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EQUICONVERGENCE THEOREMS FOR EXPANSION BY THE ROOT-FUNCTIONS OF DIRAC OPERATOR

Abstract

An one-dimensional Dirac operator is considered on a finite interval $G=(a,b)$ Component-wise uniform equiconvergence with trigonometric Fourier series of expansions by the root vector-functions of Dirac operator is studied.

Basic notion and formulation of results

In the paper, by V.A.Il'in's method [1]-[3] we study expansion of an arbitrary two-component vector-function $f(x)$ in a biorthogonal series by the root vector-functions $\{u_n(x)\}_{n=1}^{\infty}$ of the Dirac operator

$$Du = B\frac{du}{dx} + P(x)\,u,$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P(x) = diag\,(p(x)\,, q(x))\,,$$

moreover, $p(x)$ and $q(x)$ are complex-valued functions determined on an arbitrary finite interval $G=(a,b)$.

Let $L_p^2(G)$, $p \geq 1$ be a space of two-component vector-functions $f(x) = (f_1(x)\,, f_2(x))^T$ with the norm

$$\|f\|_{p,2} = \left\{ \int_G \left(|f_1(x)|^2 + |f_2(x)|^2 \right)^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}.$$

In the case $p=\infty$, $\|f\|_{\infty,2} = \sup_{x\in G} vrai\,|f(x)|$.

For $f(x) \in L_p^2(G)$, $g(x) \in L_q^2(G)$, where $p^{-1}+q^{-1}=1$, $1 \leq p \leq \infty$ there exists

$$(f,g) = \int_G \sum_{j=1}^{2} f_j(x)\,\overline{g_j(x)}dx.$$

Following [1], we'll understand the root functions of the vector-function of the operator D irrespective to the kind of boundary conditions. Namely, under the eigen vector-function of the operator D responding to the complex eigen value λ we'll understand any identically non-zero complex-valued vector-function $\overset{0}{u}(x)$ that is absolutely continuous on any closed subinterval of the interval G and almost everywhere on G satisfies the equation $D\overset{0}{u} = \lambda \overset{0}{u}$.

Similarly, under the associated function of order $l, l \geq 1$ responding to the λ and eigen function $\overset{o}{u}(x)$ we'll understand any complex-function $\overset{l}{u}(x)$ that is absolutely continuous on any closed subinterval of the interval G and almost everywhere in G satisfies the equation $D\overset{l}{u} = \lambda\overset{l}{u} - \overset{l-1}{u}$.

Let the system $\{u_n(x)\}_{n=1}^{\infty}$ that corresponds to the system of eigen values $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n = \rho_n + i v_n$ be closed and minimal in $L_p^2(G)$, $p \geq 1$ and m_n be the order of the associated function $u_n(x)$.

Introduce the following partial sums:

$$\sigma_\nu(x, f) = \left(\sigma_\nu^1(x, f), \sigma_\nu^2(x, f)\right)^T, \quad S_\nu(x, f) = \left(S_\nu(x, f_1), S_\nu(x, f_2)\right)^T,$$

where

$$\sigma_\nu^i(x, f) = \sum_{|\rho_n| \leq \nu} (f, v_n)\, u_n^i(x), \quad S_\nu(x, f_i) = \frac{1}{\pi}\int_G \frac{\sin\nu(x-y)}{x-y} f_i(y)\, dy,$$

$i = 1, 2, \nu > 0$, $u_n(x) = \left(u_n^1(x), u_n^2(x)\right)^T$, $f(x) = (f_1(x), f_2(x))^T$, $\{v_n(x)\}_{n=1}^{\infty}$ is a biorthogonally conjugated system to $\{u_n(x)\}_{n=1}^{\infty}$.

Formulate the basic result of the paper.

Theorem. *Let $P(x) \equiv 0$ and $\{u_n(x)\}_{n=1}^{\infty}$ be an arbitrary closed in $L_p^2(G)$, $p \geq 1$ and a minimal system consisting of the root functions of the operator D. If the following conditions are fulfilled:*

1.

$$|Jm\lambda_n| \leq const, \quad n = 1, 2, ...; \tag{1}$$

2.

$$\sum_{\tau \leq \operatorname{Re}\lambda_n \leq \tau+1} 1 \leq const, \quad \forall\tau \in (-\infty, +\infty); \tag{2}$$

3. for any compact $K \subset G$ there exists a constant $C_0(K)$ such that

$$\|u_n\|_{p,2,K} \|v_n\|_{q,2} \leq C_0(K), \quad n = 1, 2, ..., \tag{3}$$

where $\|\cdot\|_{p,2,K} = \|\cdot\|_{L_p^2(K)}$, then for an arbitrary vector-function $f(x) \in L_p^2(G)$ the inequality

$$\lim_{\nu\to\infty} \sup_{x\in K} |\sigma_\nu(x, f) - S_\nu(x, f)| = 0, \tag{4}$$

is true, i.e. the i-th component of the expansion of the function $f(x)$ in a biorthogonal series by the system $\{u_n(x)\}_{n=1}^{\infty}$ uniformly equiconverges on any compact of the interval G with expansion in a trigonometric series corresponding to the i-th component $f_i(x)$ of the vector-function $f(x)$.

Corollary 1. I*f the conditions of the theorem are fulfilled, then component-wise localization principle is valid for biorthogonal expansion of an arbitrary function $f(x) \in L_p^2(G)$ by the system $\{u_n(x)\}_{n=1}^{\infty}$. Convergence or divergence of the i-th component of the indicated biorthogonal series in $x_0 \in G$ depends on the behavior* *in the small vicinity of the point x_0 only of the appropriate i-th component $f_i(x)$ of the expanded vector-function $f(x)$ (and it is independent of behavior of another component).*

Theorem 2. *Let $1 < p < \infty$ and all the conditions of theorem 1 be fulfilled. Then the equality*

$$\lim_{v\to\infty} \|\sigma_\nu(x,f) - f(x)\|_{p,2,K} = 0$$

is fulfilled for any compact $K \subset G$ and an arbitrary vector function $f(x) \in L_p^2(G)$, i.e. the system $\{u_n(x)\}_{n=1}^{\infty}$ possesses the basicity property in $L_p^2(G)$ on any compact $K \subset G$.

