameter λ. We designate through e_λ: $E(T_\lambda) = e_\lambda = 1/\lambda$ the expectation of a random variable T_λ. We assume also, that service times are independent and identically distributed random variables T_μ with arbitrary distribution function $F_\mu(t)$. Then intensity of a services flow is $\mu = 1/e_\mu$, where $e_\mu = E(T_\mu) = \int\limits_0^\infty t\, dF_\mu(t)$.

The studied system has feature, that each time after n customers served in succession, the forced vacation in work of the system, proceeding random time T_ν, occurs. The random variable T_ν is arbitrarily distributed with distribution function $F_\nu(t)$ and with an expectation $e_\nu = E(T_\nu) = \int\limits_0^\infty t\, dF_\nu(t)$.

Customers who at the moment of the announcement of a vacation are in queue, remain in system and wait on renewal of service. Newcomers customers are accepted in queue also during a vacation at presence of free places in queue.

So, unlike usual queue with the limited number of places in queue, number of the customers, served one behind another without interruption, cannot exceed n for our queueing system.

Case $n = 1$, when the forced vacation occurs after each service, we shall not examine, because this case can be reduced to research of queue $M/G/1/m$ (look [2, p. 235]), if we conditionally consider a random variable $T_\mu + T_\nu$ as service time. Therefore we shall assume, that $n \geq 2$.

We introduce double numbering of states of queueing system: $s_{j(i)}$ designates a state j of a level i. Here j is a number of the customers in system; values $i = \overline{1,n}$ correspond to states of system when the i-th customer after the period of idle time of system is served; value $i = n+1$ corresponds to states of system during the forced vacation; $s_{0(1)}$ is a state of free system. So, we have all together $n(m+1)+m+2$ states: $s_{j(1)}$ $(j = \overline{0,m+1})$, $s_{j(i)}$ $(i = \overline{2,n};\ j = \overline{1,m+1})$, $s_{j(n+1)}$ $(j = \overline{0,m})$.

We assume, that at an initial moment of time the system is in a state $s_{0(1)}$. Let $N(t)$ be the number of a state in which the system is at time t. For process $N(t)$ we shall construct Markov chain which allows to study evolution of queue in time. We designate as t_k $(k = 1,2,\ldots)$ the moment of end of service of k-th served customer, and as $N_k = N(t_k + 0)$ number of the state in which the system passes after the end of service of this customer. Here we specify: if after the end of service of k-th customer the forced vacation occurs then for the moment t_k we accept not the moment of the end of service, but the moment of the end of a vacation.

We shall be convinced that the sequence $\{N_k\}$ forms Markov chain.

Let Δ_k is the number of the customers who have arrived to system in service time T_μ of k-th served customer, $\tilde{\Delta}_k$ is the number of the customers who have arrived to system in time $T_\mu + T_\nu$ equal to the sum of a service time of k-th customer and time of the forced vacation after service of this customer. We shall analyze transitions from a state in a state for one step in sequence $\{N_k\}$.

If $N_{k-1} = 0(1)$ (double numbering) or $N_{k-1} = 1(i)$ $(i = \overline{1,n-1})$ and $\Delta_k = 0$, then $N_k = 0(1)$. If $N_{k-1} = C_{k-1}(1)$, then $N_k = C_k(2)$ where

$$C_k = \begin{cases} \Delta_k, & C_{k-1} = 0, \quad 0 < \Delta_k \le m; \\ m, & C_{k-1} = 0, \quad \Delta_k > m; \\ C_{k-1} + \Delta_k - 1, & C_{k-1} > 0, \quad 0 < C_{k-1} + \Delta_k - 1 \le m; \\ m, & C_{k-1} > 0, \quad C_{k-1} + \Delta_k - 1 > m. \end{cases} \tag{1}$$

If $N_{k-1} = C_{k-1}(i)$ $(i = \overline{2,n-1})$, then $N_k = C_k(i+1)$, where

$$C_k = \min\{C_{k-1} + \Delta_k - 1\,;\, m\}. \tag{2}$$

If $N_{k-1} = C_{k-1}(n)$, then $N_k = C_k(1)$, where

$$C_k = \min\left\{C_{k-1} + \tilde{\Delta}_k - 1\,;\, m\right\}. \tag{3}$$

Since the rest of time from the moment of end of service of $(k-1)$-th customer or from the moment of end of the forced vacation after service of this customer till the moment of arrival in system of next customer is exponential and, therefore, does not depend on behavior of system till the moment t_{k-1}, then random variables Δ_k and $\tilde{\Delta}_k$ do not depend on the past, therefore equality (1) – (3) prove Markov property of sequence $\{N_k\}$.