Theorem 3. *Let the functions $p(x)$ and $q(x)$ belong to the class $L_p(x)$, $p > 2$, condition (1) be fulfilled.*

$$m = \sup_n m_n < \infty, \tag{5}$$

and the system consisting of the root vector-functions of the operator D forms an unconditional basis in the space $L_2^2(G)$.

Then for any compact $K \subset G$ and arbitrary function $f(x) \in L_2^2(G)$ equality (4) is fulfilled, i.e. the i-th component of the expansion of the vector-function $f(x) \in L_2^2(G)$ in biorthogonal series by the system $\{u_n(x)\}$ uniformly equiconverges on any compact of the interval G with expansion in trigonometric series of the appropriate i-th component $f_i(x)$ of the vector-function $f(x)$.

Corollary 2. *Under the conditions of theorem 3, the component-wise localization principle is valid for biorthogonal expansion of an arbitrary function $f(x) \in L_2^2(G)$ by the system $\{u_n(x)\}_{n=1}^{\infty}$.*

Comparsion of theorem 3 with theorem 4 of the paper [4] leads to the following statement.

Theorem 4. *Let the functions $p(x)$ and $q(x)$ belong to the class $L_p(G)$, $p > 2$, conditions (1), (2) be fulfilled and*

$$\|u_n\|_{2,2} \|v_n\|_{2,2} \le c, \quad n = 1, 2, ..., \tag{6}$$

where $\{v_n(x)\}_{k=1}^{\infty}$ is a biorthogonally conjugated system to $\{u_n(x)\}_{n=1}^{\infty}$ in $L_2^2(G)$ consisting of the root functions of the formally conjugated operator $D^* = Bd/dx + \overline{P(x)}$.

Then equality (4) is fulfilled for any compact $K \subset G$ and arbitrary function $f(x) \in L_2^2(G)$ and the component-wise localization principle is valid for biorthogonal expansion of this function by the system $\{u_n(x)\}_{n=1}^{\infty}$.

Corollary 3. *Statements of theorem 4 are valid for the complete orthonormed in $L_2^2(G)$ system $\{u_n(x)\}$ of eigen functions of the self-adjoint Dirac operator with real-valued $p(x)$, $q(x) \in L_p(G)$, $p > 2$.*

Auxiliary statements

Statement 1. (a mean value formula). *If the matrix-function $P(x)$ belongs to the class $L_1^{loc}(G)$ and the points $x-t$, x, $x+t$ are in domain G, the formulae D*

$$\frac{\overset{l}{u}(x-t)+\overset{l}{u}(x+t)}{2}=\overset{(l)}{u}\cos\lambda t+\sum_{j=1}^{l}\frac{(-1)^j}{j!}t^j\cos\left(\lambda t+\frac{\pi}{2}j\right)\overset{l-j}{u}+$$

$$+\frac{1}{2}\sum_{j=0}^{l}\frac{(-1)^j}{j!}\int_0^t(t-r)^j\left\{\sin\left(\lambda(t-r)+\frac{\pi}{2}j\right)\times\right.$$

$$\times\left[P(x-r)\overset{l-j}{u}(x-r)+P(x+r)\overset{l-j}{u}(x+r)\right]+$$

$$\left.+B\cos\left(\lambda(t-r)+\frac{\pi}{2}j\right)\left[P(x+r)\overset{l-j}{u}(x+r)-P(x-r)\overset{l-j}{u}(x-r)\right]\right\}dr. \qquad (7)$$

is valid for the root vector-functions $\overset{l}{u}(x)$, $l\geq 0$ of the operator D.

Notice that formula (7) is derived from the formulae (2)-(3) of the paper [4]. There with, the mathematical induction method is applied.

Introduce the following parameter dependent integrals:

$$\Phi_{n1}^j(r,R,\nu)=\int_r^R(t-r)^j\frac{\sin\nu t}{t}\sin\left(\lambda_n(t-r)+\frac{\pi}{2}j\right)dt,$$

$$\Phi_{n2}^j(r,R,\nu)=\int_r^R(t-r)^j\frac{\sin\nu t}{t}\cos\left(\lambda_n(t-r)+\frac{\pi}{2}j\right)dt,$$

$0<r<R<\infty$, $v>0$, $\lambda_n=\rho_n+i\nu_n$, $j=\overline{0,m_n}$, $n\in N$.

Statement 2. *For any $a\in(0,1]$, the estimates*

$$\left|\Phi_{nk}^j\right|\leq C_j(\alpha,R)\,ch\,(2R|\nu_n|)\left\{\begin{array}{c}\left|\nu-|\rho_n|\right|^{-\alpha}r^{-\alpha}\ \ for\ \ \left|\nu-|\rho_n|\right|\geq 1\\ \max\{|\ln r|,|\ln R|\}\ \ for\ \ \left|\nu-|\rho_n|\right|<1\end{array}\right.. \qquad (8)$$

are fulfilled for the integrals $\Phi_{nk}^j(r,R,\nu)$, $k=1,2$.

Proof. Consider the case $k=1$, because for $k=2$ the proof of estimation (8) is performed quite similarly. For definiteness consider the case $j=2m$, $m=0,1,...,[m_n/2]$.

Let $\left|\nu-|\rho_n|\right|\geq 1$.

$$\left|\Phi_{n1}^j\right|\leq|\cos\lambda_n r|\left|\int_r^R(t-r)^{2m}\frac{\sin\nu t}{t}\sin\lambda_n t dt\right|+$$

$$+|\sin\lambda_n r|\left|\int_r^R(t-r)^{2m}\frac{\sin\nu t}{t}\cos\lambda_n t dt\right|=$$

$$= \left|A^1_{2m}\left(r,R,\nu,\lambda_n\right)\right|\left|\cos\lambda_n r\right| + \left|A^2_{2m}\left(r,R,\nu,\lambda_n\right)\right|\left|\sin\lambda_n r\right| \qquad (9)$$

Transform the integral A^1_{2m}. Obviously,

$$A^1_{2m} = \int\limits_r^R (t-r)^{2m}\frac{\sin\nu t}{t}\sin\rho_n t ch\nu_n t dt +$$

$$+i\int\limits_r^R (t-r)^{2m}\frac{\sin\nu t}{t}\cos\rho_n t sh\nu_n t dt = \int\limits_r^R (t-r)^{2m}\frac{\sin\nu t}{t}\sin\rho_n t dt +$$

$$+\sum_{k=1}^{\infty}\frac{\nu_n^{2k}}{2\,(2k)!}\int\limits_r^R t^{2k-1}(t-r)^{2m}\left(\cos\left(\nu-\rho_n\right)t-\cos\left(\nu+\rho_n\right)t\right)dt +$$

$$+i\sum_{k=1}^{\infty}\frac{\nu_n^{2k-1}}{2\,(2k-1)!}\int\limits_r^R t^{2k-2}(t-r)^{2m}\left(\sin\left(\nu-\rho_n\right)t-\sin\left(\nu+\rho_n\right)t\right)dt =$$

$$= \int\limits_r^R (t-r)^{2m}\frac{\sin\nu t}{t}\sin\rho_n t dt + B^1_m + iB^2_m. \qquad (10)$$

Conducting integration by parts in the expressions B^1_m and B^2_m,we get

$$\left|B^k_m\right| \le C\left(R,m\right)ch\left(R\nu_n\right)\left|\nu-\left|\rho_n\right|\right|^{-1},\ k=1,2. \qquad (11)$$

Obviously,

$$\int\limits_r^R (t-r)^{2m}\frac{\sin\nu t}{t}\sin\rho_n t dt = \frac{1}{2}\int\limits_r^R (t-r)^{2m}\frac{\cos\left(\nu-\rho_n\right)t}{t}dt -$$

$$-\frac{1}{2}\int\limits_r^R (t-r)^{2m}\frac{\cos\left(\nu+\rho_n\right)t}{t}dt.$$

Integrating by parts and applying the inequality $\left|\sin x\right| \le \left|x\right|^{1-\alpha}$, $\alpha\in(0,1]$, we get

$$\left|\int\limits_r^R (t-r)^{2m}\frac{\cos\left(\nu\pm\rho_n\right)t}{t}dt\right| \le \left|\frac{\sin\left(\nu\pm\rho_n\right)t}{\left(\nu\pm\rho_n\right)t}(t-r)^{2m}\Big|_r^R\right| +$$

$$+\left|\int\limits_r^R\frac{\sin\left(\nu\pm\rho_n\right)t}{\nu\pm\rho_n}\left(t^{-2}(t-r)^{2m}+2m(t-r)^{2m-1}t^{-1}\right)dt\right| \le$$

$$\le \begin{cases} C\left(m,R\right)\left|\nu-\left|\rho_n\right|\right|^{-1} \text{ for } m=\overline{1,[m_n/2]}, \\ C\left(\alpha\right)\left|\nu-\left|\rho_n\right|\right|^{-\alpha}r^{-\alpha} \text{ for } m=0. \end{cases}$$

Considering this and estimation (11) in (10), we find

$$\left|A^1_{2m}\right| \le \begin{cases} C(R,m)\, ch\,(R\nu_n)\, |\nu - |\rho_n||^{-1} \text{ for } m=\overline{1,[m_n/2]}, \\ C(R,\alpha)\, ch\,(R\nu_n)\, |\nu - |\rho_n||^{-\alpha}\, r^{-\alpha} \text{ for } m=0. \end{cases}$$

It is proved similarly that

$$\left|A^2_{2m}\right| \le \begin{cases} C(R,m)\, ch\,(R\nu_n)\, |\nu - |\rho_n||^{-1} \text{ for } m=\overline{1,[m_n/2]}, \\ C(R,\alpha)\, ch\,(R\nu_n)\, |\nu - |\rho_n||^{-\alpha}\, r^{-\alpha} \text{ for } m=0. \end{cases}$$

Consequently, it follows from inequality (9) that

$$\left|\Phi^j_{n1}\right| \le ch\,(2R\nu_n) \begin{cases} C(R,m)\, |v - |\rho_n||^{-1} \text{ for } m=\overline{1,[m_n/2]}, \\ C(R,\alpha)\, |v - |\rho_n||^{-\alpha}\, r^{-\alpha} \text{ for } m=0. \end{cases}$$

The first part of estimation (8) for $\Phi^j_{n1}(r,R,\nu)$ is established.The second part of estimation (8) for the integral $\Phi^j_{n1}(r,R,v)$ follows from the inequality

$$\left| \int_r^R (t-r)^j \frac{\sin \nu t}{t} \sin\left(\lambda_n (t-r) + \frac{\pi}{2}\right) dt \right| \le$$

$$\le ch\,(R\nu_n) \int_r^R \frac{dt}{t} \le 2ch\,(R\nu_n) \max\left\{ |\ln R|, |\ln r| \right\}.$$

Statement 2 is proved.