The set of states of Markov chain N_k consists of $nm+1$ states: $s_{j(1)}$ $(j = \overline{0,m})$ and $s_{j(i)}$ $(i = \overline{2,n};\ \ j = \overline{1,m})$.

Direct transition from states of a level i $s_{j(i)}$ for $j = \overline{2,m}$, $i = \overline{1,n-1}$ is possible only in states of a level $i+1$, and direct transition from states of a level n is possible only in states of the first level. Direct transition in a state $s_{0(1)}$ from all states,

except states $s_{0(1)}$ and $s_{1(i)}$ $(i=\overline{1,n})$ is impossible. Therefore under the formula of a total probability for $k=1,2,\ldots$ we obtain equalities:

$$P\{N_{k+1}=0\,(1)\}=P\{N_{k+1}=0\,(1)\,|N_k=0\,(1)\}\,P\{N_k=0\,(1)\}+$$

$$+\sum_{s=1}^{n}P\{N_{k+1}=0\,(1)\,|N_k=1\,(s)\}\,P\{N_k=1\,(s)\};$$

$$P\{N_{k+1}=j\,(1)\}=$$

$$=\sum_{s=1}^{m}P\{N_{k+1}=j\,(1)\,|N_k=s\,(n)\}\,P\{N_k=s\,(n)\}\ \left(j=\overline{1,\,m}\right); \tag{4}$$

$$P\{N_{k+1}=j\,(2)\}=$$

$$=\sum_{s=0}^{m}P\{N_{k+1}=j\,(2)\,|N_k=s\,(1)\}\,P\{N_k=s\,(1)\}\quad\left(j=\overline{1,\,m}\right);$$

$$P\{N_{k+1}=j\,(i)\}=$$

$$=\sum_{s=1}^{m}P\{N_{k+1}=j\,(i)\,|N_k=s\,(i-1)\}\,P\{N_k=s\,(i-1)\}\quad\left(j=\overline{1,\,m};\ \ i=\overline{3,\,n}\right).$$

Transitive probabilities $p^{(k)}_{j(i)s(l)}=p_{j(i)s(l)}=P\{N_{k+1}=j\,(i)\,|N_k=s\,(l)\}$ do not depend on number k of the served customer, and these probabilities are determined in view of values of random variables $\Delta_k=\Delta$, $\tilde{\Delta}_k=\tilde{\Delta}$. It testifies to uniformity of Markov chain $\{N_k\}$.

Using the ergodic theorem for homogeneous nonreducible Markov chain with finite set of states ([3, p. 61] or [4, p. 67]), we can prove existence of limits

$$\pi_{j(i)}=\lim_{k\to\infty}P\{N_k=j\,(i)\}\quad\left(i=1,\ \ j=\overline{0,\,m};\quad i=\overline{2,\,n},\ j=\overline{1,\,m}\right).$$

After transition to limit in equalities (4) we receive system of the linear algebraic equations for definition of probabilities $\pi_{j(i)}$:

$$\pi_{0(1)}=\alpha_0\left(\pi_{0(1)}+\sum_{k=1}^{n-1}\pi_{1(k)}\right)+\alpha_0\beta_0\pi_{1(n)};$$

$$\pi_{j(1)}=\sum_{k=1}^{j+1}\pi_{k(n)}\sum_{l=0}^{j+1-k}\alpha_l\beta_{j+1-k-l}\qquad\left(j=\overline{1,\ m-1}\right);$$

$$\pi_{m(1)}=\sum_{k=1}^{m}\pi_{k(n)}\left(1-\sum_{l=0}^{m-k}\alpha_l\sum_{s=0}^{m-k-l}\beta_s\right);$$

$$\pi_{j(2)}=\alpha_j\pi_{0(1)}+\sum_{k=1}^{j+1}\alpha_{j+1-k}\pi_{k(1)}\qquad\left(j=\overline{1,\ m-1}\right); \tag{5}$$

$$\pi_{m(2)}=\left(1-\sum_{k=0}^{m-1}\alpha_k\right)\pi_{0(1)}+\sum_{k=1}^{m}\left(1-\sum_{l=0}^{m-k}\alpha_l\right)\pi_{k(1)};$$

$$\pi_{j(i)} = \sum_{k=1}^{j+1} \alpha_{j+1-k} \pi_{k(i-1)} \qquad \left(j = \overline{1,\ m-1}\,;\quad i = \overline{3,\ n}\right);$$

$$\pi_{m(i)} = \sum_{k=1}^{m} \left(1 - \sum_{l=0}^{m-k} \alpha_l\right) \pi_{k(i-1)} \qquad \left(i = \overline{3,\ n}\right).$$