Give two estimations related with Dirichlet's discontinuous factor [5], [2]:

$$\left| \frac{2}{\pi} \int_0^R \frac{\sin \nu t}{t} \cos \lambda_n t dt - \delta(\nu, \lambda_n) \right| \le C(R) \frac{ch\,(\nu_n R)}{1 + |\nu - |\rho_n||}, \tag{12}$$

$$\left| S_{R_0} \left[\frac{2}{\pi} \int_0^R \frac{\sin \nu t}{t} \cos \lambda_n t dt \right] - \delta(\nu, \lambda_n) \right| \le C(R_0) \frac{ch\,(2R_0\nu_n)}{1 + (\nu - |\rho_n|)^2}, \tag{13}$$

where

$$\delta(\nu, \lambda_n) = \begin{cases} 1 \text{ for } \nu > |\rho_n|, \\ 1/2 \text{ for } \nu = |\rho_n|, \\ 0 \text{ for } \nu < |\rho_n|, \end{cases}$$

$$S_{R_0}[f] = \frac{2}{R_0} \int_{R_{0/2}}^{R_0} f(R)\, dR, \quad \lambda_n = \rho_n + i\nu_n.$$

Denote

$$A^j(\nu, \lambda_n R) = \int_0^R t^{j-1} \sin \nu t \cos\left(\lambda_n t + \frac{\pi}{2} j\right) dt, \quad j = \overline{1, m_n}.$$

Integrating by parts, we easily prove

Statement 3. *For* $|\nu_n| \leq const$ *the estimation*

$$\left|A^j(\nu, \lambda_n, R)\right| \leq C_j(R)\left(1 + |\nu - |\rho_n||\right)^{-1}.$$

is valid for the integrals $A^j(v, \lambda_n, R)$.

Let $a = (a_1, a_2)^T, b = (b_1, b_2)^T$ be arbitrary vectors from E^2. Obviously, $ab^T = (a_i b_j)_{i,j=1}^2$, and for an arbitrary vector $c = (c_1, c_2)^T$ the inequality

$$\left|ab^T c\right| \leq |a|\,|b|\,|c|.$$

is fulled.

Consequently, the norm of the matrix ab^T doesn't exceed the quantity, i.e.

$$\left\|ab^T\right\|_{E^2 \to E^2} \leq |a|\,|b|. \tag{14}$$

Introduce the denotation

$$Q^{\pm}(P, u_{n-j}, x, r) = P(x+r)\,u_{n-j}(x+r) \pm P(x-r)\,u_{n-j}(x-r).$$

Proof of theorems 1-3

Proof of theorem 1. Let a fixed, connected compact $K \subset G$ and the numbers $R > 0$ and $R_0 > 0$ be such that $R_0/2 \leq R \leq R_0$, $2R_0 \leq dist(K, \partial G)$. Introduce for each $x \in K$ the function

$$\Psi(x, y, R, \nu) = \begin{cases} \frac{1}{\pi}\sin\nu(x-y)/(x-y) & \text{for } |x-y| \leq R \\ 0 \text{ for } |x-y| > R, \end{cases}$$

where $y \in G$, $\nu > 0$.

Denote $W(x, y, R, \nu) = diag(\psi, \psi)$;

$$\widehat{W}(R_0) = \widehat{W}(x, y, R_0, \nu) = S_{R_0}\left[W(x, y, R, \nu)\right];$$

$$W_n(R) = \int\limits_G W(x, y, R, \nu)\,u_n(y)\,dy, \quad x \in K.$$

Obviously, $\widehat{W}_n(R_0) = S_{R_0}[W_n(R)]$. Taking into account the definition of the matrix $W(x, y, R, \nu)$, the function $\psi(x, y, R, v)$ and the mean value formula (7) for $P(x) \equiv 0$, we find

$$\widehat{W}_n(R_0) = \delta(\nu, \lambda_n)\,u_n(x) + Q_n(x, \nu, R_0),$$

where

$$Q_n(x, \nu, R_0) = \left\{\frac{2}{\pi}S_{R_0}\left[\int\limits_0^R \frac{\sin\nu t}{t}\cos\lambda_n t dt\right] - \delta(\nu, \lambda_n)\right\}u_n(x) +$$

$$+\frac{2}{\pi}\sum_{j=1}^{m_n}\frac{(-1)^j}{j!}S_{R_0}\left[\int_0^R t^{j-1}\sin\nu t\cos\left(\lambda_n t+\frac{\pi}{2}j\right)dt\right]u_{n-j}(x).$$

Prove that the series

$$\sum_{n=1}^{\infty}\left\|\left\|Q_n(x,v,R_0)\overline{v_n(\cdot)}^T\right\|_{E^2\to E^2}\right\|_{L_q(G)} \tag{15}$$

converges and its sum is uniformly bounded with respect to $x\in K$, where $p^{-1}+q^{-1}=1$.

By inequalities (1), (13) and (14) we have

$$\sum_{n=1}^{\infty}\left\|\left\|\left\{\frac{2}{\pi}S_{R_0}\left[\int_0^R\frac{\sin\nu t}{t}\cos\lambda_n tdt\right]-\delta(\nu,\lambda_n)\right\}u_n(x)\overline{v_n(\cdot)}^T\right\|_{E^2\to E^2}\right\|_{L_q(G)}\le$$

$$\le\sum_{n=1}^{\infty}\left\|\left|\frac{2}{\pi}S_{R_0}\left[\int_0^R\frac{\sin\nu t}{t}\cos\lambda_n tdt\right]-\delta(\nu,\lambda_n)\right||u_n(x)||v_n(\cdot)|\right\|_{L_q(G)}\le$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{1+(v-|\rho_n|)^2}|u_n(x)|\,\|v_n\|_{q,2}. \tag{16}$$

Applying the estimation (see. [4])

$$|u_n(x)|\le C(K)(1+|Jm\lambda_n|)^{1/p}\|u_n\|_{p,2},x\in K, \tag{17}$$

and considering conditions (1), (2), (3), we establish convergence of the series in the right hand side of (16) and boundedness of its sum uniformly with respect to $x\in K$.

By inequalities (1), (14) and the following inequality (see [2])

$$\left|S_{R_0}\left[\int_0^R t^{j-1}\sin\nu t\cos\left(\lambda_n t+\frac{\pi}{2}j\right)dt\right]\right|\le$$

$$\le C(R_0)\left(1+(\nu-|\rho_n|)^2\right)^{-1},\quad j=\overline{1,m_n}.$$

we find

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\left\|\left\|\sum_{j=1}^{m_n}\frac{(-1)^j}{j!}S_{R_0}\left[\int_0^R t^{j-1}\sin\nu t\cos\left(\lambda_n t+\frac{\pi}{2}j\right)dt\right]\times\right.\right.$$

$$\left.\left.\times u_{n-j}(x)\overline{v_n(\cdot)}^T\right\|_{E^2\to E^2}\right\|_{L_q(G)}\le$$

$$\le C\sum_{n=1}^{\infty}\left(1+(\nu-|\rho_n|)^2\right)^{-1}\sum_{j=1}^{m_n}\frac{1}{j!}\left\|\left\|u_{n-j}(x)\overline{v_n(\cdot)}^T\right\|_{E^2\to E^2}\right\|_{L_q(G)}\le$$

$$\le C\sum_{n=1}^{\infty}\left(1+(\nu-|\rho_n|)^2\right)^{-1}\sum_{j=1}^{m_n}|u_{n-j}(x)|\,\|v_n\|_{q,2}. \tag{18}$$

Apply the anti-apriori estimation (see [4])

$$|u_{n-j}(x)| \leq C(K)(1+|Jm\lambda_n|)^{j+\frac{1}{p}} \|u_n\|_{p,2}, x \in K \tag{19}$$

and take into account conditions (1)-(3). As a result we get that right hand side of (18) is majorized by the convergent number series

$$C\sum_{k=1}^{\infty}\frac{1}{1+k^2}.$$

Consequently, series (15) converges and its sum has order $O(1)$ uniformly with respect to $x \in K$. Hence, it follows the convergence of each series

$$\sum_{n=1}^{\infty} Q_n^i(x,y,\nu,R_0)\overline{v_n^j}(y), \quad i,j=1,2. \tag{20}$$

in $L_q(G)$ in $y \in K$ is uniform with respect to $x \in K$ and their sums are respected uniformly with restricted to $x \in K$. Denote the sums of series (20) by $\Omega_{ij}(x,y,\nu,R_0)$, $i,j=1,2$, and consider the matrix $\Omega(x,y) = (\Omega_{ij}(x,y,v,R_0))_{i,j=1}^2$. It is easy to notice that

$$\int_G \Omega(x,y)\,u_n(y)\,dy =$$

$$= \int_G \left(\widehat{W}(R_0) - \sum_{|\mathrm{Re}\,\lambda_n|\leq\nu} \delta(\nu,\lambda_n)\,u_n(x)\,\overline{v_n(y)}^T\right) u_n(y)\,dy. \tag{21}$$

Let $f(x) \in L_p^2(G)$ be an arbitrary function. By the closeness of the system $\{u_n(x)\}_{n=1}^{\infty}$ we can represent the function $f(x)$ in the form

$$f(x) = \sum_{k=1}^{N(\varepsilon,f)} c_k u_k(x) + g(x), \quad \|g\|_{p,2} < \frac{\varepsilon}{C(\nu)},$$

where $C(v)$ is determined from the inequality

$$\left\|\left\|\Omega(x,\cdot) - \widehat{W}(R_0) + \sum_{|\mathrm{Re}\,\lambda_n|\leq\nu} \delta(\nu,\lambda_n)\,u_n(x)\,\overline{v_n(\cdot)}^T\right\|_{E^2\to E^2}\right\|_{L_q(G)} \leq$$

$$\leq C(\nu) = const\cdot\nu, \quad x \in K,$$

that follows from the definition of matrices $\Omega(x,y)$, $\widehat{W}(R_0)$ from conditions (1)-(3) and inequalities (14),(17).

Then, allowing for equality (21) we have

$$\left|\int_G \Omega(x,y)\,f(y)\,dy - \int_G \left(\widehat{W}(R_0) - \sum_{|\mathrm{Re}\,\lambda_n|\leq\nu} \delta(\nu,\lambda_n)\,u_n(x)\,\overline{v_n(y)}^T\right) f(y)\,dy\right| \leq$$

$$\leq C(\nu)\|g\|_{p,2} < \varepsilon$$

Consequently, for any $f(x) \in L_p^2(G)$, the equality

$$\int_G \Omega(x,y) f(y)\,dy =$$

$$= \int_G \left(\widehat{W}(R_0) - \sum_{|\operatorname{Re}\lambda_n| \leq \nu} \delta(\nu,\lambda_n) u_n(x) \overline{v_n(y)}^T \right) f(y)\,dy, \; x \in K$$

is valid.

Hence, by uniform with respect to $x \in K$ boundedness of the elements of the matrix $\Omega(x,y)$, definition of $\delta(v,\lambda_n)$, conditions 2), 3) we find

$$\left| \int_G \left(\widehat{W}(R_0) - \sum_{|\operatorname{Re}\lambda_n| \leq \nu} u_n(x) \overline{v_n(y)}^T \right) f(y)\,dy \right| \leq C(K)\|f\|_{p,2}. \tag{22}$$

Since

$$\left\| \widehat{W}(R_0) - W(x,y,R,\nu) \right\|_{E^2 \to E^2} \leq C(R_0),$$

from (22) we get

$$\sup_{x \in K} |\sigma_\nu(x,f) - S_\nu(x,f)| \leq C_1(K)\|f\|_{p,2}.$$

Allowing for belonging of $u_n(x)$ to $W_2^1(G)$ and closeness of the system $\{u_n(x)\}_{n=1}^\infty$ in $L_p^2(G)$, from this inequality we derive relation (4) in the standard way (see [3], p. 1870). Theorem 1 is proved.