It is necessary to solve system (5) with a normalization condition

$$\sum_{j=0}^{m} \pi_{j(1)} + \sum_{i=2}^{n} \sum_{j=1}^{m} \pi_{j(i)} = 1.$$

Such designations are used in (5): a_j is a probability of realization of j events in the stationary Poisson stream of intensity λ in time T_μ, β_j is a similar probability for time T_ν, that is

$$\alpha_j = \int_0^\infty \frac{(\lambda t)^j}{j!} e^{-\lambda t} dF_\mu(t), \qquad \beta_j = \int_0^\infty \frac{(\lambda t)^j}{j!} e^{-\lambda t} dF_\nu(t)\,.$$

We introduce designations: $p_{j(i)}(t) = P\left\{N(t) = j(i)\right\}$, $p_{j(i)} = \lim\limits_{t\to\infty} p_{j(i)}(t)$. We shall show, that by means of solutions of system (5), we can find some of steady-state probabilities $p_{j(i)}$, and also steady-state values of characteristics of considered queueing system.

Let's analyze work of system on very big time interval T. If $N(T)$ is the number of the customers arrived in system in time T (in view of the lost customers), $N_{serv}(T)$ is the number of the served customers for this time, then the approximate equality is executed

$$T \approx e_\lambda N(T) \approx \left(e_\lambda \pi_{0(1)} + e_\nu \sum_{k=1}^{m} \pi_{k(n)} + e_\mu\right) N_{serv}(T). \tag{6}$$

Equality (6) is carried out with greater accuracy when the time interval T increases. Having expressed from (6) relations $N_{serv}(T)/N(T)$, after transition to limit at $T \to \infty$, we can find steady-state probability of service of customer, arrived on an input of system,

$$P_{serv} = \lim_{T\to\infty} \frac{N_{serv}(T)}{N(T)} = \frac{e_\lambda}{e_\lambda \pi_{0(1)} + e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)} + e_\mu} =$$

$$= \frac{1}{\pi_{0(1)} + \lambda\left(e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)} + e_\mu\right)} = \frac{1}{Z},$$

where $Z = \pi_{0(1)} + \lambda\left(e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)} + e_\mu\right)$.

Taking into account, that $T_{0(1)}(T) \approx e_\lambda \pi_{0(1)} N_{serv}(T)$ is a total (in time T) time of stay of system in a state $s_{0(1)}$, we find steady-state probability of stay of system

in this state by means of limiting transition

$$p_{0(1)} = \lim_{T\to\infty} \frac{T_{0(1)}(T)}{T} = \frac{e_\lambda \pi_{0(1)}}{e_\lambda \pi_{0(1)} + e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)} + e_\mu} = \frac{\pi_{0(1)}}{Z}.$$

We calculate average time of stay of system in each of states $s_{1(1)}, \ldots, s_{1(n)}$ as an expectation of minimal of two random variables T_λ and T_μ:

$$E(T_{1(i)}) = E\left(\min\{T_\lambda,\ T_\mu\}\right) = \int\limits_0^\infty t\lambda e^{-\lambda t}\left(1 - F_\mu(t)\right) dt + \int\limits_0^\infty t p_\mu(t) e^{-\lambda t} dt =$$

$$= \frac{1}{\lambda} - \int\limits_0^\infty \lambda t e^{-\lambda t} F_\mu(t) dt + \frac{\alpha_1}{\lambda} = \frac{1}{\lambda} - \left(\frac{\alpha_1}{\lambda} + \frac{\alpha_0}{\lambda}\right) + \frac{\alpha_1}{\lambda} = \frac{1-\alpha_0}{\lambda} \qquad \left(i = \overline{1,\ n}\right).$$

Here $p_\mu(t)$ is a distribution density of probabilities of a random variable T_μ, and integral $\int\limits_0^\infty \lambda t e^{-\lambda t} F_\mu(t) dt$ is calculated by integration by parts.

So, a total (in time T) time of stay of system in a state $s_{1(i)}$ is

$$T_{1(i)}(T) \approx \frac{(1-\alpha_0)\pi_{1(i)}}{\lambda} N_{serv}(T) \qquad \left(i = \overline{1,\ n}\right),$$

therefore

$$p_{1(i)} = \lim_{T\to\infty} \frac{T_{1(i)}(T)}{T} = \frac{(1-\alpha_0)\pi_{1(i)}}{\lambda\left(e_\lambda \pi_{0(1)} + e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)} + e_\mu\right)} = \frac{(1-\alpha_0)\pi_{1(i)}}{Z} \ \left(i = \overline{1,\ n}\right).$$

We carry out similar limiting transition and we calculate steady-state probability of occurrence of the forced vacation in work of system

$$P_{vac} = \sum_{k=0}^{m} p_{k(n+1)} = \frac{e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)}}{e_\lambda \pi_{0(1)} + e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)} + e_\mu} = \frac{\lambda e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)}}{Z}. \tag{7}$$

Using the received formulas for steady-state probabilities, we can find an average of the occupied servers (operating ratio of single-server queueing system)

$$\bar{k} = 1 - p_{0(1)} - \sum_{k=0}^{m} p_{k(n+1)} = 1 - \frac{\pi_{0(1)} + \lambda e_\nu \sum\limits_{k=1}^{m} \pi_{k(n)}}{Z} = \frac{\lambda e_\mu}{Z}$$

and the probability of that queue is equal m

$$\sum_{k=1}^{n} p_{m+1\,(k)} + p_{m\,(n+1)} = 1 - P_{serv} = \frac{Z-1}{Z}.$$

3. Case, when $m = 1$. When maximal queue is equal 1 ($m = 1$) we can to calculate still some steady-state probabilities of states, and also average length of queue $\bar{r}$ and an average waiting time $\bar{t}_r$.

Really, as $P_{serv} = p_{0(1)} + p_{0\,(n+1)} + \sum\limits_{k=1}^{n} p_{1\,(k)}$, and $p_{0\,(n+1)} + p_{1\,(n+1)} = P_{vac}$, and last probability is calculated under the formula (7), therefore

$$p_{0\,(n+1)} = P_{serv} - \left(p_{0(1)} + \sum_{k=1}^{n} p_{1\,(k)} \right), \qquad p_{1\,(n+1)} = P_{vac} - p_{0\,(n+1)}.$$

From a normalization condition $P_{serv} + p_{1\,(n+1)} + \sum\limits_{k=1}^{n} p_{2\,(k)} = 1$ we can calculate probability of presence of queue during work of system: $\sum\limits_{k=1}^{n} p_{2\,(k)} = 1 - P_{serv} - p_{1\,(n+1)}$. The average length of queue is equal

$$\bar{r} = 1 - P_{serv} = \frac{e_\nu \pi_{1(n)} + e_\mu - \frac{1-\pi_{0(1)}}{\lambda}}{e_\lambda \pi_{0(1)} + e_\nu \pi_{1(n)} + e_\mu}.$$

Using the formula for $\bar{r}$, we find average waiting time

$$\bar{t}_r = \lim_{T\to\infty} \frac{\bar{r}\,T}{N_{serv}(T)} = e_\mu + e_\nu \pi_{1(n)} - \frac{1-\pi_{0(1)}}{\lambda}.$$

In case of $m = 1$ the system of the algebraic equations (5) with a normalization condition takes simple form

$$\pi_{0(1)} = \alpha_0 \left(\pi_{0(1)} + \sum_{k=1}^{n-1} \pi_{1(k)} \right) + \alpha_0 \beta_0 \pi_{1(n)}\,;$$

$$\pi_{1(1)} = (1 - \alpha_0\beta_0)\,\pi_{1(n)}\,; \qquad \pi_{1(2)} = (1-\alpha_0)\left(\pi_{0(1)} + \pi_{1(1)}\right)\,;$$

$$\pi_{1(i)} = (1-\alpha_0)\,\pi_{1(i-1)} \quad \left(i = \overline{3,\,n}\right)\,; \qquad \pi_{0(1)} + \sum_{k=1}^{n} \pi_{1(k)} = 1,$$

and we can write solutions of this system in the compact form:

$$\pi_{0(1)} = \frac{1 - (1-\alpha_0\beta_0)(1-\alpha_0)^{n-1}}{(1-\alpha_0)^{n-1}} \cdot \pi_{1(n)}; \qquad \pi_{1(1)} = (1-\alpha_0\beta_0)\pi_{1(n)};$$

$$\pi_{1(k)} = \frac{\pi_{1(n)}}{(1-\alpha_0)^{n-k}} \quad \left(k = \overline{2,\,n-1}\right); \qquad \pi_{1(n)} = \frac{\alpha_0(1-\alpha_0)^{n-1}}{1-(1-\alpha_0)^n}.$$