Statement of theorem 2 follows from the statement of theorem 1 and basicity of the trigonometric system in $L_p(G)$, $1 < p < \infty$.

Proof of theorem 3. Since the system $\{u_n(x)\}_{n=1}^\infty$ forms an unconditional basis in $L_2^2(G)$, the systems $\left\{u_n(x)\|u_n\|_{2,2}^{-1}\right\}_{n=1}^\infty$ and $\left\{v_n(x)\|u_n\|_{2,2}\right\}_{n=1}^\infty$ form a Riesz basis in $L_2^2(G)$. We can represent the function $f(x) \in L_2^2(G)$ in the form

$$f(x) = \sum_{n=1}^\infty \left(f, v_n\|u_n\|_{2,2}\right) u_n(x)\|u_n\|_{2,2}^{-1}.$$

Fix an arbitrary, connected compact $K \subset G$ and the number R satisfying the condition $0 < 2R \leq dist(K,\partial G)$. We'll compare the partial sum $\sigma_\nu(x,f)$ with $\widetilde{S_\nu}(x,f) = \left(\widetilde{S_\nu}(x,f_1), \widetilde{S_{v\nu}}(x,f_2)\right)$, where

$$\widetilde{S_v}(x,f_i) = \frac{1}{\pi} \int_{|x-y| \leq R} \frac{\sin\nu(x-y)}{x-y} f_i(y)\,dy, \quad x \in K, \quad i = 1,2.$$

Since the difference $S_\nu(x,f_i) - \widetilde{S}_\nu(x,f_i)$ tends to zero uniformly with respect to $x \in K$ as $v \to +\infty$, for proving theorem 3 it suffices to establish estimation (4) for $\widetilde{S}_\nu(x,f)$.

Allowing for expansion of the function $f(x)$, for $\widetilde{S}_\nu(x,f)$ it holds the representation

$$\widetilde{S}_\nu(x,f)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(f,\upsilon_n\|u_n\|_{2,2}\right)\int_0^R\frac{u_n(x-t)+u_n(x+t)}{2\|u_n\|_{2,2}}\frac{\sin\nu t}{t}dt. \qquad (23)$$

Considering the mean value formula (7), transform the integral contained in the representation (23)

$$\frac{2}{\pi}\|u_n\|_{2,2}^{-1}\int_0^R\frac{u_n(x-t)+u_n(x+t)}{2}\frac{\sin\nu t}{t}dt=\frac{2}{\pi}\frac{u_n(x)}{\|u_n\|_{2,2}}\int_0^R\frac{\sin\nu t}{t}\cos\lambda_n t dt+$$

$$+\frac{2}{\pi}\sum_{j=1}^{m_n}\frac{(-1)^j}{j!}\frac{u_{n-j}(x)}{\|u_n\|_{2,2}}\int_0^R t^{j-1}\sin\nu t\cos\left(\lambda_n t+\frac{\pi}{2}j\right)dt+$$

$$+\frac{1}{\pi}\sum_{j=0}^{m_n}\frac{(-1)^j}{j!}\|u_n\|_{2,2}^{-1}\int_0^R\frac{\sin\nu t}{t}\int_0^t(t-r)^j\sin\left(\lambda_n(t-r)+\frac{\pi}{2}j\right)\times$$

$$\times\left[P(x-r)u_{n-j}(x-r)+P(x+r)u_{n-j}(x+r)\right]drdt+$$

$$+\frac{1}{\pi}\sum_{j=0}^{m_n}\frac{(-1)^j}{j!}\|u_n\|_{2,2}^{-1}B\int_0^R\frac{\sin\nu t}{t}\int_0^t(t-r)^j\cos\left(\lambda_n(t-r)+\frac{\pi}{2}j\right)\times$$

$$\times\left[P(x+r)u_{n-j}(x+r)-P(x-r)u_{n-j}(x-r)\right]drdt=$$

$$=\frac{u_n(x)}{\|u_n\|_{2,2}}\delta(\nu,\lambda_n)+\frac{u_n(x)}{\|u_n\|_{2,2}}\left[\frac{2}{\pi}\int_0^R\frac{\sin\nu t}{t}\cos\lambda_n t dt-\delta(v,\lambda_n)\right]+$$

$$+\frac{2}{\pi}\sum_{j=1}^{m_n}\frac{(-1)^j}{j!}\frac{u_{n-j}(x)}{\|u_n\|_{2,2}}A^j(\nu,\lambda_n,R)+$$

$$+\frac{1}{\pi}\sum_{j=0}^{m_n}\frac{(-1)^j}{j!}\|u_n\|_{2,2}^{-1}\left\{\int_0^R\left[P(x-r)u_{n-j}(x-r)+P(x+r)u_{n-j}(x+r)\right]\times\right.$$

$$\times\left(\int_r^R(t-r)^j\frac{\sin\nu t}{t}\sin\left(\lambda_n(t-r)+\frac{\pi}{2}j\right)dt\right)dr+$$

$$+B\int_0^R\left[P(x+r)u_{n-j}(x+r)-P(x-r)u_{n-j}(x-r)\right]\times$$

$$\left.\times\left(\int_r^R(t-r)^j\frac{\sin\nu t}{t}\cos\left(\lambda_n(t-r)+\frac{\pi}{2}j\right)dt\right)dr\right\}=$$