4. Examples of the solutions of system (5) for $m \geq 2$. In case of $m = n = 2$ the system of the algebraic equations (5) with a normalization condition takes the form

$$\pi_{0(1)} = \alpha_0 \left(\pi_{0(1)} + \pi_{1(1)}\right) + \alpha_0\beta_0\pi_{1(2)}\,;$$

$$\pi_{1(1)} = (\alpha_1\beta_0 + \alpha_0\beta_1)\,\pi_{1(2)} + \alpha_0\beta_0\pi_{2(2)}\,;$$

$$\pi_{2(1)} = (1-\alpha_0(\beta_0+\beta_1)-\alpha_1\beta_0)\,\pi_{1(2)} + (1-\alpha_0\beta_0)\,\pi_{2(2)}\,;$$

$$\pi_{1(2)} = \alpha_1\left(\pi_{0(1)}+\pi_{1(1)}\right) + \alpha_0\pi_{2(1)}\,;$$

$$\pi_{2(2)} = (1-\alpha_0-\alpha_1)\left(\pi_{0(1)}+\pi_{1(1)}\right) + (1-\alpha_0)\,\pi_{2(1)}\,;$$

$$\pi_{0(1)}+\pi_{1(1)}+\pi_{2(1)}+\pi_{1(2)}+\pi_{2(2)} = 1.$$

Solving this system, we receive

$$\pi_{0(1)} = \alpha_0^2\left((\alpha_0-\alpha_1-2)\,\beta_0 + \left(\alpha_0-\alpha_0^2-\alpha_1\right)\beta_0^2 - \alpha_0\beta_1\right)/Z_{22};$$

$$\pi_{1(1)} = \alpha_0\left((1-\alpha_0)\,(\alpha_0-\alpha_1-1)\,\beta_0 + \left(\alpha_0^2-\alpha_0+\alpha_1\right)\alpha_0\beta_0^2 - (1-\alpha_0)\,\alpha_0\beta_1\right)/Z_{22};$$

$$\pi_{2(1)} = \left(\alpha_0\,(1+\beta_0+\alpha_1\beta_1) + \beta_0\left(\alpha_1^2-\alpha_0^2\right)-1\right)/Z_{22};$$

$$\pi_{1(2)} = \alpha_0\left(\alpha_0\,(1+\beta_0) - \beta_0\left(\alpha_0^2+\alpha_1\right)-1\right)/Z_{22};$$

$$\pi_{2(2)} = \left(\alpha_0\,(2+\alpha_1\beta_1) + \alpha_0^2\,(\beta_0\,(\alpha_1-1)-\beta_1-1) + \alpha_0^3\,(\beta_0+\beta_1) + \alpha_1^2\beta_0 - 1\right)/Z_{22};$$

$$Z_{22} = 2\alpha_0\,(1-\alpha_1\beta_0+\alpha_1\beta_1) + \alpha_0^2\,(\beta_0\,(\alpha_1-1)-2\beta_1) + \alpha_0^3\beta_1 + 2\alpha_1^2\beta_0 - 2.$$

If $m=2,\ \ n=3$, then from (5) we receive such system:

$$\pi_{0(1)} = \alpha_0\left(\pi_{0(1)}+\pi_{1(1)}+\pi_{1(2)}\right) + \alpha_0\beta_0\pi_{1(3)}\,;\pi_{1(1)} =$$

$$= (\alpha_1\beta_0+\alpha_0\beta_1)\,\pi_{1(3)} + \alpha_0\beta_0\pi_{2(3)}\,;$$

$$\pi_{2(1)} = (1-\alpha_0(\beta_0+\beta_1)-\alpha_1\beta_0)\,\pi_{1(3)} + (1-\alpha_0\beta_0)\,\pi_{2(3)}\,;$$

$$\pi_{1(2)} = \alpha_1\left(\pi_{0(1)}+\pi_{1(1)}\right) + \alpha_0\pi_{2(1)}\,;$$

$$\pi_{2(2)} = (1-\alpha_0-\alpha_1)\left(\pi_{0(1)}+\pi_{1(1)}\right) + (1-\alpha_0)\,\pi_{2(1)}\,;$$

$$\pi_{1(3)} = \alpha_1\pi_{1(2)} + \alpha_0\pi_{2(2)}\,;\pi_{2(3)} = (1-\alpha_0-\alpha_1)\,\pi_{1(2)} + (1-\alpha_0)\,\pi_{2(2)}\,;$$

$$\pi_{0(1)}+\pi_{1(1)}+\pi_{2(1)}+\pi_{1(2)}+\pi_{2(2)}+\pi_{1(3)}+\pi_{2(3)} = 1$$

with solutions

$$\pi_{0(1)} = \frac{\alpha_0^2\beta_0}{Z_{23}}\left(\alpha_0(3+4\alpha_1)-2+\alpha_0^3-3\alpha_1-2\alpha_1^2-\alpha_0^2(2+\alpha_1)\right)+$$

$$+\frac{\alpha_0^2}{Z_{23}}\left(\alpha_0\beta_1\left(2\alpha_0-\alpha_0^2-2\alpha_1-1\right)-\beta_0^2\left(\alpha_0^2-\alpha_0+\alpha_1\right)^2-1\right);$$

$$\pi_{1(1)} = \frac{\alpha_0\beta_0}{Z_{23}}\left(2\alpha_0(1+\alpha_1)^2+\alpha_0^3(2+\alpha_1)-2\alpha_0^2(1+2\alpha_1)-\alpha_1-\alpha_1^2-\alpha_0^4-1\right)+$$

$$+\frac{\alpha_0^2}{Z_{23}}\left(\beta_1\left(2\alpha_0(1+\alpha_1)+\alpha_0^3-2\alpha_0^2-\alpha_1-1\right)+\beta_0^2\left(\alpha_0^2-\alpha_0+\alpha_1\right)^2\right);$$

$$\pi_{2(1)} = \frac{1}{Z_{23}}\left(\alpha_1^3\beta_0-\alpha_0^3\beta_1-1-\alpha_0^2\left(\beta_0(2\alpha_1+1)+\beta_1(\alpha_1-1)\right)\right)+$$

$$+\frac{\alpha_0}{Z_{23}}\left((1+\beta_0)(1+\alpha_1)+\alpha_1^2(\beta_1-\beta_0\right);$$

$$\pi_{1(2)} = \frac{\alpha_0}{Z_{23}}\left(\alpha_0(1+\beta_0+\alpha_1\beta_0-\alpha_1\beta_1)-\alpha_1\beta_0(1+\alpha_1)-\right.$$

$$-\alpha_0^2(\beta_0+\alpha_1\beta_0-\beta_1)-\alpha_0^3\beta_1-1\big);$$

$$\pi_{2(2)}=\frac{1}{Z_{23}}\big(\alpha_0^2\big(\alpha_0(1+2\beta_0+\alpha_1\beta_1)++\beta_0(\alpha_1^2-\alpha_1-1)-\alpha_1\beta_1-2\big)+$$

$$+\beta_0(\alpha_1^3-\alpha_0^4)-1\big)+\frac{\alpha_0}{Z_{23}}\big(2+\alpha_1(1+\beta_0)+\alpha_1^2(\beta_1-\beta_0)\big);$$

$$\pi_{2(3)}=\frac{\alpha_0^3}{Z_{23}}\big(\alpha_0^2(\beta_0+\beta_1)+$$

$$+\alpha_0\left(\beta_0(\alpha_1-2)-2\beta_1-1\right)+2+\beta_0+\beta_1(1+2\alpha_1))+$$

$$+\frac{1}{Z_{23}}\big(\alpha_0^2\big(2\alpha_1^2\beta_0-\alpha_1(2+\beta_0+2\beta_1)-2\big)+$$

$$+\alpha_0\big(2+2\alpha_1+\alpha_1^2(\beta_1-\beta_0)\big)+\alpha_1^3\beta_0-1\big);$$

$$Z_{23}=\alpha_0^5\beta_1+\alpha_0^4\left(\beta_0(\alpha_1-1)-3\beta_1\right)+\alpha_0^3\left(1+(2-3\alpha_1)\beta_0+\beta_1(2+3\alpha_1)\right)+$$

$$+\alpha_0^2\left(\beta_0(3\alpha_1^2-1)-6\alpha_1\beta_1-2\right)+3\alpha_0\left(1+\alpha_1+\alpha_1^2(\beta_1-2\beta_0)\right)+3\alpha_1^3\beta_0-3.$$