$$= \frac{u_n(x)}{\|u_n\|_{2,2}}\delta(\nu,\lambda_n) + \frac{u_n(x)}{\|u_n\|_{2,2}}\left[\frac{2}{\pi}\int_0^R \frac{\sin\nu t}{t}\cos\lambda_n t dt - \delta(\nu,\lambda_n)\right] +$$

$$+\frac{2}{\pi}\sum_{j=1}^{m_n}\frac{(-1)^j}{j!}\frac{u_{n-j}(x)}{\|u_n\|_{2,2}}A^j(\nu,\lambda_n,R) + \frac{1}{\pi}\sum_{j=0}^{m_n}\frac{(-1)^j}{j!}\|u_n\|_{2,2}^{-1}\times$$

$$\times\left\{\int_0^R Q^+(P,u_{n-j},x,r)\,\Phi_{n1}^j(r,R,\nu)\,dr + \right.$$

$$\left. +B\int_0^R Q^-(P,u_{n-j},x,r)\,\Phi_{n2}^j(r,R,\nu)\,dr\right\}.$$

Considering this in equality (23), taking into account $|\nu_n| \le const$ and inequality (12), we get

$$\left|\widetilde{S}_\nu(x,f) - \sigma_\nu(x,f)\right| \le \frac{1}{2}\sum_{|\rho_n|=v}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|\frac{|u_n(x)|}{\|u_n\|_{2,2}} +$$

$$+C(R)\sum_{n=1}^{\infty}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|\frac{|u_n(x)|}{(1+|\nu-|\rho_n||)\|u_n\|_{2,2}} +$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|\sum_{j=1}^{m_n}\frac{(-1)^j}{j!}\frac{|u_{n-j}(x)|}{\|u_n\|_{2,2}}\left|A^j(\nu,\lambda_n,R)\right| +$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|\sum_{j=0}^{m_n}\frac{1}{j!\,\|u_n\|_{2,2}}\times$$

$$\times\left|\int_0^R Q^+(P,u_{n-j},x,r)\,\Phi_{n1}^j(r,R,\nu)\,dr\right| + \frac{1}{\pi}\sum_{n=1}^{\infty}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|\sum_{j=0}^{m_n}\frac{1}{j!\,\|u_n\|_{2,2}}\times$$

$$\times\left|\int_0^R Q^-(P,u_{n-j},x,r)\,\Phi_{n2}^j(r,R,\nu)\,dr\right| = S_1+S_2+S_3+S_4+S_5. \qquad (24)$$

The sums $S_i, i=\overline{1,5}$ should be estimated. For estimating the sum S_1 we apply the Bessel inequality estimations (17), (1) and (2).

$$S_1 = \frac{1}{2}\sum_{\rho_n=\nu}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|\left|u_n(x)\|u_n\|_{2,2}^{-1}\right| \le$$

$$\le \frac{1}{2}\left(\sum_{\rho_n=\nu}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|^2\right)^{\frac{1}{2}}\left(\sum_{\rho_n=\nu}\left|u_n(x)^2\right|\|u_n\|_{2,2}^{-2}\right)^{\frac{1}{2}} \le$$

$$\le C(K)\|f\|_{2,2}\left(\sum_{\rho_n=\nu}1\right)^{1/2} \le C(K)\|f\|_{2,2}\,.$$

By the Bessel inequality, estimates (17), (1) and (2) we get

$$S_2 \le C(K)\|f\|_{2,2}\left(\sum_{n=1}^{\infty}\frac{|u_n(x)|^2\|u_n\|_{2,2}^{-2}}{(1+|\nu-|\rho_n||)^2}\right)^{1/2}\le$$

$$\le C(K)\|f\|_{2,2}\left(\sum_{n=1}^{\infty}(1+|\nu-|\rho_n||)^{-2}\right)^{1/2}\le$$

$$\le C(K)\|f\|_{2,2}\left(\sum_{k=0}^{\infty}\left(\frac{1}{1+k}\right)^2\sum_{k\le|\nu-|\rho_n||\le k+1}1\right)^{1/2}\le$$

$$\le C(K)\|f\|_{2,2}\left(\sum_{j=1}^{\infty}j^{-2}\right)^{1/2}\le C(K)\|f\|_{2,2}\,.$$

Applying statement 3, estimations (19), (1), (2) and the Bessel inequality, we get

$$S_3 \le C\sum_{n=1}^{\infty}\left|\left(f,\nu_n\|u\|_{2,2}\right)\right|\times$$

$$\times\left(\sum_{j=1}^{m_n}\frac{C(K)}{j!}(1+|Jm\lambda_n|)^{j+\frac{1}{2}}C_j(R)(1+|\nu-|\rho_n||)^{-1}\right)\le$$

$$\le C(K)\sum_{n=1}^{\infty}\left|\left(f,v_n\|u\|_{2,2}\right)\right|(1+|v-|\rho_n||)^{-1}\le C(K)\|f\|_{2,2}\,.$$

The sums S_4 and S_5 are estimated by a unique scheme. Therefore we estimate only the sum S_4. From the espression $Q^+(P,u_{n-j},x,r)$ allowing for (19) for $p=2$ we find that for $x\in K$, $0\le r\le R$

$$\left|Q^+(P,u_{n-j},x,r)\right|\le CQ(x,r)\|u_n\|_{2,2}\,. \tag{25}$$

where $Q(x,r)=|P(x-r)|+|q(x-r)|+|P(x+r)|+|q(x+r)|$.

Obviously, for $x\in K$

$$\left(\int_0^R Q^p(x,r)\,dr\right)^{\frac{1}{p}}\le C\left(\|p\|_p+\|q\|_p\right), \tag{26}$$

where $\|\cdot\|_p=\|\cdot\|_{L_p(G)}$.