For $m=3,\ n=2$ we have the system

$$\pi_{0(1)}=\alpha_0\left(\pi_{0(1)}+\pi_{1(1)}\right)+\alpha_0\beta_0\pi_{1(2)};$$

$$\pi_{1(1)}=\left(\alpha_1\beta_0+\alpha_0\beta_1\right)\pi_{1(2)}+\alpha_0\beta_0\pi_{2(2)};$$

$$\pi_{2(1)}=\left(\alpha_0\beta_2+\alpha_1\beta_1+\alpha_2\beta_0\right)\pi_{1(2)}+\left(\alpha_0\beta_1+\alpha_1\beta_0\right)\pi+\alpha_0\beta_0\pi_{3(2)};$$

$$\pi_{1(2)}=\alpha_1\left(\pi_{0(1)}+\pi_{1(1)}\right)+\alpha_0\pi_{2(1)};$$

$$\pi_{2(2)}=\alpha_2\left(\pi_{0(1)}+\pi_{1(1)}\right)+\alpha_1\pi_{2(1)}+\alpha_0\pi_{2(1)};$$

$$\pi_{3(2)}=\left(1-\alpha_0-\alpha_1-\alpha_2\right)\left(\pi_{0(1)}+\pi_{1(1)}\right)+\left(1-\alpha_0-\alpha_1\right)\pi_{2(1)}+\left(1-\alpha_0\right)\pi_{3(1)};$$

$$\pi_{0(1)}+\pi_{1(1)}+\pi_{2(1)}+\pi_{3(1)}+\pi_{1(2)}+\pi_{2(2)}+\pi_{3(2)}=1.$$

Solving this system, we receive

$$\pi_{0(1)}=\frac{\alpha_0^3\beta_0^2}{Z_{32}}\big(1+\alpha_0^2+4\alpha_1+\alpha_1^2-\alpha_0(2+\alpha_1+\alpha_2)+$$

$$+\beta_0\left(\alpha_0^2-\alpha_0^3+\alpha_1^2-\alpha_0(\alpha_1+\alpha_2)\right)\big)+$$

$$+\frac{\alpha_0^3}{Z_{32}}\left(\alpha_0^2\beta_1^2+\beta_0\left(1+\alpha_0\beta_1(2+\alpha_1)-\alpha_0^2(\beta_1+\beta_2)\right)\right);$$

$$\pi_{1(1)}=\frac{\alpha_0^2\beta_0^2}{Z_{32}}\left(\alpha_1(2+\alpha_1)-\alpha_0^3+\alpha_0^2(2+\alpha_1+\alpha_2)-\alpha_0(1+4\alpha_1+\alpha_1^2+\alpha_2\right)+$$

$$+\frac{\alpha_0^3}{Z_{32}}\left(\beta_0^3\left(\alpha_0(\alpha_1+\alpha_2)-\alpha_0^2+\alpha_0^3-\alpha_1^2\right)-(\alpha_0-1)\alpha_0\beta_1^2\right)+$$

$$+\frac{\alpha_0^2\beta_0(\alpha_0-1)}{Z_{32}}\left(\alpha_0^2(\beta_1+\beta_2)-\alpha_0\beta_1(1+\alpha_1)-1\right);$$

$$\pi_{2(1)}=\frac{\alpha_0}{Z_{32}}\left(\beta_0^2\left(\alpha_0^2-\alpha_0^3-\alpha_1^2(1+\alpha_1)-\alpha_0(\alpha_1+\alpha_2-\alpha_1\alpha_2)\right)-\right.$$

$$\left.-\alpha_0\beta_1(\alpha_0+\alpha_0\alpha_1\beta_1-1)\right)+$$

$$+\frac{\alpha_0\beta_0}{Z_{32}}\left(1+\alpha_1-\alpha_0(2+\alpha_1+\alpha_1\beta_1+\alpha_1^2\beta_1)+\alpha_0^2(1+\alpha_1\beta_2)\right);$$

$$\pi_{3(1)}=\frac{\alpha_0}{Z_{32}}\left(\alpha_1^2\beta_0^2(1-2\alpha_2)+\beta_0(\alpha_1^2+\alpha_1^3\beta_1-2\alpha_2-1)-3\alpha_1\beta_1-1\right)+$$

$$+\frac{\alpha_0^2}{Z_{32}}\left(\beta_0^2(\alpha_2+\alpha_2^2-\alpha_1)+\right.$$

$$\left.+\beta_0\left(2+\alpha_2+\alpha_1(\beta_1-\alpha_2\beta_1-1)-\alpha_1^2\beta_2\right)+2\alpha_1\beta_1+\alpha_1^2\beta_1^2-\beta_2\right)+$$

$$+\frac{1}{Z_{32}}\left(\alpha_0^3\left(\beta_0^2(2\alpha_1-1)-\beta_0(1+\beta_1-\alpha_2\beta_2)-\alpha_2\beta_1^2+\beta_2\right)+\right.$$

$$\left.+(\alpha_1^2\beta_0-1)^2+\alpha_0^4\beta_0(\beta_0+\beta_1)\right);$$

$$\pi_{1(2)}=\frac{\alpha_0^2\beta_0}{Z_{32}}\left(1+\alpha_0^2+2\alpha_1-\alpha_0(2+\alpha_1)\right)+$$

$$+\frac{\alpha_0^2}{Z_{32}}\left(\beta_0^2\left(\alpha_0^2-\alpha_0^3+\alpha_1^2-\alpha_0(\alpha_1+\alpha_2)\right)-\alpha_0\beta_1(\alpha_0-1)\right);$$

$$\pi_{2(2)}=\frac{\alpha_0}{Z_{32}}\left(1+\alpha_1\beta_0-\alpha_1^2\beta_0-\alpha_1^3\beta_0^2+\alpha_0^4\beta_0(\beta_0+\beta_1)+\right.$$

$$\left.+\alpha_0^3\left(\beta_0^2(\alpha_1-1)-\beta_0(1+\beta_1)+\beta_2\right)\right)+$$

$$+\frac{\alpha_0^2}{Z_{32}}\left(\alpha_1\alpha_2\beta_0^2-2\alpha_1\beta_1-1-\beta_0(1+2\alpha_1+\alpha_2+\alpha_1^2\beta_1)\right)+$$

$$+\frac{\alpha_0^3}{Z_{32}}\left(\alpha_2\beta_0^2+\alpha_1\beta_1+\beta_0(2+\alpha_2+\alpha_1\beta_1+\alpha_2\beta_1)-\beta_2\right);$$

$$\pi_{3(2)}=\frac{1}{Z_{32}}\left((\alpha_1^2\beta_0-1)^2-\alpha_0\left(2+2\alpha_2\beta_0+2\alpha_1^2\beta_0(\alpha_2\beta_0-1)+\right.\right.$$

$$\left.\left.+3\alpha_1\beta_1-\alpha_1^3\beta_0\beta_1\right)\right)+\frac{\alpha_0^4}{Z_{32}}\left(\alpha_0\left(\beta_0(\beta_2-\beta_1)-\beta_1^2\right)+\right.$$

$$\left.+\alpha_2\beta_0^2+(\beta_1-\beta_2)(1+\beta_0)+\beta_1^2-\alpha_1\beta_0(\beta_0+\beta_1)\right)+$$

$$+\frac{\alpha_0^2}{Z_{32}}\left(1+\alpha_2^2\beta_0^2+\beta_1(1+4\alpha_1)+\alpha_1^2\beta_1^2-\right.$$

$$\left.-\beta_2-\beta_0\left(\alpha_1^2\beta_2+\alpha_1(2+\alpha_2\beta_1)-2\alpha_2-1\right)\right)+$$

$$+\frac{\alpha_0^3}{Z_{32}}\left(\beta_0^2(\alpha_1^2+\alpha_2-\alpha_1)+(2+\alpha_1)\beta_1+(\alpha_1+\alpha_2)\beta_1^2-2\beta_2\right)+$$

$$+\frac{\alpha_0^3\beta_0}{Z_{32}}\left(1+\alpha_2(1+\beta_1-\beta_2)-\alpha_1(1+\beta_2)\right);$$

$$Z_{32}=2\alpha_0\left(1+2\alpha_2\beta_0+\alpha_1^2\beta_0(2\alpha_2\beta_0-1)-\alpha_1(\beta_0-3\beta_1)+\alpha_1^3\beta_0(\beta_0-\beta_1)\right)+$$

$$+\alpha_0^4\left(\alpha_2\beta_0^2+2\beta_1^2+\beta_0(\beta_1-\alpha_1\beta_1-2\beta_2)\right)+\alpha_0^5(\beta_0\beta_2-\beta_1^2)+$$

$$+\alpha_0^3\left(\beta_0^2(2\alpha_2-2\alpha_1+\alpha_1^2)+2\left(\beta_1-\beta_2+\beta_1^2(\alpha_1+\alpha_2)\right)-\right.$$

$$\left.-\beta_0\left(2\alpha_2\beta_2+2\alpha_1(\beta_1+\beta_2)-1\right)\right)+$$

$$+2\alpha_0^2\left(\beta_0^2(\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2)+\beta_1(1+2\alpha_1+\alpha_1^2\beta_1)-\beta_2\right)-$$

$$-2\alpha_0^2\beta_0\left(\alpha_1^2(\beta_1+\beta_2)+\alpha_1(2+\alpha_2\beta_1)-\alpha_2-1\right)+2\left(\alpha_1^2\beta_0-1\right)^2.$$

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