Allowing for inequalities (25), (26), for the sum S_4 we find

$$S_4\le C\sum_{n=1}^{\infty}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|\sum_{j=0}^{m_n}\int_0^R Q(x,r)\,\Phi_{n1}^j(r,R,v)\,dr\le$$

$$\le C\sum_{n=1}^{\infty}\left|\left(f,v_n\|u_n\|_{2,2}\right)\right|\sum_{j=0}^{m_n}\left(\|p\|_p+\|q\|_p\right)\left(\int_0^R\left|\Phi_{n1}^j(r,R,v)\right|^q dr\right)^{1/q}\le$$

$$\leq C\sum_{n=1}^{\infty}\left|\left(f,v_n\left\|u_n\right\|_{2,2}\right)\right|\sum_{j=0}^{m_n}\left(\int_0^R\left|\Phi_{n1}^j\left(r,R,v\right)\right|^q dr\right)^{1/q}$$

Here we apply the Bessel inequality and statement 2 for $\alpha\in(1/2,(p-1)/p)$, $p>2$. As a result we have:

$$S_4\leq C\left\|f\right\|_{2,2}\left\{\sum_{|\nu-|\rho_n||<1}\left(\int_0^R\left(\max\left\{\left|\ln r\right|,\left|\ln R\right|\right\}\right)^q dr\right)^{2/q}+\right.$$

$$\left.+\sum_{|\nu-|\rho_n||\geq 1}\left(\int_0^R r^{-\alpha q}dr\right)^{2/q}\frac{1}{\left|\nu-|\rho_n|\right|^{2\alpha}}\right\}^{1/2}\leq$$

$$\leq C\left\|f\right\|_{2,2}\left\{\left(\int_0^R\left(\max\left\{\left|\ln r\right|,\left|\ln R\right|\right\}\right)^q dr\right)^{2/q}\sum_{|\nu-|\rho_n||\leq 1}1+\right.$$

$$\left.+\left(\int_0^R r^{-\alpha q}dr\right)^{2/q}\sum_{|\nu-|\rho_n||\geq 1}\left|\nu-|\rho_n|\right|^{-2\alpha}\right\}^{1/2}.$$

Since $0<\alpha q<1$, by condition (2) we get

$$S_4\leq C\left\|f\right\|_{2,2}\left\{1+\sum_{k=1}^{\infty}\sum_{|\nu-|\rho_n||\geq 1}\left|\nu-|\rho_n|\right|^{-2\alpha}\right\}\leq$$

$$\leq C\left\|f\right\|_{2,2}\left\{1+\sum_{k=1}^{\infty}\frac{1}{k^{2\alpha}}\sum_{k<|\nu-|\lambda_n||\leq k+1}1\right\}\leq C\left\|f\right\|_{2,2}.$$

The estimation

$$S_5\leq C\left\|f\right\|_{2,2}.$$

is established in the same way.

Consequently, by the obtained estimations for the sums S_i, $i=\overline{1,5}$ it follows that for any function $f(x)\in L_2^2(G)$, the estimation

$$\left|\widetilde{S}_\nu(x,f)-\sigma_\nu(x,f)\right|\leq C_1(K)\left\|f\right\|_{2,2}\qquad(27)$$

is fulfilled uniformly with respect to $x\in K$.

Derive relation (4) for $\widetilde{S}_\nu(x,f)$ from estimation (27). It follows from closeness of the system $\{u_n(x)\}$ in $L_2^2(G)$ that for any $\varepsilon>0$ there exist the constants C_k, $k=\overline{1,n(\varepsilon,f)}$ such that

$$\left\|f-\sum_{k=1}^{n(\varepsilon,f)}C_k u_k(x)\right\|_{2,2}<\frac{\varepsilon}{2C_1(K)}.$$

Denote

$$g(x)=\sum_{k=1}^{n(\varepsilon,f)}C_k u_k(x).$$

Then,

$$\left\|\widetilde{S}_\nu(\cdot,f)-\sigma_\nu(\cdot,f)\right\|_{C(K)}=$$

$$=\left\|\widetilde{S}_\nu(\cdot,f-g)+\widetilde{S}_\nu(\cdot,g)-\sigma_\nu(\cdot,f-g)-\sigma_\nu(\cdot,g)\right\|_{C(K)}\leq$$

$$\leq\left\|\widetilde{S}_\nu(\cdot,f-g)-\sigma_\nu(\cdot,f-g)\right\|_{C(K)}+\left\|\sigma_\nu(\cdot,g)-\widetilde{S}_\nu(\cdot,g)\right\|_{C(K)}.$$

Hence, by estimation (27) and equality $\sigma_\nu(f,g)=g(x)$, for sufficiently large ν we get

$$\left\|\widetilde{S}_\nu(\cdot,f)-\sigma_\nu(\cdot,f)\right\|_{C(K)}\leq C_1(K)\|f-g\|_{2,2}+$$

$$+\left\|g(x)-\widetilde{S}_\nu(\cdot,g)\right\|_{C(K)},\qquad g=(g_1,g_2)^T.$$

Difference $\widetilde{S}_\nu(x,g_i)-g_i(x)$, $i=1,2$, tends to zero as $\nu\to\infty$ uniformly with respect to $x\in K$ since $g_i(x)\in W_2^1(G)$, $i=1,2$. Consequently, for $\nu\geq\nu_0>0$ (ν_0 is a sufficiently large number)

$$\left\|\widetilde{S}_\nu(\cdot,f)-\sigma_\nu(\cdot,f)\right\|_{C(K)}\leq C_1(K)\frac{\varepsilon}{2C_1(K)}+\frac{\varepsilon}{2}=\varepsilon$$

Theorem 3 is proved.

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